Lipschitz geometry of abnormal definable surface germs

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The goal: outer Lipschitz classification

A set $X \subset \mathbb{R}^n$ definable in a polynomially bounded o-minimal structure (e.g., semialgebraic or subanalytic) inherits two metrics: the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y. X is **normally embedded** if these two metrics on X are equivalent.

A **surface germ** is a closed two-dimensional germ X at the origin. Germs X and Y are **outer (inner) Lipschitz equivalent** if there is an outer (inner) bi-Lipschitz homeomorphism $X \to Y$.

Finiteness theorems: Mostowski 85, Parusiński 94, Valette 05. Any definable family has finitely many outer Lipschitz equivalence classes.

Inner Lipschitz classification of surface germs: Birbrair 99.

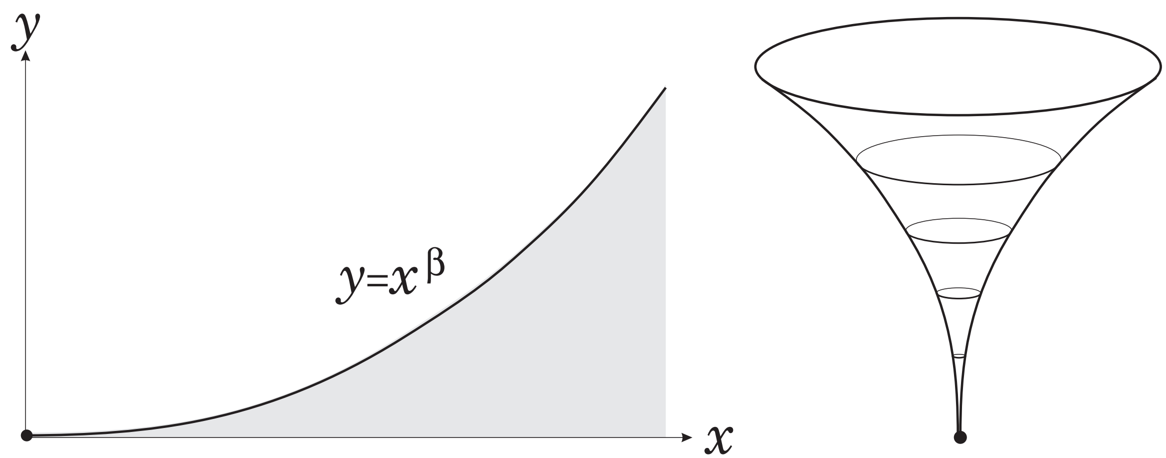
Outer Lipschitz classification of surface germs: an open problem.

Building blocks

For $\beta \in \mathbb{F}$, $\beta \geq 1$, the **standard β-Hölder triangle** is a surface germ

$$T_\beta = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}.$$

The **standard β-horn** is $C_\beta = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$.



A **β-Hölder triangle** is a germ inner Lipschitz equivalent to T_β.

A **β-horn** is a germ inner Lipschitz equivalent to C_β.

Arc spaces

An **arc** in X is a germ of a map $\gamma : [0, \epsilon) \to X$ such that $|\gamma(t)| = t$.

The **Valette link** $V(X)$ is the space of all arcs in X.

The **tangency order** $tord(\gamma, \gamma')$ of two arcs $\gamma \neq \gamma'$ is the exponent κ in $|\gamma - \gamma'| = c\, t^{\kappa} +$ (higher order terms), where $c \neq 0$.
This defines a **non-archimedean metric** on $V(X)$.
By definition, $tord(\gamma, \gamma) = \infty$.

An arc γ in X is **Lipschitz non-singular** if it is an interior arc of a normally embedded Hölder triangle $T \subset X$. Otherwise γ is **Lipschitz singular**. There are finitely many Lipschitz singular arcs in $V(X)$.

A Hölder triangle $T \subset X$ is **non-singular** if all its interior arcs are Lipschitz non-singular in X.

If $T = T(\gamma_1, \gamma_2)$ is a β-Hölder triangle bounded by arcs γ_1 and γ_2, then an arc γ in T is **generic** if $tord(\gamma, \gamma_1) = tord(\gamma, \gamma_2) = \beta$.

