The geometry of 2-dimensional indefinite improper affine spheres

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the affine normal vector field

Let $\phi : U \to \mathbb{R}^{N+1}$ be an immersion, where $\dim U = N$.

D is the canonical affine connection on $\mathbb{R}^{N+1}$,

ξ is a vector field transversal to $\phi(U)$.

ξ defines a connection ∇ on U and symmetric bilinear form h on TU by

$$D_X \phi_* Y = \phi_*(\nabla_X Y) + h(X, Y)\xi, \ \forall X, Y \in \mathcal{X}(U)$$

the affine normal vector field

h defines a volume element ν_h on U.

ξ defines a volume element $\phi^*\Theta_\xi$, where $\Theta_\xi(\cdot) = \det(\cdot, \xi)$

Theorem (Blaschke)

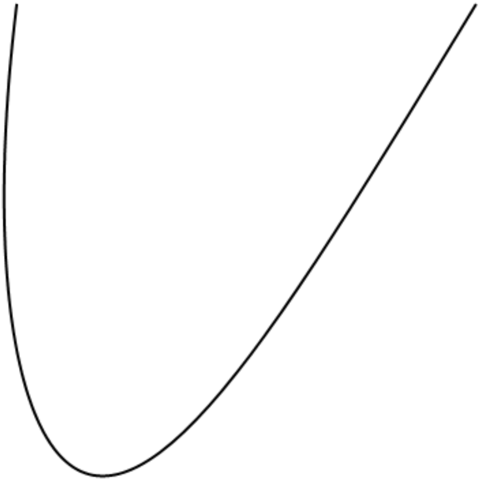
There exists a unique, up to sign, transversal vector field ξ such that $\nu_h = \phi^*\Theta_\xi$ and furthermore $\nabla(\phi^*\Theta_\xi) = 0$.

ξ is called the affine normal, or Blaschke normal vector field to the hypersurface $\phi(U) \subset \mathbb{R}^{N+1}$.

Definition

A hypersurface $\Phi(U) \subset \mathbb{R}^{N+1}$ is an affine sphere if its affine normal lines through each point of $\Phi(U)$ intersect at one point, called its center or else are mutually parallel.
The affine sphere is improper if affine normal vectors are mutually parallel i.e. the center is infinity. Otherwise the affine sphere is called proper.

Abel Transon's work on normal affine of a curve 1841.



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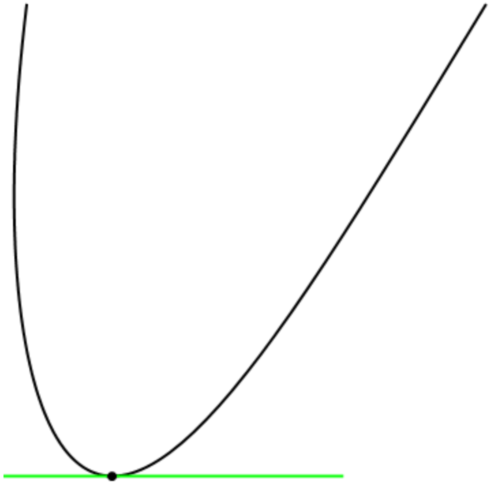

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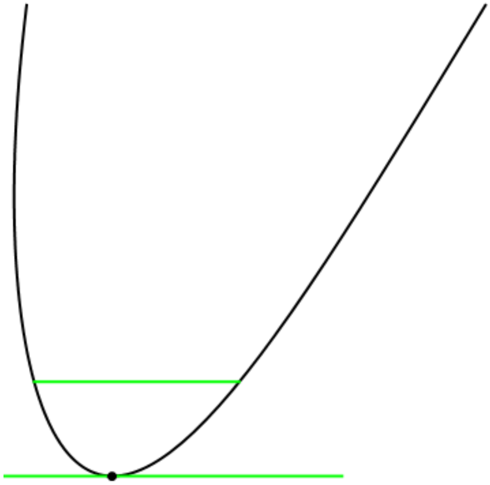

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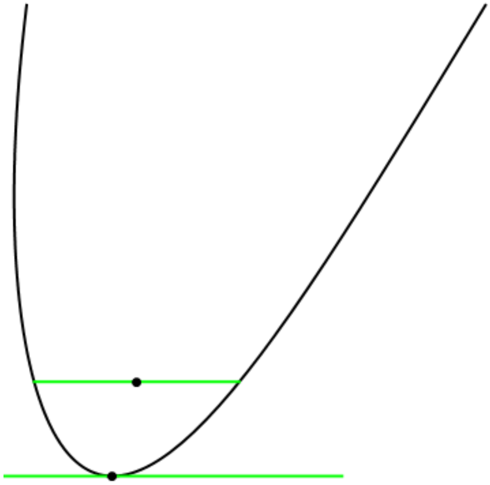

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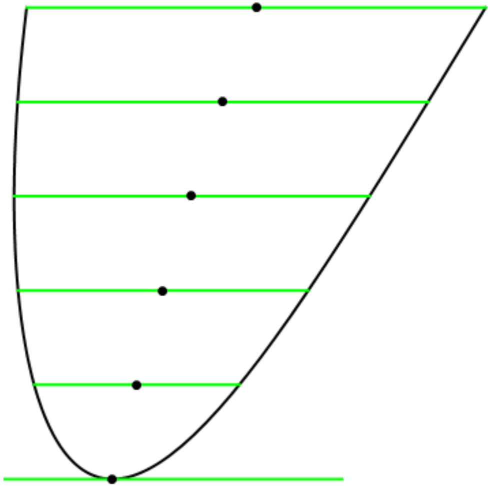

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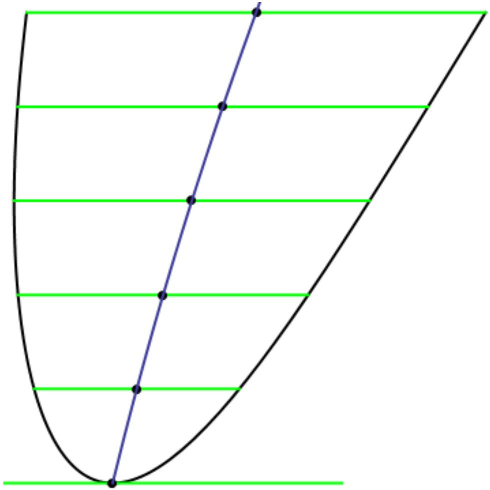
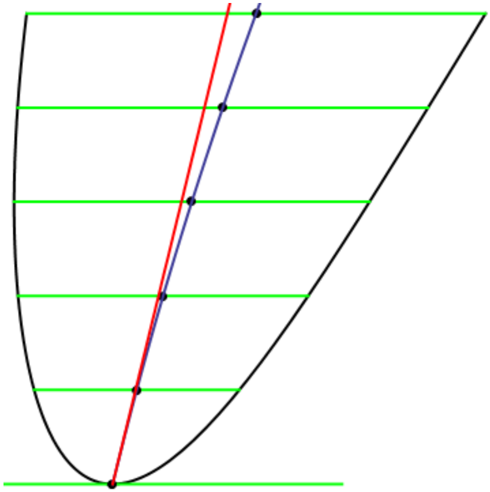


Abel Transon's work on the affine normal of a curve 1841.



Gheorghe Tzitzeica's works on affine spheres 1908-09.

Wilhelm Blaschke's book 1923.

Gaspard Monge's work 1784.

André-Marie Ampère work 1820.

Let $\xi = (0, \cdots, 0, 1)$ be a parallel vector field for the canonical connection D on $\mathbb{R}^{N+1}$.

Proposition

The graph of a smooth function $F : U \to \mathbb{R}$ ($U \subset \mathbb{R}^N$ open) is an improper affine sphere with the affine normal ξ iff

$$\det\left(\frac{\partial^2 F}{\partial x^2}\right) = c \tag{1}$$

Jörgens-Calabi-Pogorelov Theorem

Theorem

If $F : \mathbb{R}^N \to \mathbb{R}$ is a strictly convex smooth function satisfying

$$\det \left(\frac{\partial^2 F}{\partial x^2} \right) = 1 \tag{2}$$

then F is a quadratic polynomial.

Monge-Ampère equation in symplectic terms

Let V be an open subset of $\mathbb{R}^{2n}$ with the canonical symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dx_{i+n}$.

Let $F : V \to \mathbb{R}$ be a smooth function and let Y_F be the Hamiltonian vector field of F.

Proposition

F satisfies the classical Monge-Ampère equation (1) iff

$$\det(DY_F) = c, \tag{3}$$

where DY_F denotes the jacobian matrix of the map $x \mapsto Y_F(x)$.

Improper affine spheres in symplectic terms

Let $U \subset \mathbb{R}^{2n}$ be an open subset. Let an immersion $\phi : U \to \mathbb{R}^{2n+1}$ transversal to $\xi = (0, \cdots, 0, 1) \in \mathbb{R}^{2n} \times \mathbb{R}$, where the latter $\mathbb{R}^{2n}$ carries the symplectic form ω.

$\phi(r) = (x(r), f(r)) \in \mathbb{R}^{2n} \times \mathbb{R}$, where $x(r) \in V \subset \mathbb{R}^{2n}$ is locally invertible and $f(r) = F(x(r))$, for some $F : V \to \mathbb{R}$.

Let $Y_F(x)$ be the Hamiltonian vector field of F defined by equation and let $y(r) = Y_F(x(r))$.

$A(r) : T_r U \to T_r U$ is defined by

$$Dy(r) = Dx(r) \cdot A(r). \tag{4}$$

Proposition

$\phi(U)$ is an IAS with the affine normal $(0, ..., 0, 1)$ if and only if $\det A = c$, for some constant c.

