Characteristic-free approach to unfoldings (and other results)

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In the 40's Whitney studied maps of C^∞ manifolds. When a map is not an immersion/submersion, one tries to deform it locally, in hope to make it 'generic'. This approach has led to the rich theory of stable maps, developed by Thom, Mather and many others.
The main 'engine' was vector field integration. This chained the whole theory to the C^∞, or $\mathbb{R}/\mathbb{C}$-analytic setting.

I will present a purely algebraic approach, studying maps of germs of Noetherian schemes, in any characteristic. The relevant groups of equivalence admit 'good' tangent spaces. One has the theory of unfoldings (triviality and versality). Then I will discuss the new results on stable maps and theorems of Mather-Yau/Gaffney-Hauser.

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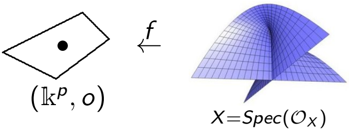

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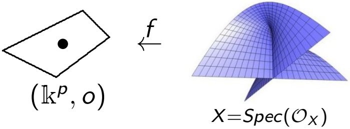

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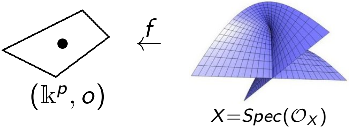

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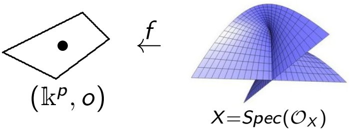

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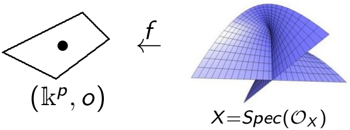

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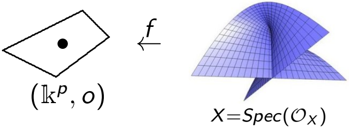

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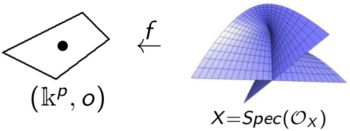

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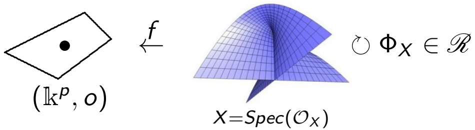

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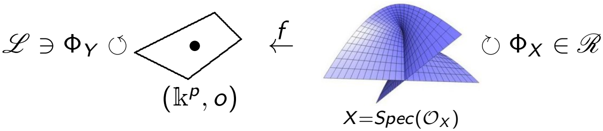

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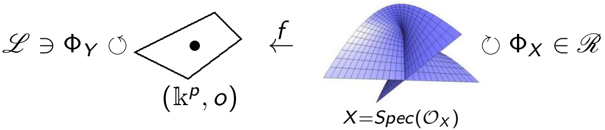

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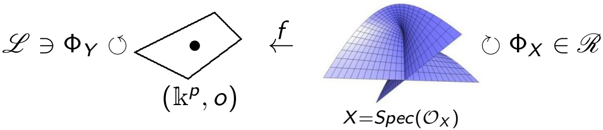

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Here $g_t \in \mathscr{G}_t$ is an unfolding of identity. E.g. $(\mathscr{R}_t)$ $x \to x + t \cdot (\dots)$

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Here $g_t \in \mathscr{G}_t$ is an unfolding of identity. E.g. $(\mathscr{R}_t)$ $x \to x + t \cdot (\dots)$

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2.*Suppose F is infinitesimally trivial and $\mathscr{G}$-separable. Then F is trivial.*

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This fails if $char(\Bbbk)>0$. A modified version in [Greuel-Pham.19]. A stronger version:

Theorem (2022). ($\Bbbk$ an infinite field. $\mathcal{O}_X \in \Bbbk[[x]], \Bbbk\{x\}, \Bbbk\langle x\rangle$)
Suppose an ideal $\mathfrak{a} \subset \mathcal{O}_X$ satisfies: $\mathfrak{a}^2 \cdot \mathcal{O}_X^{\oplus p} \subseteq (x) \cdot \mathfrak{a} \cdot T_{\mathscr{R}^e} f + (x) \cdot (f) \cdot \mathcal{O}_X^{\oplus p}$.

Mather-Yau/Gaffney-Hauser results $(\mathcal{O}_X \in \Bbbk[[x]]/_J, \Bbbk\{x\}/_J, \Bbbk\langle x\rangle/_J)$

Theorem (2022). ($char(\Bbbk) = 0$, $\Bbbk = \bar{\Bbbk}$**)**
1. The $\mathscr{K}$-type of $f \in \mathrm{Maps}(X, (\Bbbk^p, o))$ is determined by $[T^1_{\mathscr{K}} f]$.
2. If $V(f) \subset X$ has an isolated singularity then the $\mathscr{K}$-type is determined by $[T^1_{\mathscr{K}^e} f]$.

Here: • $T^1_{\mathscr{K}^e} f = \mathrm{Maps}(X, (\Bbbk^p, o))/_{T_{\mathscr{K}^e} f} \cong \mathcal{O}_X^{\oplus p}/_{T_{\mathscr{R}^e} f + (f) \cdot \mathcal{O}_X^{\oplus p}}$.
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And a similar result for $\mathscr{A}$-equivalence.

Mather-Yau/Gaffney-Hauser results $(\mathcal{O}_X \in \Bbbk[[x]]/J, \Bbbk\{x\}/J, \Bbbk\langle x\rangle/J)$

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Thanks for your attention!