Zones

A **zone** $Z \subset V(X)$ is a "**connected**" set of arcs: for any two arcs $\gamma \neq \gamma'$ in Z, there is a non-singular Hölder triangle $T \subset X$ bounded by γ and γ' such that $V(T) \subset Z$.

The **order** $ord(Z)$ of a zone Z is the infimum of the tangency orders of arcs in Z. A **singular** zone $Z = \{\gamma\}$ has order ∞.

A zone Z is **normally embedded** if every Hölder triangle T such that $V(T) \subset Z$ is normally embedded.

A zone Z of order β is **weakly normally embedded** if every zone $Z' \subset Z$ of order $\beta' > \beta$ is normally embedded.

A zone Z is **closed** if there are arcs γ and γ' in Z such that $tord(\gamma, \gamma') = ord(Z)$, otherwise Z is **open**.

A zone Z is **perfect** if, for any $\gamma \neq \gamma'$ in Z, there is a Hölder triangle T such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T.

Normal and abnormal zones

A Lipschitz non-singular arc γ in a surface germ X is **abnormal** if there are normally embedded non-singular Hölder triangles T and T' in X such that $\gamma = T \cap T'$ and $T \cup T'$ is not normally embedded. Otherwise, γ is **normal**.

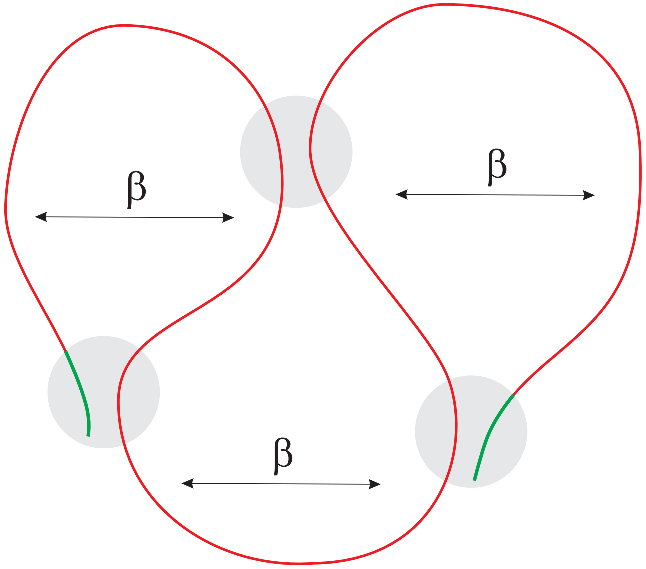
A zone $Z \subset V(X)$ is **abnormal** (resp., **normal**) if all arcs in Z are abnormal (resp., normal). An abnormal (resp., normal) zone is **maximal** if it is not contained in a larger abnormal (resp., normal) zone.

Theorem (AG, Souza 21) **For any surface germ X, there is a canonical partition of $V(X)$ into finitely many maximal abnormal and normal zones. All maximal normal zones are normally embedded. All maximal abnormal zones are closed perfect and weakly normally embedded.**

Snakes and snake zones

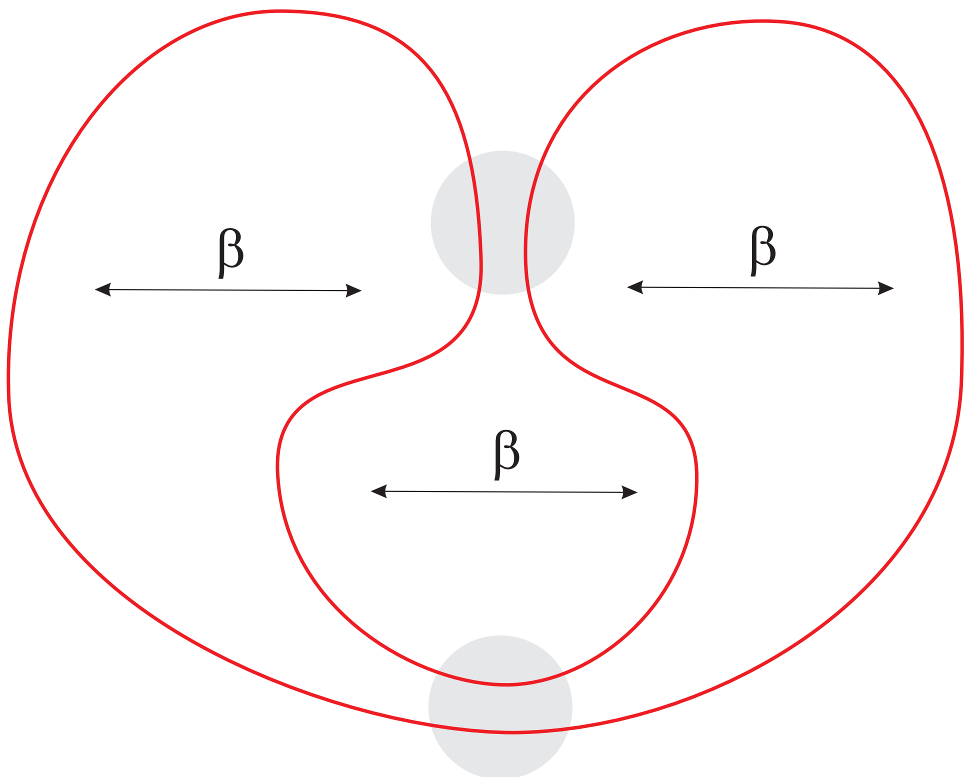
A β-**snake** is a non-singular β-Hölder triangle T such that the set of all **abnormal** arcs in T **is the same as** the set of all **generic** arcs.