The symplectic structructure of TV and contact structure of $TV \times \mathbb{R}$

Let V be an open subset of $\mathbb{R}^{2n}$ and let ω be the canonical symplectic form on V.

$$\flat : TV \ni v \mapsto \omega(v, \cdot) \in T^*V$$

Let α be the canonical Liouville 1-form on T^*V. $\Omega = \flat^* d\alpha$ is a symplectic form on TV and $\theta = dz + \flat^*\alpha$ is a contact form on $TV \times \mathbb{R}$, where z is a coordinate on $\mathbb{R}$.

Let $F : V \to \mathbb{R}$ be a smooth function. Let Y_F be the Hamiltonian vector field of F

Proposition

A map $\tilde{L} : V \ni x \mapsto (x, Y_F(x), F(x)) \in TV \times \mathbb{R}$ is a Legendrian immersion to the contact space $(TV \times \mathbb{R}, \{\theta = 0\})$.

Center-chord improper affine spheres

Let U_1, U_2 be open subsets of $\mathbb{R}^n$, $s = (s_1, ..., s_n) \in U_1$ and $t = (t_1, ..., t_n) \in U_2$

Let $\beta : U_1 \to (\mathbb{R}^{2n}, \omega)$, $\gamma : U_2 \to (\mathbb{R}^{2n}, \omega)$ be Lagrangian embeddings, $\Lambda_1 = \beta(U_1)$, $\Lambda_2 = \gamma(U_2)$.

Define $x : U_1 \times U_2 \to \mathbb{R}^{2n}$ by $x(s,t) = \frac{1}{2}(\beta(s) + \gamma(t))$.

and $y : U_1 \times U_2 \to \mathbb{R}^{2n}$ by $y(s,t) = \frac{1}{2}(\gamma(t) - \beta(s))$,

β and γ are Lagrangian, so $\omega(x_{s_i}, y_{s_j}) = \omega(x_{t_i}, y_{t_j}) = 0$.

$$\omega(x_{s_i}, y_{t_j}) = \omega(\beta_{s_i}, \gamma_{t_j}) = \omega(\gamma_{t_j}, -\beta_{s_i}) = \omega(x_{t_j}, y_{s_i}),$$

There exists function $f : U_1 \times U_2 \to \mathbb{R}$ satisfying

$$f_{s_i} = \omega(x_{s_i}, y), \quad f_{t_i} = \omega(x_{t_i}, y), \text{for } i = 1, \cdots, n.$$

Center-chord improper affine spheres

$$K_{2n} = \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} \tag{5}$$

Theorem

Assume that the tangent spaces of Λ_1 at $\beta(s)$ and of Λ_2 at $\gamma(t)$ are transversal.
Then the immersion $\phi(s,t) = (x(s,t), f(s,t))$ is an immersion with $A(r) = A(s,t) = K_{2n}$. As a consequence, $\Sigma^{2n} = Image(\phi) \subset \mathbb{R}^{2n+1}$ is an improper affine sphere with Blaschke normal $\xi = (0, \cdots, 0, 1)$

Special improper affine spheres

Let U be open subset of $\mathbb{C}^n$. Let $H : U \to \mathbb{C}$ be a holomorphic map

$$H(z) = P(s,t) + iQ(s,t), \tag{6}$$

with $z = s + it$, $z = (z_1, .., z_n)$, $s = (s_1, .., s_n)$, $t = (t_1, .., t_n)$.

$\frac{\partial Q}{\partial s} = (\frac{\partial Q}{\partial s_1}, ..., \frac{\partial Q}{\partial s_n})$ $\frac{\partial Q}{\partial t} = (\frac{\partial Q}{\partial t_1}, ..., \frac{\partial Q}{\partial t_n})$.

We define

$$x(s,t) = (x^{(1)}(s,t), x^{(2)}(s,t)) = \left(s, \frac{\partial Q}{\partial t}\right)$$

$$y(s,t) = (y^{(1)}(s,t), y^{(2)}(s,t)) = \left(t, \frac{\partial Q}{\partial s}\right).$$

$$f(s,t) = Q(s,t) - \sum_{k=1}^{n} t_k \cdot \frac{\partial Q}{\partial t_k}(s,t).$$

Special improper affine spheres

$$J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{7}$$

Theorem

At points (s,t) such that

$$\det(\frac{\partial^2 Q}{\partial t^2}) \neq 0$$

the map $\phi(s,t) = (x(s,t), f(s,t))$ is an immersion with $A(r) = A(s,t) = J_{2n}$.
$\Sigma^{2n} = Image(\phi) \subset \mathbb{R}^{2n+1}$ is an improper affine sphere with the affine normal $\xi = (0, \cdots, 0, 1)$.

Other examples

Take $n = 2$ and

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

$f(x_1, x_2, x_3, x_4) = x_1 x_3 + x_2 x_4 + x_2 x_3$ and since A is not similar to K_4 or J_4, (x, f) is neither center-chord nor special.

If one considers the product of a center-chord IAS with a special IAS, one obtains a new IAS.

Theorem (M. Craizer, W.D., P. de M. Rios)

Any germ of a simple stable Legendrian singularity is realizable as a center-chord IAS.

Theorem (M. Craizer, W.D., P. de M. Rios)

Any germ of a simple stable Legendrian singularity is realizable as a special IAS.

2-dimentional indefinite IAS

Let $\gamma, \beta \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ be regular periodic curves. Let

$$X(u,v) := \frac{\gamma(u)+\beta(v)}{2}, \quad Y(u,v) := \frac{\gamma(u)-\beta(v)}{2}.$$

and

$$f(u,v) = \frac{1}{4}\left(\int_0^u \det\left(\gamma'(t), \gamma(t)-\beta(0)\right)dt + \int_0^v \det\left(\beta'(t), \gamma(u)-\beta(t)\right)dt\right).$$

For the pair (γ, β) we define the following map:

$$\Psi \colon \mathbb{R}\times\mathbb{R} \ni (u,v) \mapsto \left(X(u,v), f(u,v)\right) \in \mathbb{R}^2 \times \mathbb{R}. \qquad (8)$$

Proposition

The map Ψ *defined in* (8) *is a parametrization of an improper affine sphere. We will call it a center-chord or indefinite improper affine sphere.*

2-dimentional indefinite IAS

$$\xi = (0,0,1),$$

$$\Gamma^1_{11} = \frac{\det\left(\gamma''(u), \beta'(v)\right)}{\det\left(\gamma'(u), \beta'(v)\right)}, \ \Gamma^2_{11} = \frac{\det\left(\gamma'(u), \gamma''(u)\right)}{\det\left(\gamma'(u), \beta'(v)\right)},$$

$$\Gamma^1_{22} = \frac{\det\left(\beta''(v), \beta'(v)\right)}{\det\left(\gamma'(u), \beta'(v)\right)}, \ \Gamma^2_{22} = \frac{\det\left(\gamma'(u), \beta''(v)\right)}{\det\left(\gamma'(u), \beta'(v)\right)},$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{12} = \Gamma^2_{21} = 0,$$

$$h = -\frac{1}{4}\det\left(\gamma'(u), \beta'(v)\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\omega_h = \frac{1}{4}\left|\det\left(\gamma'(u), \beta'(v)\right)\right| \, \mathrm{d}u \, \mathrm{d}v.$$

Singular points

Proposition

The center-chord improper affine sphere of the pair (γ, β) *is singular at* (u, v) *if and only if* $\gamma'(u), \beta'(v)$ *are parallel.*
A singular point (u, v) *is degenerate if and only if both* $\gamma(u), \beta(v)$ *are inflection points.*
A nondegenerate singular point (u, v) *is*

- *an intrinsic cuspidal edge (an* A_2*-point) if and only if* $\kappa_\gamma(u) \neq \operatorname{sgn}\left(\gamma'(u) \cdot \beta'(v)\right) \kappa_\beta(v)$.
- *an intrinsic swallowtail (an* A_3*-point) if and only if* $\kappa_\gamma(u) = \operatorname{sgn}\left(\gamma'(u) \cdot \beta'(v)\right) \kappa_\beta(v)$, *and* $\kappa'_\gamma(u) \neq \kappa'_\beta(v)$, *where* $'$ *denote the derivative with respect to the corresponding arc length parameter.*

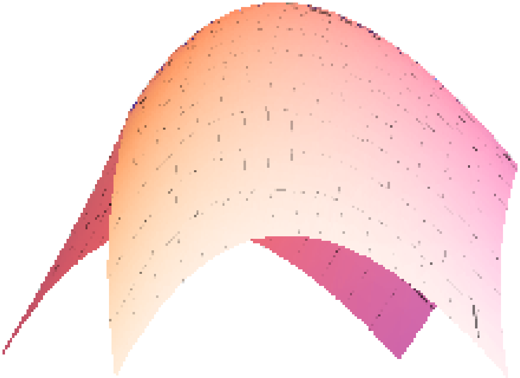
Singular curvatures

Proposition

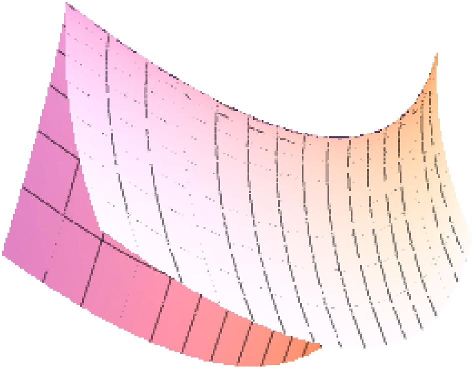
Let γ, β be arclength parametrized regular curves. Let κ denote the curvature of a space curve $\hat{\gamma} = \Psi \circ \sigma$ (the image of the singular curve). If (u, v) is a A_2-point then

$$\begin{aligned}
\kappa_s(u,v) &= \operatorname{sgn}\left(\gamma''(u) \cdot \beta''(v)\right) \kappa(u,v), \\
\kappa_s(u,v) &= \kappa(u,v) \text{ if } (u,v) \in \Sigma^+, \\
\kappa_s(u,v) &= -\kappa(u,v) \text{ if } (u,v) \in \Sigma^-.
\end{aligned}$$

Positive cuspidal egdes Σ^+



Negative cuspidal egdes Σ^-



Proposition

The following formulas hold:

$$K(u,v) = -\frac{16}{\left(4+|\gamma(u)-\beta(v)|^2\right)^2},$$

$$K\,\mathrm{d}\hat{A} = -\frac{2\det\left(\gamma'(u),\beta'(v)\right)}{\left(4+|\gamma(u)-\beta(v)|^2\right)^{3/2}}\,\mathrm{d}u\,\mathrm{d}v,$$

$$K\,\mathrm{d}A = -\frac{2\left|\det\left(\gamma'(u),\beta'(v)\right)\right|}{\left(4+|\gamma(u)-\beta(v)|^2\right)^{3/2}}\,\mathrm{d}u\,\mathrm{d}v,$$

Genericity conditions for a pair of curves

$\gamma_1 \sqcup \gamma_2 \in \mathcal{G}$ satisfies the following conditions:

- both γ_1 and γ_2 are regular curves with only non-degenerate inflection points and no undulation points, and only transversal self-crossings,
- γ_1, γ_2 have only transversal crossings,
- if $\gamma_1(s_1), \gamma_2(s_2)$ is a parallel pair, then $\gamma_1(s_1)$ or $\gamma_2(s_2)$ is not an inflection point,
- if $\gamma_1(s_1), \gamma_2(s_2)$ is a parallel pair, the points $\gamma_1(s_1), \gamma_2(s_2)$ are not inflection points, and $\kappa_{\gamma_1}(s_1) = -\mathrm{sgn}\left(\gamma_1'(s_1) \cdot \gamma_2'(s_2)\right) \kappa_{\gamma_2}(s_2)$, then $\kappa_{\gamma_1}'(s_1) \neq \kappa_{\gamma_2}'(s_2)$, where κ' denote the derivative of the curvature with respect to the corresponding arc length parameter.

- the point $\gamma_1(s_1) + \gamma_2(s_2)$ for any parallel pair $\gamma_1(s_1), \gamma_2(s_2)$ such that $\kappa_{\gamma_1}(s_1) = -\mathrm{sgn}\left(\gamma_1'(s_1) \cdot \gamma_2'(s_2)\right) \kappa_{\gamma_2}(s_2)$ does not coincide with the point $\gamma_1(u_1) + \gamma_2(v_1)$ for any other parallel pair.
- there are at most finitely many distinct pairs (and no triples etc.) of parallel pairs $\gamma_1(u_1), \gamma_2(v_1)$ and $\gamma_1(u_2), \gamma_2(v_2)$ such that $\gamma_1(u_1) + \gamma_2(v_1) = \gamma_1(u_2) + \gamma_2(v_2)$ and $\det\left(\gamma_1'(u_1), \gamma_1'(u_2)\right) \neq 0$.

Then $\mathcal{G}$ is a generic subset of $C^\infty(S_1^1 \sqcup S_2^1, \mathbb{R}^2)$ with Whitney C^∞ topology. Furthermore, IAS of a pair of curves γ_1, γ_2 such that $\gamma_1 \sqcup \gamma_2 \in \mathcal{G}$ has only cuspidal edges and swallowtails singularities.

Genericity conditions for a curve

γ in $\mathcal{G}$ satisfies the following conditions:

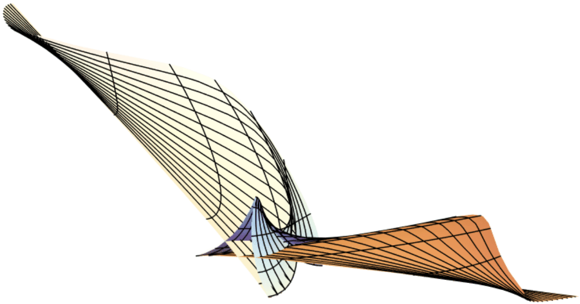
- γ is a regular curve with only non-degenerate inflexion points and no undulation points,
- γ has only transversal self crossings,
- if $\gamma(s_1), \gamma(s_2)$ is a parallel pair of γ, then $\gamma(s_1)$ or $\gamma(s_2)$ is not an inflexion point,
- if $\gamma(s_1), \gamma(s_2)$ is a parallel pair of γ, the points $\gamma(s_1), \gamma(s_2)$ are not inflexion points of γ, the dot product $\gamma'(s_1) \cdot \gamma'(s_2)$ is negative, and $\kappa_\gamma(s_1) = \kappa_\gamma(s_2)$, then $\kappa'_\gamma(s_1) \neq \kappa'_\gamma(s_1)$, where κ'_γ denote the derivative of the curvature with respect to the arc length parameter.

Genericity conditions for a curve (continuation)

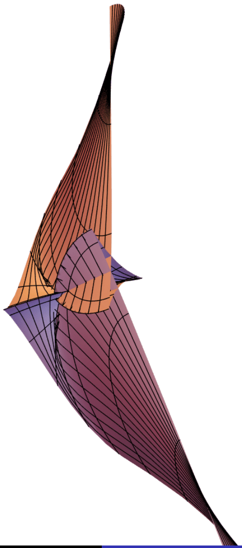
- the point $\gamma(s_1) + \gamma(s_2)$ for any parallel pair $\gamma(s_1), \gamma(s_2)$ such that $\kappa_\gamma(s_1) = -\operatorname{sgn}(\gamma'(s_1) \cdot \gamma'(s_2))\, \kappa_\gamma(s_2)$ does not coincide with the point $\gamma(u_1) + \gamma(v_1)$ for any other parallel pair.
- there are at most finitely many distinct pairs (and no triples etc.) of parallel pairs $\gamma(u_1), \gamma(v_1)$ and $\gamma(u_2), \gamma(v_2)$ such that $\gamma(u_1) + \gamma(v_1) = \gamma(u_2) + \gamma(v_2)$ and $\det(\gamma'(u_1), \gamma'(u_2)) \neq 0$.

Then $\mathcal{G}$ is a generic subset of $C^\infty(S^1, \mathbb{R}^2)$ with Whitney C^∞ topology and IAS of $\gamma \in \mathcal{G}$ has only cuspidal edges, swallowtails and $A_{4/2}$-singularities

$A_{4/2}$ singularity - the Kowalczyk-Janeczko symmetric butterfly



$A_{4/2}$ singularity - the Kowalczyk-Janeczko symmetric butterfly



Let $\gamma, \beta \in C^{\infty}(\mathbb{R}, \mathbb{R}^2)$ be regular 2π-periodic curves. Let

$$\widetilde{\Psi}(p,q) = \left(\widetilde{X}(p,q), \widetilde{f}(p,q)\right),$$

where

$$\widetilde{X}(p,q) = X(p+q, p-q), \ \ \widetilde{f}(p,q) = f(p+q, p-q).$$

Proposition

$$\widetilde{\Psi}(p+2\pi, q) = \widetilde{\Psi}(p,q) + \frac{1}{2}\left(0, 0, \widetilde{A}_\gamma - \widetilde{A}_\beta\right),$$

$$\widetilde{\Psi}(p, q+2\pi) = \widetilde{\Psi}(p,q) + \frac{1}{2}\left(0, 0, \widetilde{A}_\gamma + \widetilde{A}_\beta\right),$$

where $\widetilde{A}_\gamma$ denotes the oriented area bounded by γ.

Corollary

If $\widetilde{A}_\gamma$ and $\widetilde{A}_\beta$ are incommensurable, then the set

$$\mathcal{D} = \Psi\left(\mathbb{R} \times \mathbb{R}\right) \cap \{\Psi(u_0, v_0) + t(0,0,1) | t \in \mathbb{R}\}$$

is dense in $\{\Psi(u_0, v_0) + t(0,0,1) | t \in \mathbb{R}\}$ for any $(u_0, v_0) \in \mathbb{R} \times \mathbb{R}$.

Periodicity

Let $\gamma \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ be a regular 2π-periodic curve and $\beta = \gamma$.

Proposition

$$
\begin{aligned}
\widetilde{\Psi}(p+2\pi, q) &= \widetilde{\Psi}(p, q), \\
\widetilde{\Psi}(p, q+2\pi) &= \widetilde{\Psi}(p, q) + \left(0, 0, \widetilde{A}_\gamma\right), \\
\widetilde{X}(p, q) &= \widetilde{X}(p, -q) \\
\widetilde{f}(p, q) &+ \widetilde{f}(p, -q) = 0 \\
\widetilde{X}(p, q) &= \widetilde{X}(p+\pi, \pi - q) \\
\widetilde{f}(p, q) &+ \widetilde{f}(p+\pi, \pi - q) = \frac{1}{2} A_\gamma
\end{aligned}
$$

A center-chord IAS

Assume that $\alpha(u) = (\cos(u), \sin(u))$, i.e., L is the unit circle in the plane.