If X is a surface germ, then a maximal abnormal zone $Z \subset V(X)$ of order β is called a **snake zone** if there is a β-Hölder triangle $T \subset X$ such that $Z \subset V(T)$.



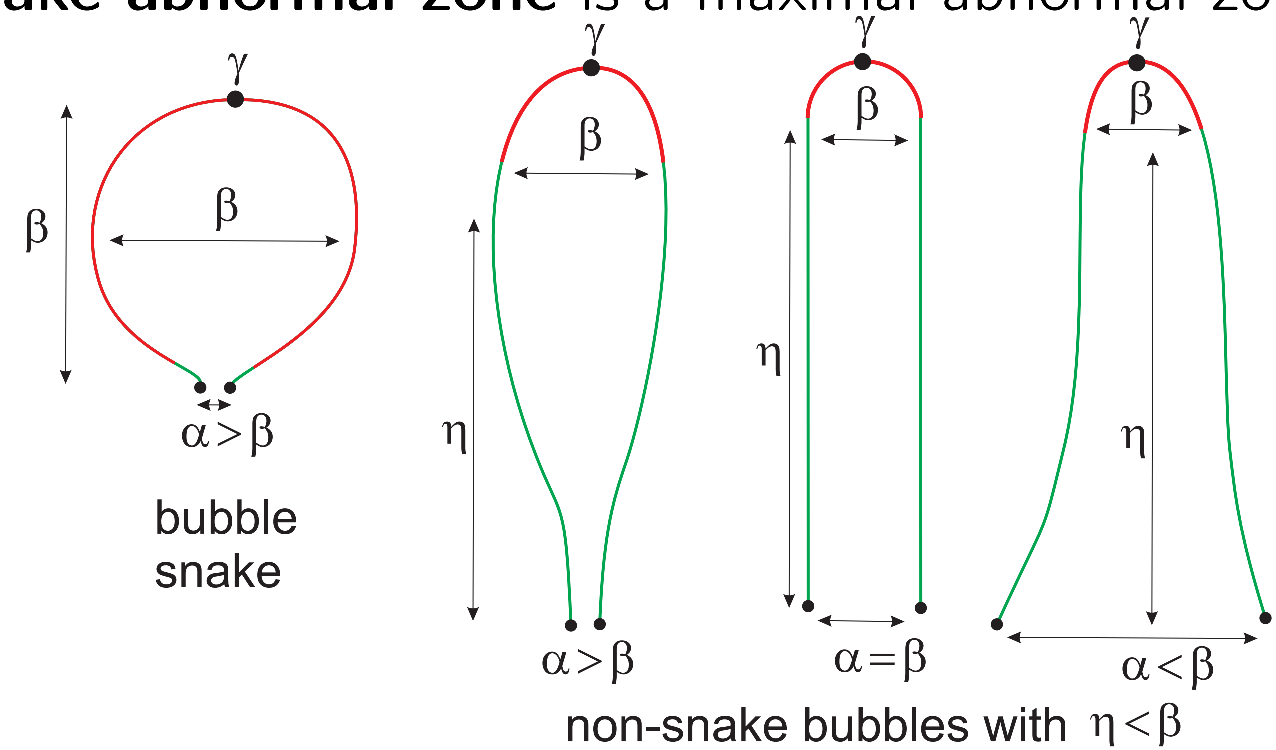
Circular snakes and zones

A **circular** β**-snake** is a β-horn C such that all arcs in $V(C)$ are abnormal. If C is a circular β-snake, then $V(C)$ is called a **circular** β**-snake zone**.