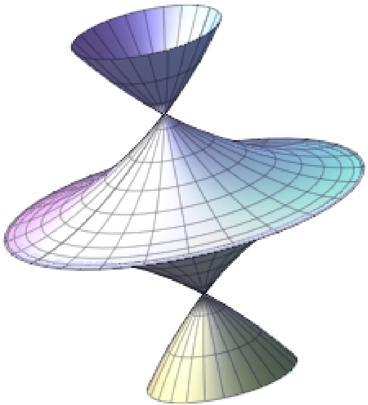
Then

$$X(u,v) = \cos\left(\frac{u-v}{2}\right)\left(\cos\left(\frac{u+v}{2}\right), \sin\left(\frac{u+v}{2}\right)\right)$$

$$Y(u,v) = \sin\left(\frac{u-v}{2}\right)\left(-\sin\left(\frac{u+v}{2}\right), \cos\left(\frac{u+v}{2}\right)\right)$$

$$f(u,v) = \frac{1}{4}\left(v - u + \sin(u-v)\right).$$

A center-chord IAS



A special IAS

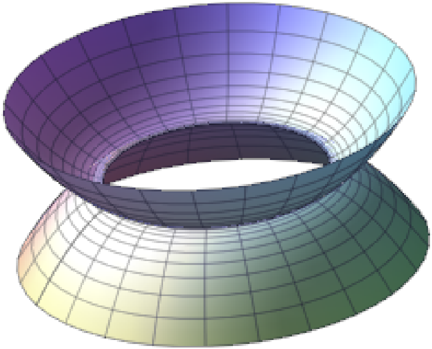
Consider $\gamma(z) = (\cos(z), \sin(z))$. Then L is the unit circle in the plane and

$$X(s,t) = \cosh(t)\,(\cos(s), \sin(s))\,,$$

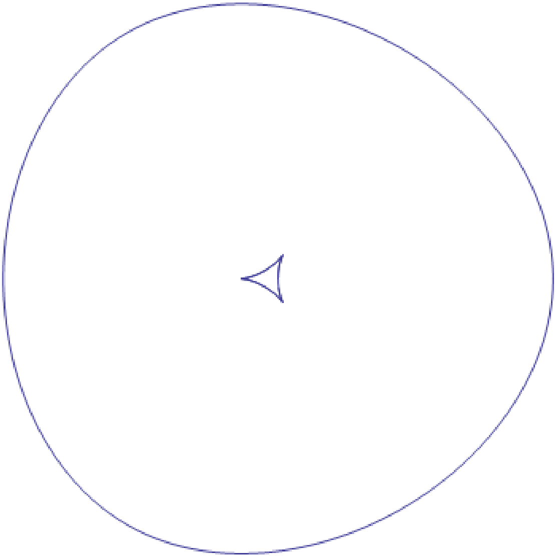
$$Y(s,t) = \sinh(t)\,(-\sin(s), \cos(s))\,,$$