Bubbles: bubble snakes and non-snake bubbles

An η-**bubble** is a non-singular η-Hölder triangle $T = T(\gamma_1, \gamma_2)$, where $tord(\gamma_1, \gamma_2) > \eta$, partitioned by an arc γ into two pancakes.
A **bubble snake** is a β-bubble that is also a β-snake.
A **non-snake bubble** T is a bubble that does not contain a snake.
A **non-snake abnormal zone** is a maximal abnormal zone in T.

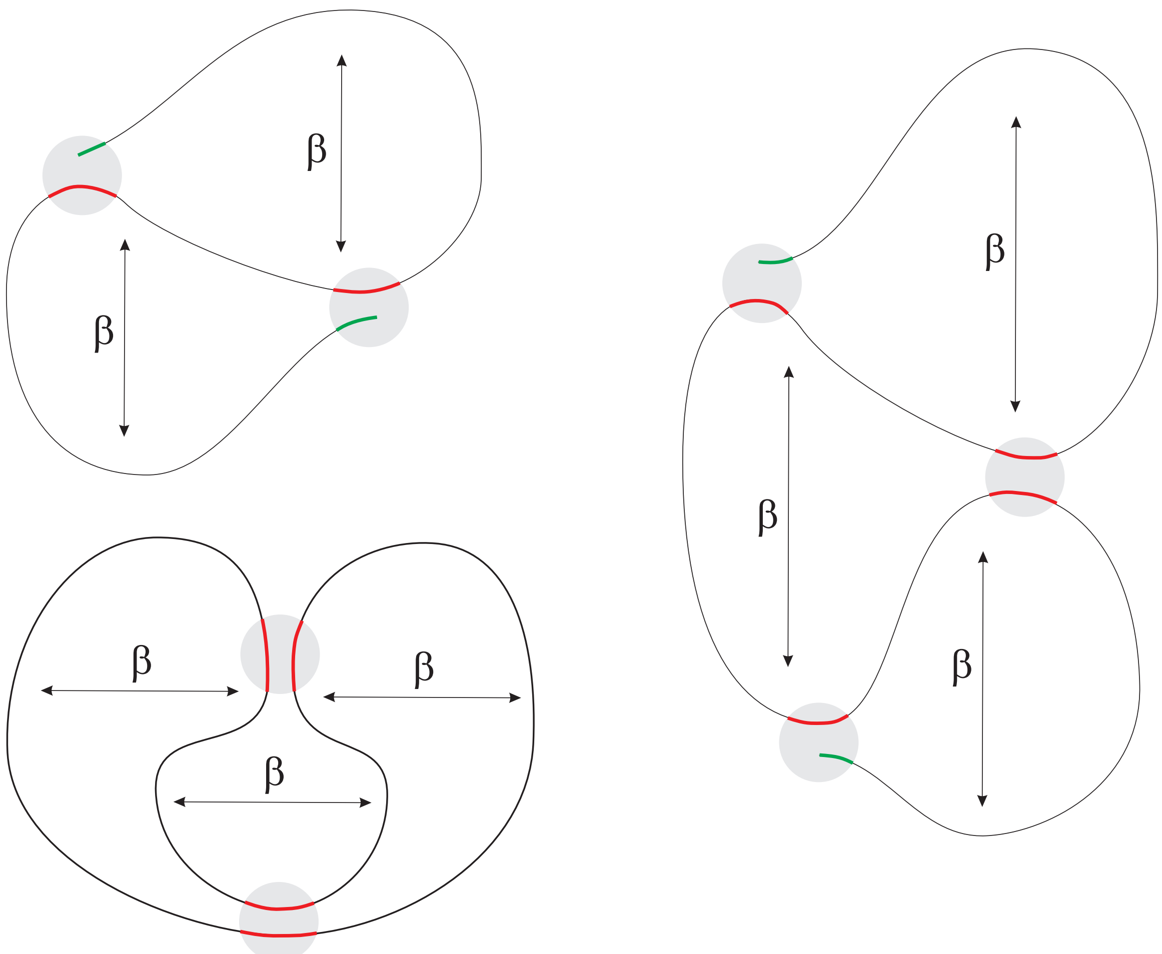


Theorem (AG, Souza 21) Let X be a surface germ. Then **each maximal abnormal β-zone** in $V(X)$ **is either a β-snake zone, or a circular β-snake zone, or a non-snake abnormal β-zone** $Z \subset V(T)$ where T is a non-snake η-bubble for some $\eta < \beta$.

Non-snake abnormal zones are closed perfect, normally embedded.

Each maximal β-snake zone, and each circular β-snake zone, has a canonical partition into finitely many normally embedded β-zones: closed perfect **segments** and open perfect **nodal zones**.

Segments, **nodal zones** (red) and **boundary nodal zones** (green). Shaded disks are the **nodes** of a β-snake: nodal zones N and N' belong to the same **node** if $tord(N, N') > \beta$.