$$f(s,t) = \frac{1}{4}\,(\sinh(2t) - 2t)\,.$$

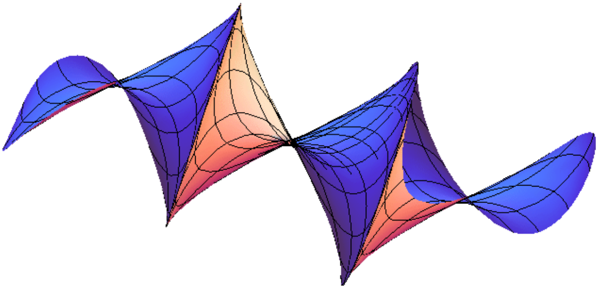
A special IAS



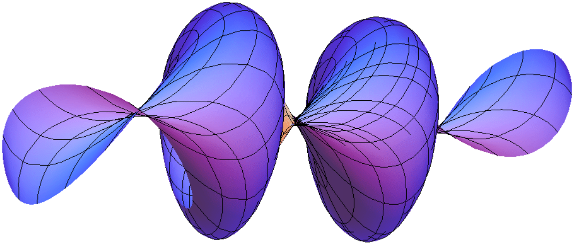
A center-chord IAS



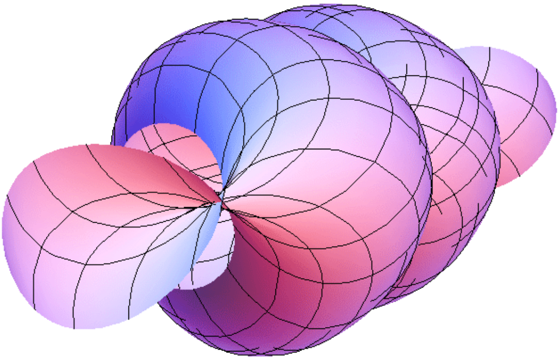
A center-chord IAS



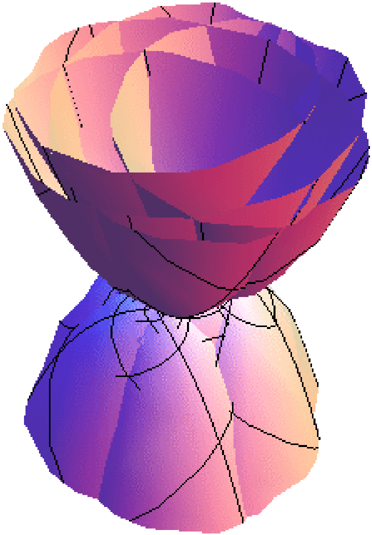
A center-chord IAS



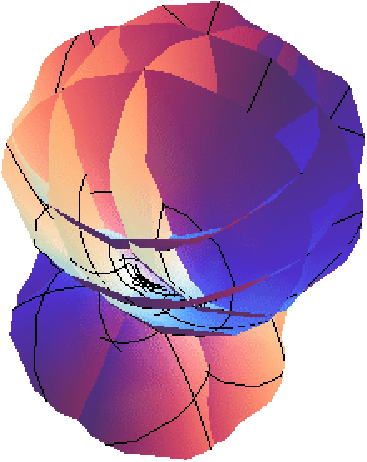
A center-chord IAS



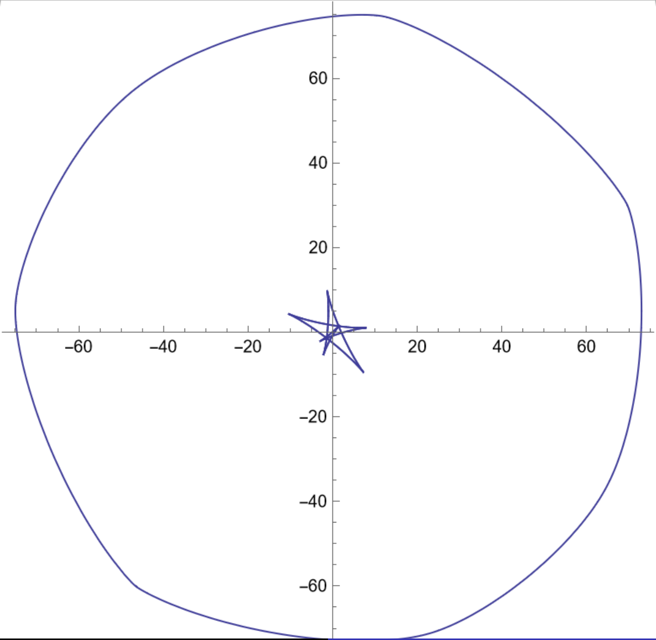
A special IAS



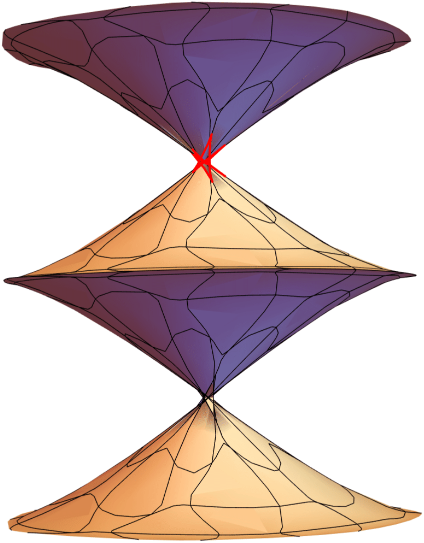
A special IAS



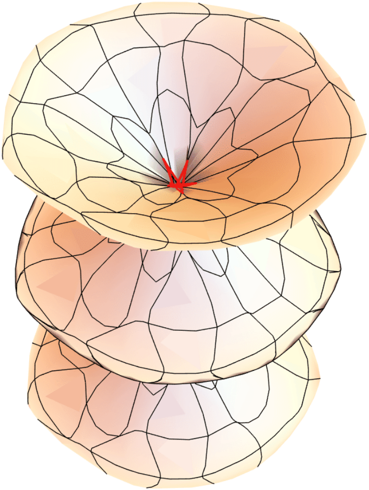
A center-chord IAS



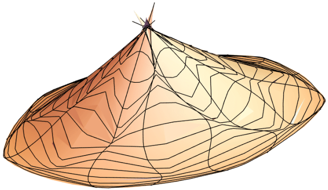
A center-chord IAS



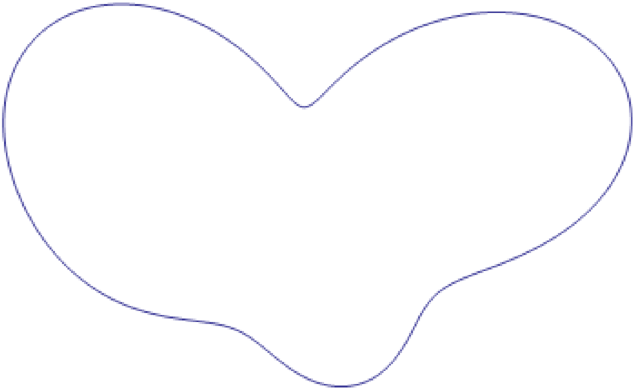
A center-chord IAS



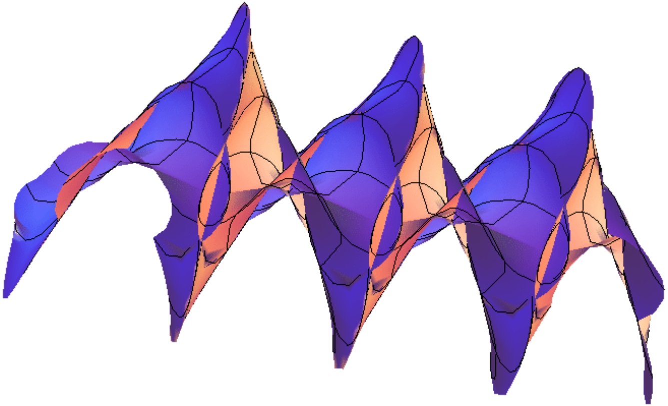
A center-chord IAS



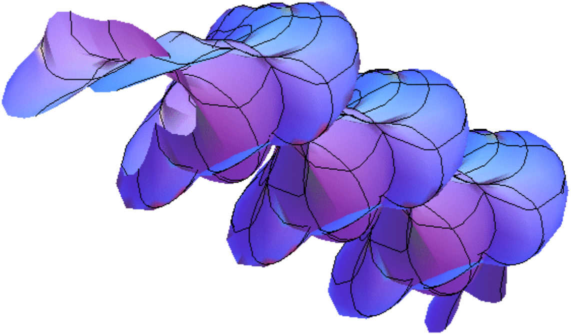
A nonconvex curve



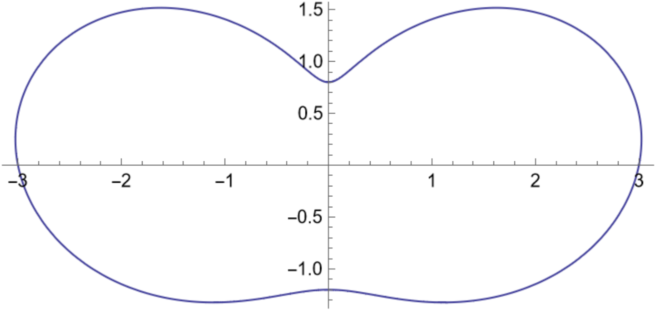
A center-chord IAS



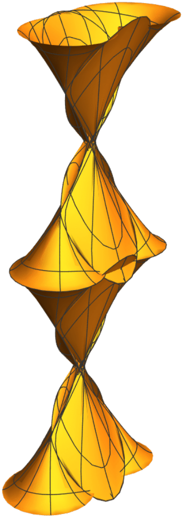
A center-chord IAS



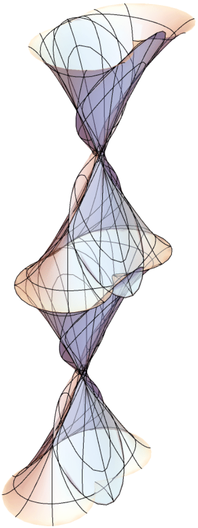
A center-chord IAS



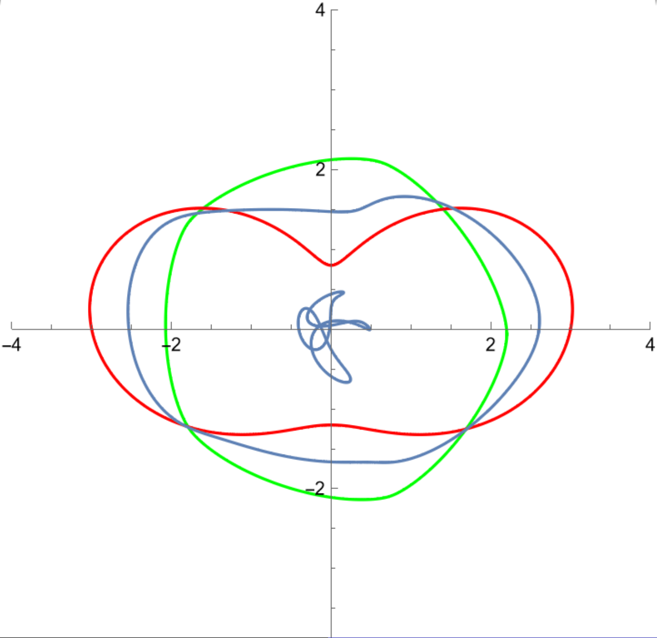
A center-chord IAS



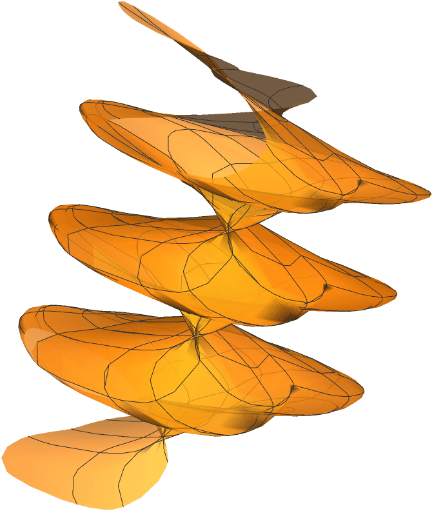
A center-chord IAS



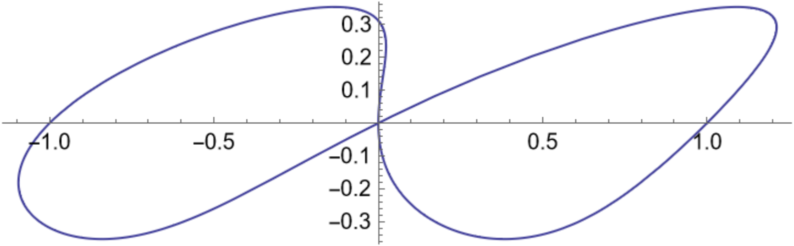
A center-chord IAS



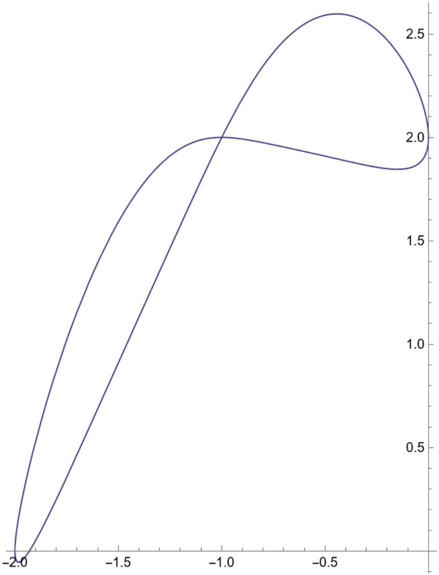
A center-chord IAS



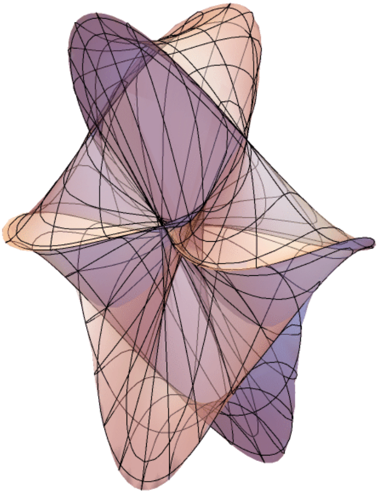
A center-chord IAS



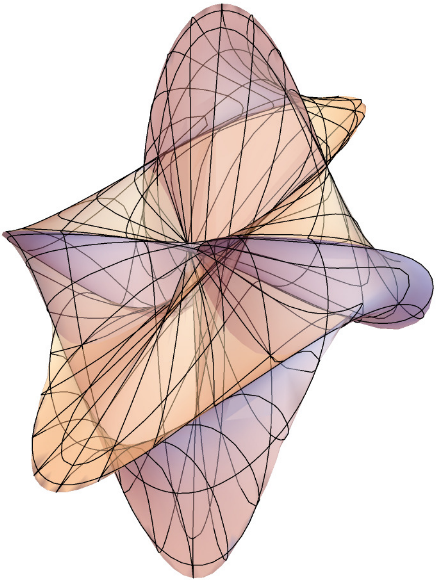
A center-chord IAS



A center-chord IAS



A center-chord IAS



Gauss-Bonnet formulas

Theorem

Let Φ be a parametrisation CC IAS whose set of singular points Σ admits at most peaks. If $M \subset \mathbb{R} \times \mathbb{R}$ be a closed bounded region enclosed by a curve transversal to Σ, then

$$2\pi\chi(M) =$$

$$\int_M K dA + 2\int_{\Sigma^+} \kappa d\tau - 2\int_{\Sigma^-} \kappa d\tau$$

$$+ \int_{\partial M \cap M^+} \hat{\kappa}_g d\tau - \int_{\partial M \cap M^-} \hat{\kappa}_g d\tau - \sum_{p \in null(\Sigma \cap \partial M)} (2\alpha_+(p) - \pi),$$

$$\int_M K d\hat{A} + \int_{\partial M} \hat{\kappa}_g d\tau =$$

$$2\pi\left(\chi(M^+) - \chi(M^-)\right) + 2\pi\left(\# P^+ - \# P^-\right)$$

$$+\pi\left(\#(\Sigma \cap \partial M)^+ - \#(\Sigma \cap \partial M)^-\right) + \pi\left(\# P_{\partial M^+} - \# P_{\partial M^-}\right).$$

Gauss-Bonnet formulas

$d\tau$ is the arc length measure,
Σ^+ (respectively Σ^-) is the positively (respectively negatively) curved cuspidal edges of Σ,
P^+ (respectively P^-) is the set of positive (respectively negative) peaks in $M \setminus \partial M$,
$(\Sigma \cap \partial M)^+$ (respectively $(\Sigma \cap \partial M)^-$, null$(\Sigma \cap \partial M)$) is the set of positive (respectively negative, null) singular points in $\Sigma \cap \partial M$,
$P_{\partial M^+}$ (respectively $P_{\partial M^-}$) is the set of peaks in the positive (respectively negative) boundary.