Combinatorics of snakes: snake names

A word $W = [w_1, \cdots w_m]$ is a **snake name** if
1) Each letter appears in W at least twice,
2) For $1 < k < m$, there is a subword $W_k = [w_j \cdots w_k \cdots w_\ell]$ of W such that $w_j = w_\ell$ and W_k has no other repeated letters.

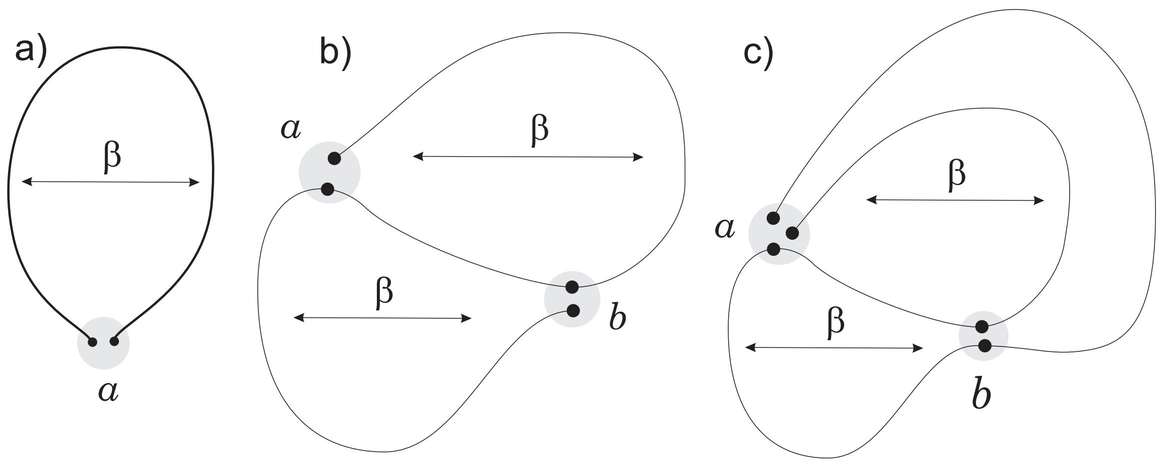
Example: The word $[a\,b\,c\,d\,a\,c\,b\,d]$ is a snake name with $W_2 = W_3 = W_4 = [a\,b\,c\,d\,a]$ and $W_5 = W_6 = W_7 = [d\,a\,c\,b\,d]$, but the word $[a\,b\,a\,c\,d\,c\,b\,d]$ is not a snake name, since it does not have a subword W_3 for the second entry of the letter a.

To each oriented snake one can assign a snake name as follows:
1) Distinct letters are assigned to all nodes of the snake,
2) The letters assigned to the nodes to which nodal zones of the snake belong are written in the order in which the nodal zones appear in the snake.

The number of distinct letters in W is the number of nodes of the snake, the length of W is the number of nodal zones of the snake.