Gauss-Bonnet formulas

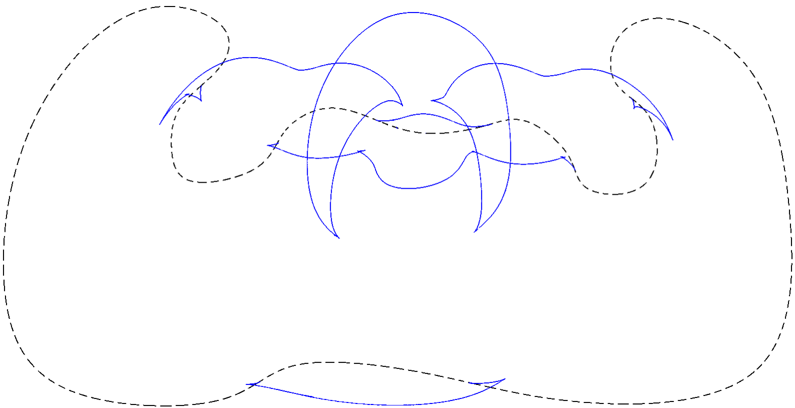
Theorem

Let γ be a generic curve. Then CCIAS satisfies the following formula

$$\int_M KdA + 2\int_{\Sigma\cap M^\circ} \kappa_s d\tau + 2\int_\gamma |\kappa_\gamma| ds = 0,$$

where s is the arc-length parameter of γ.

Infexion points on the Wigner caustic on shell



Rysunek: A curve M with 8 inflexion points (the dashed line) and branches of the Wigner caustic between inflexion points of M.

Inflexion points on the Wigner caustic on shell

Theorem (W.D., M. Zwierzyński)

Let M be a generic regular closed curve. Let $S^1 \ni s \mapsto f(s) \in \mathbb{R}^2$ be a parameterization of M and let C be a branch of the Wigner caustic which connects two inflexion points $f(t_1)$ and $f(t_2)$ of M. Then the number of inflexion points of C and the number of inflexion points of the arc $f\big((t_1, t_2)\big)$ are even.

Symplectic and contacts forms

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

is the canonical symplectic form in $\mathbb{R}^{2n}$.

$$\Omega = \sum_{i=1}^{n} dq_i \wedge d\dot{p}_i + d\dot{q}_i \wedge dp_i$$

is the canonical symplectic form in $T\mathbb{R}^{2n}$.

$$\theta = dz - \sum_{i=1}^{n} \dot{p}_i dq_i + \dot{q}_i dp_i \qquad (9)$$

is the canonical contact form in $T\mathbb{R}^{2n} \times \mathbb{R}$.

Lagrangian and Legendrian maps

$(x, y) : U \to (T\mathbb{R}^{2n}, \Omega)$ is a Lagrangian immersion,
$(x, y, f) : U \to (T\mathbb{R}^{2n} \times \mathbb{R}, \{\theta = 0\})$ is a Legendrian immersion,

$$\pi : T\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \ni (q, p, \dot{q}, \dot{p}) \mapsto (q, p) \in \mathbb{R}^{2n},$$

$$\tilde{\pi} : T\mathbb{R}^{2n} \times \mathbb{R} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R} \ni (q, p, \dot{q}, \dot{p}, z) \mapsto (q, p, z) \in \mathbb{R}^{2n} \times \mathbb{R}.$$

The Lagrangian map is

$$\pi \circ (x, y) : U \to \mathbb{R}^{2n}.$$

The Legendrian map is

$$\tilde{\pi} \circ (x, y, f) : U \to \mathbb{R}^{2n} \times \mathbb{R}.$$

Generating functions

A generating function of the Lagrangian submanifold $\mathcal{L}$ and the Legendrian submanifold $\tilde{\mathcal{L}}$ is a function

$$g : \mathbb{R}^n \times \mathbb{R}^n \ni (q, \dot{q}) \mapsto g(q, \dot{q}) \in \mathbb{R},$$

satisfying

$$\mathcal{L} = \{(q, p, \dot{q}, \dot{p}) : \frac{\partial g}{\partial q} = \dot{p}, \frac{\partial g}{\partial \dot{q}} = p\}. \tag{10}$$

and

$$\tilde{\mathcal{L}} = \{(q, p, \dot{q}, \dot{p}, z) : \frac{\partial g}{\partial q} = \dot{p}, \frac{\partial g}{\partial \dot{q}} = p, z = g(q, \dot{q}) - \dot{q} \cdot p\}. \tag{11}$$

Generating families

A generating family of the Lagrangian map $\pi \circ L$ and the Legendrian map $\tilde{\pi} \circ \tilde{L}$ is a function $G : \mathbb{R}^n \times \mathbb{R}^{2n} \ni (\beta, q, p) \mapsto G(\beta, q, p) \in \mathbb{R}$ satisfying

$$\mathcal{L} = \{(q, p, \dot{q}, \dot{p}) : \exists \beta : \frac{\partial G}{\partial \beta} = 0, \frac{\partial G}{\partial q} = \dot{p}, -\frac{\partial G}{\partial p} = \dot{q}\},$$

and

$$\tilde{\mathcal{L}} = \{(q, p, \dot{q}, \dot{p}, z) : \exists \beta : \frac{\partial G}{\partial \beta} = 0, \frac{\partial G}{\partial q} = \dot{p}, -\frac{\partial G}{\partial p} = \dot{q}, z = G(\beta, q, p)\}.$$

A generating family can be obtained from a generating function by

$$G(\beta, q, p) = g(q, \beta) - p \cdot \beta.$$

Generating families for center-chord IASs on shell

For a center-chord IAS, where L is defined by $(u, dS(u))$,

$$g_{cc}(q, \dot{q}) = \frac{1}{2}S(q + \dot{q}) - \frac{1}{2}S(q - \dot{q})$$

is a generating function and

$$G_{cc}(\beta, q, p) = \frac{1}{2}S(q + \beta) - \frac{1}{2}S(q - \beta) - p \cdot \beta.$$

is a generating family.

$G_{cc}(\beta, q, p)$ is an odd function of β.

Generating families for special IASs on shell

For a special IAS defined by the holomorphic function H taking $\mathbb{R}^n$ to $\mathbb{R}$,

$$g_{sp}(q, \dot{q}) = Q(q, \dot{q})$$

is a generating function on shell and the generating family for $\phi_{sp}(L)$ on shell is given by

$$G_{sp}(\beta, q, p) = Q(q, \beta) - p \cdot \beta, \tag{12}$$

where Q is the imaginary part of H.

If $H(\mathbb{R}^n) \subset \mathbb{R}$ then $\overline{H(z)} = H(\bar{z})$. It implies that $Q(q, -\beta) = -Q(q, \beta)$.

$G_{sp}(\beta, q, p)$ is an odd function of β.

Z_2-symmetric in fibers Lagrangian submanifolds

Definition

A Lagrangian submanifold L of $T\mathbb{R}^{2n}$ is $\mathbb{Z}_2$-*symmetric* in the fibers if for every point (κ, λ) in L the point $(-\kappa, \lambda)$ belongs to L. The Lagrangian map $\pi|_L : L \to \mathbb{R}^{2n}$ is $\mathbb{Z}_2$-*symmetric* if the Lagrangian submanifold L is $\mathbb{Z}_2$-symmetric in the fibers.

$\mathcal{D}_{k+m}$ is the group of diffeomorphism-germs $(\mathbb{R}^k \times \mathbb{R}^m, 0) \to (\mathbb{R}^k \times \mathbb{R}^m, 0)$, $\mathcal{D}_k^{odd}$ is the subgroup of odd diffeomorphism-germs $(\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ i.e. $\Phi \in \mathcal{D}_k^{odd}$ if $\Phi(-x) \equiv -\Phi(x)$.

Definition

Two odd generating family-germs F, G of $\mathbb{Z}_2$-symmetric Lagrangian submanifold-germs are *fibred $\mathcal{R}^{odd}$-equivalent* if there exists a odd (in variables) fibred diffeomorphism-germ $\Psi \in \mathcal{D}_{k+n}$ i. e.

$$\Psi(x, \lambda) \equiv (\Phi(x, \lambda), \Lambda(\lambda)) \text{ and } \Phi|_{\mathbb{R}^k \times \{\lambda\}} \in \mathcal{D}_k^{odd} \text{ for every } \lambda,$$

such that

$$F = G \circ \Psi.$$

Theorem on equivalence

Theorem

$\mathbb{Z}_2$-symmetric Lagrangian map-germs are $\mathbb{Z}_2$-symmetrically Lagrangian equivalent if and only if their odd generating families are fibred $\mathcal{R}^{odd}$-equivalent.

Singularities of odd functions

Definition

A smooth function-germ f at 0 on $\mathbb{R}^m$ is **even** if $f(-x) \equiv f(x)$ and it is **odd** if $f(-x) \equiv -f(x)$.

$\mathcal{E}_m^{even}$ is the ring of even smooth function-germs $f : (\mathbb{R}^m, 0) \to \mathbb{R}$

$\mathcal{E}_m^{odd}$ is the set of odd smooth function-germs $g : (\mathbb{R}^m, 0) \to (\mathbb{R}, 0)$, which has a module structure over $\mathcal{E}_m^{even}$.

Singularities of odd functions

Definition

A diffeomorphism-germ $\Phi : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is **odd** if $\Phi(-x) \equiv -\Phi(x)$. Denote by $\mathcal{D}_m^{odd}$ the group of odd diffeomorphism-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$.

Definition

Let $f, g \in \mathcal{E}_m^{odd}$. We say that f and g are $\mathcal{R}^{odd}$**-equivalent** if there exists $\Phi \in \mathcal{D}_m^{odd}$ such that $f = g \circ \Phi$.

$L\mathcal{R}^{odd}g$ is the tangent space to the $\mathcal{R}^{odd}$-orbit of g at g.

Singularities of odd functions

Proposition

Let $g \in \mathcal{E}_m^{odd}$. The tangent space $L\mathcal{R}^{odd}g$ to the $\mathcal{R}^{odd}$-orbit of g at g is the $\mathcal{E}_m^{even}$-module generated by $\left\{x_j \frac{\partial g}{\partial x_i} : i, j = 1, \cdots, m\right\}$.

Definition

A function-germ $F \in \mathcal{E}_{m+k}$ is an **odd deformation** of $f \in \mathcal{E}_m^{odd}$ if $F|_{\mathbb{R}^m \times \{0\}} = f$ and for any fixed $\lambda \in \mathbb{R}^k$ the function-germ $F|_{\mathbb{R}^m \times \{\lambda\}} \in \mathcal{E}_m^{odd}$. The space $\mathbb{R}^k$ is called the **base** of the odd deformation F and k is its **dimension**.

Singularities of odd functions

Theorem (W. D., Miriam Manoel, Pedro de M. Rios)

Let $g \in \mathcal{E}_m^{odd}$. Then
(a) A k-parameter deformation G of g is $\mathcal{R}^{odd}$-versal if and only if

$$\mathcal{E}_m^{odd} = \mathcal{E}_m^{even}\left\{x_j \frac{\partial g}{\partial x_i} : i, j \leq m\right\} + \mathbb{R}\left\{\frac{\partial G}{\partial \lambda_\ell}|_{\mathbb{R}^m \times \{0\}} : \ell \leq k\right\}.$$

(b) If $W \subset \mathcal{E}_m^{odd}$ is a finite dimensional vector space such that $\mathcal{E}_m^{odd} = L\mathcal{R}^{odd} g \oplus W$, and if $h_1, \ldots, h_s \in \mathcal{E}_m^{odd}$ is a basis for W, then $G(x, \lambda) \equiv g(x) + \sum_{j=1}^{s} \lambda_j h_j(x)$ is a $\mathcal{R}^{odd}$-miniversal deformation of g.

Singularities of odd functions

Definition

Odd deformations $F, G \in \mathcal{E}_{m+k}$ are **fibred $\mathcal{R}^{odd}$-equivalent** if there exists a fibred diffeomorphism-germ $\Psi \in \mathcal{D}_{m+k}$ s.t. $\Psi(x,\lambda) \equiv (\Phi(x,\lambda), \Lambda(\lambda))$, $\Phi|_{\mathbb{R}^m \times \{\lambda\}} \in \mathcal{D}_m^{odd}$, $\forall \lambda \in \mathbb{R}^k$, and $F = G \circ \Psi$.

$\mathcal{M}_m^{k(odd)}$ is the $\mathcal{E}_m^{even}$-submodule of $\mathcal{E}_m^{odd}$ generated by $x_1^{k_1} \cdots x_m^{k_m}, \forall k_1, \cdots, k_m \geq 0$, s.t. $k_1 + \cdots + k_m = k$.

Singularities of odd functions

Proposition

$g \in \mathcal{E}_m^{odd}$ is finitely $\mathcal{R}^{odd}$-determined if and only if $\mathcal{M}_m^{k(odd)} \subset L\mathcal{R}^{odd} g$ for some odd positive integer k .

Proposition

If $g \in \mathcal{E}_m^{odd}$ is a germ of a submersion than g is $\mathcal{R}^{odd}$-equivalent to the following germ $(x_1, \cdots, x_m) \mapsto x_1$.

Theorem

Let $g \in \mathcal{E}_m^{odd}$ with a singular point at 0. If $m \geq 3$, then g is not $\mathcal{R}^{odd}$-simple.

Singularities of odd functions

Theorem (W.D., Miriam Manoel, Pedro de M. Rios)

Let $g \in \mathcal{E}_1^{odd}$. Then g is $\mathcal{R}^{odd}$-simple if, and only if, g is $\mathcal{R}^{odd}$-equivalent to one of the following function-germs at 0:

$$A_{2k/2}: \ x \mapsto x^{2k+1}, \ \text{for } k = 1, 2, \cdots$$

Corollary

For $k \geq 1$, $\mathcal{R}^{odd}$-miniversal deformation of $A_{2k/2}$ is

$$G(x, \lambda_1, \cdots, \lambda_k) \ = \ x^{2k+1} + \sum_{j=1}^{k} \lambda_j x^{2j-1} \ .$$

Theorem (W.D., Miriam Manoel, Pedro de M. Rios)

Let $g \in \mathcal{E}_2^{odd}$. Then g is $\mathcal{R}^{odd}$-simple if, and only if, g is $\mathcal{R}^{odd}$-equivalent to one of the following function-germs at 0:

$$D^{\pm}_{2k/2}: \ (x_1, x_2) \mapsto x_1^2 x_2 \pm x_2^{2k-1}, \ \text{for } k = 2, 3, \cdots$$

$$E_{8/2}: \ (x_1, x_2) \mapsto x_1^3 + x_2^5,$$

$$J^{\pm}_{10/2}: \ (x_1, x_2) \mapsto x_1^3 \pm x_1 x_2^4,$$

$$E_{12/2}: \ (x_1, x_2) \mapsto x_1^3 + x_2^7.$$

Singularities of odd functions

The $\mathcal{R}^{odd}$-miniversal deformation of the odd-simple map-germs are given by:

$$D^{\pm}_{2k/2} \ : \ F(x_1, x_2, \lambda_1, \cdots, \lambda_k) \equiv$$

$$x_1^2 x_2 \pm x_2^{2k-1} + \lambda_1 x_1 + \sum_{i=2}^{k} \lambda_i x_2^{2i-3}.$$

$$E_{8/2} \ : \ F(x_1, x_2, \lambda_1, \cdots, \lambda_4) \equiv$$

$$x_1^3 + x_2^5 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1 x_2^2 + \lambda_4 x_2^3.$$

$$J^{\pm}_{10/2} \ : \ F(x_1, x_2, \lambda_1, \cdots, \lambda_5) \equiv$$

$$x_1^3 \pm x_1 x_2^4 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1^2 x_2 + \lambda_4 x_2^2 x_1 + \lambda_5 x_2^3.$$

$$E_{12/2} \ : \ F(x_1, x_2, \lambda_1, \cdots, \lambda_6) \equiv$$

$$x_1^3 + x_2^7 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1 x_2^2 + \lambda_4 x_2^3 + \lambda_5 x_1 x_2^4 + \lambda_6 x_2^5.$$

Relation between the generating families.

Assume L is the graph of the analytic function $dS(s)$, where

$$S(s) = \sum_{k=0}^{\infty} a_k s^k, \quad a_k \in \mathbb{R}.$$

Take then

$$H(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then

$$g_{cc}(q, \dot{q}) = -i g_{sp}(q, i\dot{q}).$$

If

$$g_{cc}(q, \dot{q}) = \sum_{j=0}^{\infty} b_{2j+1}(q) \dot{q}^{2j+1}.$$

then

$$g_{sp}(q, \dot{q}) = \sum_{j=0}^{\infty} (-1)^j b_{2j+1}(q) \dot{q}^{2j+1},$$

The realization theorem

$$G_{cc}(\beta, q, p) = g_{cc}(q, \beta) - p \cdot \beta.$$

$$G_{sp}(\beta, q, p) = g_{sp}(q, \beta) - p \cdot \beta.$$

Theorem

$\mathcal{R}^{odd}$-versal deformations of $A_{2/2}$, $A_{4/2}$ (for $m = 1$) and $D^{\pm}_{4/2}$, $D^{\pm}_{6/2}$ $D^{\pm}_{8/2}$, $E_{8/2}$ (for $m = 2$) are realizable as generating families G_{cc} and G_{sp}.

$\mathcal{R}^{odd}$-versal deformations of $A_{2k/2}$ for $k > 2$ (and for $m = 1$) and $D_{2k/2}$ for $k > 4$, $J^{\pm}_{10/2}$ and $E_{12/2}$ (for $m = 2$) are not realizable as generating families G_{cc} or G_{sp}.

The realization theorem

$$G_{cc}(\beta, q, p) = g_{cc}(q, \beta) - p \cdot \beta.$$

$$G_{sp}(\beta, q, p) = g_{sp}(q, \beta) - p \cdot \beta.$$

Theorem

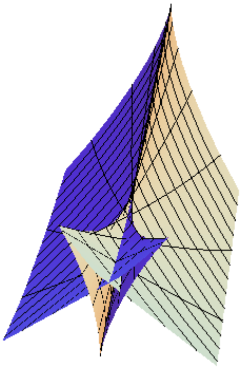
$\mathcal{R}^{odd}$-versal deformations of $A_{2/2}$, $A_{4/2}$ (for $m = 1$) and $D^{\pm}_{4/2}$, $D^{\pm}_{6/2}$ $D^{\pm}_{8/2}$, $E_{8/2}$ (for $m = 2$) are realizable as generating families G_{cc} and G_{sp}.
$\mathcal{R}^{odd}$-versal deformations of $A_{2k/2}$ for $k > 2$ (and for $m = 1$) and $D_{2k/2}$ for $k > 4$, $J^{\pm}_{10/2}$ and $E_{12/2}$ (for $m = 2$) are not realizable as generating families G_{cc} or G_{sp}.