Example: Three snakes with the snake names

a) $[a\, a]$, **b)** $[a\, b\, a\, b]$, **c)** $[a\, b\, a\, b\, a]$

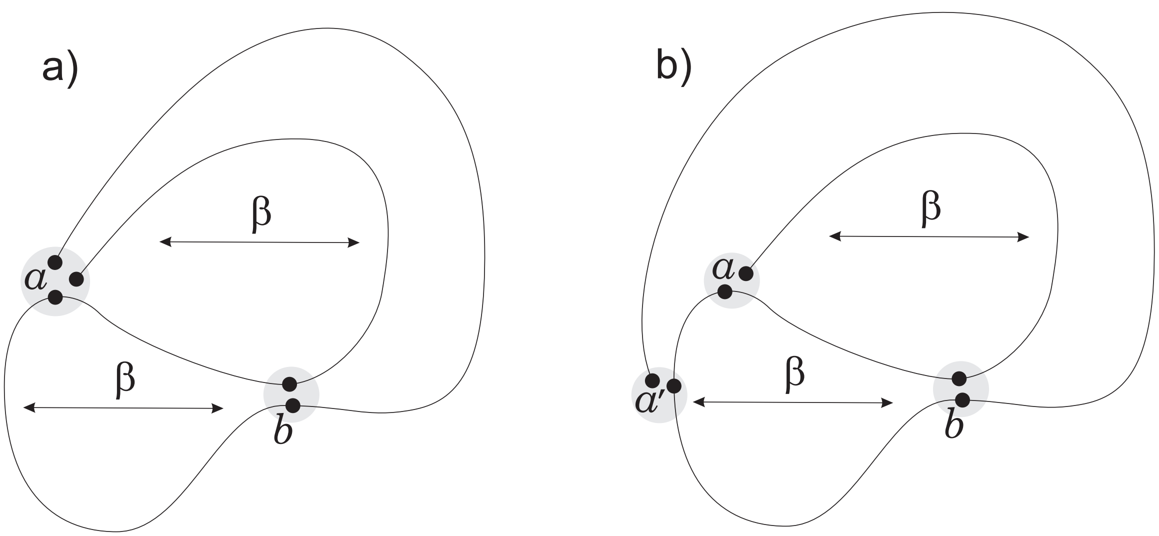


Conversely, if W is a snake name, then there is a snake with the snake name W.

A word W is **binary** if each letter appears in W exactly twice.
A snake is **binary** if its snake name is a binary word.

Example: Snakes names $[a\,a]$ and $[a\,b\,a\,b]$ are binary, $[a\,b\,a\,b\,a]$ is not.

A non-binary snake can be reduced to a binary one by splitting its non-binary nodes:



Recursion for the number of binary snake names

For a binary snake name $W = [w_1 \cdots w_{2m}]$, let j be the second entry of the letter w_1 in W, and let w_k be the first entry of W such that the subword $[w_2 \cdots w_k]$ of W contains a repeated letter.
Let $M_m(j,k)$ be the number of binary snake names of length $2m$ with parameters j and k, and let $M_m = \sum_{j,k} M_m(j,k)$ be the number of all binary snake names of length $2m$.

Then $M_1 = 1$, $M_2 = M_2(3,4) = 1$ and, for $m \geq 2$,

$$M_{m+1}(j,k) = M_{m,A}(j,k) + M_{m,B}(j,k),$$

where

$$M_{m,A}(j,k) = \sum_{l=k-1}^{m+2} M_m(k-2,l) \quad \text{and}$$

$$M_{m,B}(j,k) = (2m-k+1)M_m(j-1,k-1).$$

Binary snake names and standard Young tableaux

Let $W = [w_1 \cdots w_{2m+2}]$ be a binary snake name. We assign to W a Young tableau $\lambda(W)$ with two rows of length m as follows:
For $i = 2, \ldots, 2m+1$, insert $i-1$ into the first row of $\lambda(W)$ if w_i is the first entry of a letter in W, and into the second row otherwise. In particular, an empty tableau is assigned to the snake name $W = [a\,a]$ of length 2.

Proposition (AG, Souza 21) For each binary snake name W, the tableau $\lambda(W)$ is a **Standard Young Tableau (SYT)**.

For each SYT λ of shape (m, m), there is a unique **inversion free** snake name W of length $2m + 2$ such that $\lambda = \lambda(W)$.
An **inversion** in a binary word is a pattern $[\cdots x \cdots y \cdots y \cdots x \cdots]$.

Corollary The number of inversion free snake names of length $2m + 2$ is the **Catalan number**

$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$