Realizations of $A_{2/2}$ and $A_{4/2}$ singularities

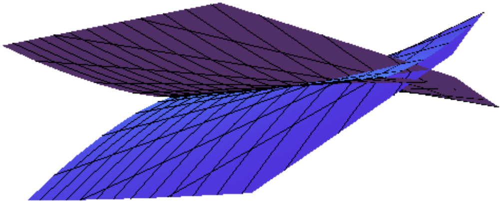
Proposition

1. *If $S^{(3)}(0) \neq 0$, G_{cc} and G_{sp} are $\mathcal{R}^{odd}$-equivalent to the $\mathcal{R}^{odd}$ versal deformation of $A_{2/2}$.*
2. *If $S^{(3)}(0) = 0$, $S^{(4)}(0) \neq 0$, $S^{(5)}(0) \neq 0$, G_{cc} and G_{sp} are $\mathcal{R}^{odd}$-equivalent to the $\mathcal{R}^{odd}$ versal deformation of $A_{4/2}$.*

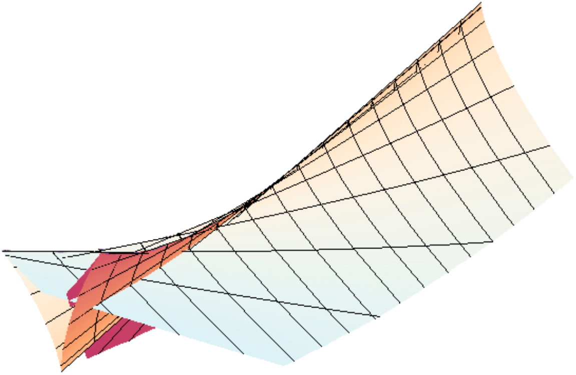
$A_{4/2}$-singularity of a special IAS - the Janeczko-Roberts symmetric butterfly



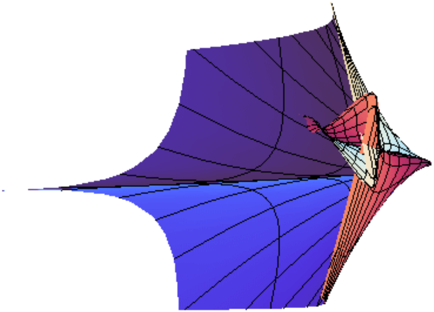
$A_{4/2}$-singularity of a special IAS
- the Janeczko-Roberts symmetric butterfly



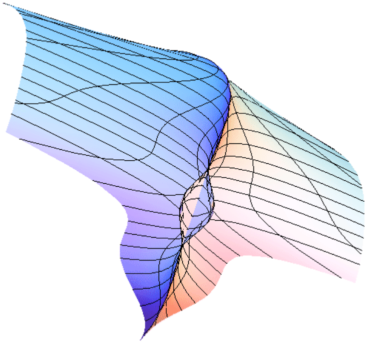
$A_{4/2}$-singularity of a special IAS
- the Janeczko-Roberts symmetric butterfly



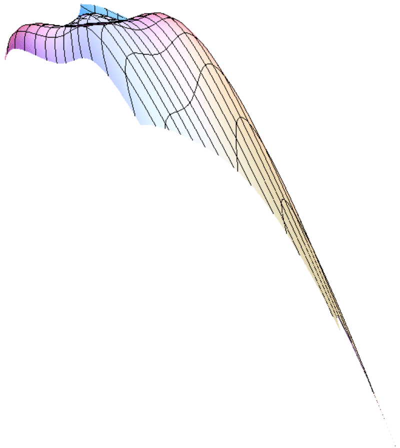
$A_{4/2}$-singularity of a center-chord IAS - the Janeczko-Roberts symmetric butterfly



$A_{4/2}$-singularity of a center-chord IAS



$A_{4/2}$-singularity of a center-chordl IAS



Realizations of $D^{\pm}_{4/2}$ singularities

$$S_{i,j} = \frac{\partial^{i+j} S}{\partial q_1^i \partial q_2^j}(0,0),$$

$$j_0^3 S = \frac{1}{6} S_{3,0} q_1^3 + \frac{1}{2} S_{2,1} q_1^2 q_2 + \frac{1}{2} S_{1,2} q_1 q_2^2 + \frac{1}{6} S_{0,3} q_2^3.$$

$\Delta(j_0^3 S)$ is the discriminant of $j_0^3 S$

Proposition

Assume $\Delta(j_0^3 S) \neq 0$.

1. *If* $\Delta(j_0^3 S) > 0$, G_{cc} *and* G_{sp} *are* $\mathcal{R}^{odd}$*-equivalent to the* $\mathcal{R}^{odd}$ *versal deformation of* $D^{-}_{4/2}$.
2. *If* $\Delta(j_0^3 S) < 0$, G_{cc} *and* G_{sp} *are* $\mathcal{R}^{odd}$*-equivalent to the* $\mathcal{R}^{odd}$ *versal deformation of* $D^{+}_{4/2}$.

Notation

$$r_1 = \frac{S_{2,1}S_{1,2} - S_{3,0}S_{0,3}}{2(S_{3,0}S_{1,2} - S_{2,1}^2)}; \quad r_2 = \frac{S_{3,0}^2 S_{0,3} - 4S_{3,0}S_{2,1}S_{1,2} + 3S_{2,1}^3}{S_{3,0}S_{1,2} - S_{2,1}^2}$$

$$\sigma_{0,n} = \frac{\sum_{k=0}^{n} \binom{n}{k} S_{k,n-k} r_1^k}{(S_{3,0}r_1 - r_2)^n}, \quad n = 5, 7,$$

$$\tilde{r}_1 = \frac{S_{2,1}S_{1,2} - S_{3,0}S_{0,3}}{2(S_{0,3}S_{2,1} - S_{1,2}^2)}; \quad \tilde{r}_2 = \frac{S_{0,3}^2 S_{3,0} - 4S_{0,3}S_{2,1}S_{1,2} + 3S_{1,2}^3}{S_{0,3}S_{2,1} - S_{1,2}^2},$$

$$\sigma_{n,0} = \frac{\sum_{k=0}^{n} \binom{n}{k} S_{k,n-k} \tilde{r}_1^k}{(S_{0,3}\tilde{r}_1 - \tilde{r}_2)^n}, \quad n = 5, 7,$$

$$\delta_1 = S_{3,0}S_{1,2} - S_{2,1}^2; \quad \delta_2 = S_{0,3}S_{2,1} - S_{1,2}^2.$$

Realizations of $D^{\pm}_{6/2}$ singularities

Lemma

If $\Delta(j_0^3 S) = 0$, then $\delta_i \leq 0$, $i = 1, 2$.

Proposition

Assume $\Delta(j_0^3 S) = 0$.

1. *If $\delta_1 \cdot \sigma_{0,5} < 0$ or $\delta_2 \cdot \sigma_{5,0} < 0$, G_{cc} is $\mathcal{R}^{odd}$-equivalent to the $\mathcal{R}^{odd}$ versal deformation of $D^+_{6/2}$, while G_{sp} is $\mathcal{R}^{odd}$-equivalent to the $\mathcal{R}^{odd}$ versal deformation of $D^-_{6/2}$.*
2. *If $\delta_1 \cdot \sigma_{0,5} > 0$ or $\delta_2 \cdot \sigma_{5,0} > 0$, G_{cc} is $\mathcal{R}^{odd}$-equivalent to the $\mathcal{R}^{odd}$ versal deformation $D^-_{6/2}$, while G_{sp} is $\mathcal{R}^{odd}$-equivalent to the $\mathcal{R}^{odd}$ versal deformation of $D^+_{6/2}$.*

Proposition

Assume $\Delta(j^3_0 S) = 0$.

1. *If* $\delta_1 < 0$, $\sigma_{0,5} = 0$ *and* $\sigma_{0,7} > 0$ *or* $\delta_2 < 0$, $\sigma_{5,0} = 0$ *and* $\sigma_{7,0} > 0$, G_{cc} *and* G_{sp} *are* $\mathcal{R}^{odd}$*-equivalent to the* $\mathcal{R}^{odd}$ *versal deformation of* $D^{+}_{8/2}$.
2. *If* $\delta_1 < 0$, $\sigma_{0,5} = 0$ *and* $\sigma_{0,7} < 0$ *or* $\delta_2 < 0$, $\sigma_{5,0} = 0$ *and* $\sigma_{7,0} < 0$, G_{cc} *and* G_{sp} *are* $\mathcal{R}^{odd}$*-equivalent to the* $\mathcal{R}^{odd}$ *versal deformation of* $D^{-}_{8/2}$.

Realizations of $E_{8/2}$ singularities

Proposition

Assume $\Delta(j_0^3 S) = 0$. *If*

$$\delta_1 = 0, \quad S_{3,0} \neq 0, \sum_{k=0}^{5} \binom{5}{k} S_{k,5-k}(-S_{2,1})^k (S_{3,0})^{5-k} \neq 0,$$

or

$$\delta_2 = 0, \quad S_{0,3} \neq 0, \sum_{k=0}^{5} \binom{5}{k} S_{k,5-k}(-S_{1,2})^k (S_{0,3})^{5-k} \neq 0,$$

then G_{cc} *and* G_{sp} *are* $\mathcal{R}^{odd}$*-equivalent to the* $\mathcal{R}^{odd}$ *versal deformation of* $E_{8/2}$.

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Thank You