Three Hypotheses on the Łojasiewicz Exponent

GKLW Workshop in Singularity Theory

A special session dedicated to the memory of Stanisław Łojasiewicz

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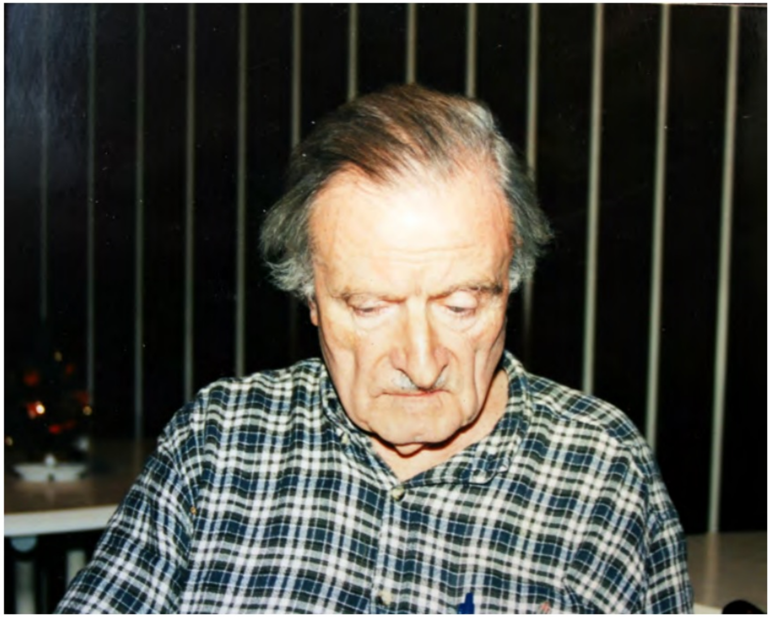

Stanisław Łojasiewicz (9 X 1926 – 14 XI 2002)

Introduction

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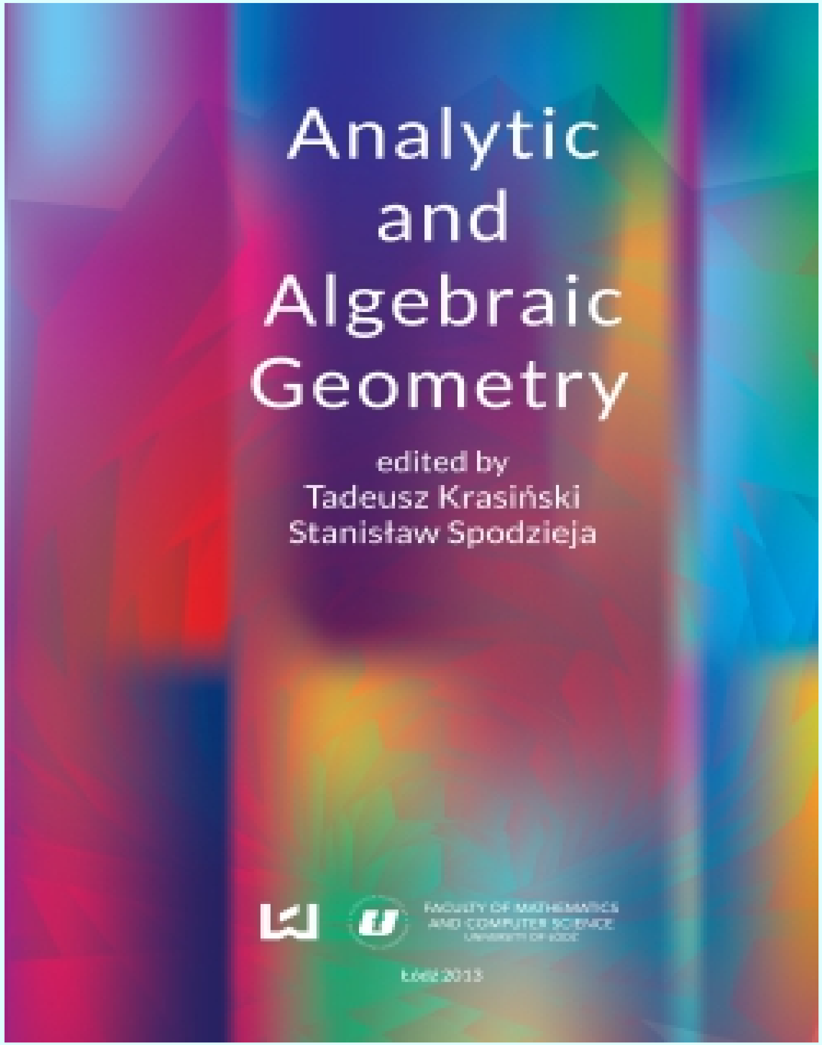


Introduction

GEOMETRIC DESINGULARIZATION OF CURVES IN MANIFOLDS *) **)

STANISŁAW ŁOJASIEWICZ

1. Introduction

The article does not pretend to any originality. In the literature there exists a number of descriptions of desingularizations in the case of curves. Deciding for this description the author think it is worth looking in details into this fascinating topic in an easily accessible case, namely – in the effects of multi blowings-up for curves in manifolds and for coherent sheaves on 2-dimensional manifolds.

All the needed facts from analytic geometry can be find in the author's books [L1], [L2].

2. The canonical blowing-up of $\mathbb{C}^n$ at 0

The *blow-up of* $\mathbb{C}^n$ *at* 0 is

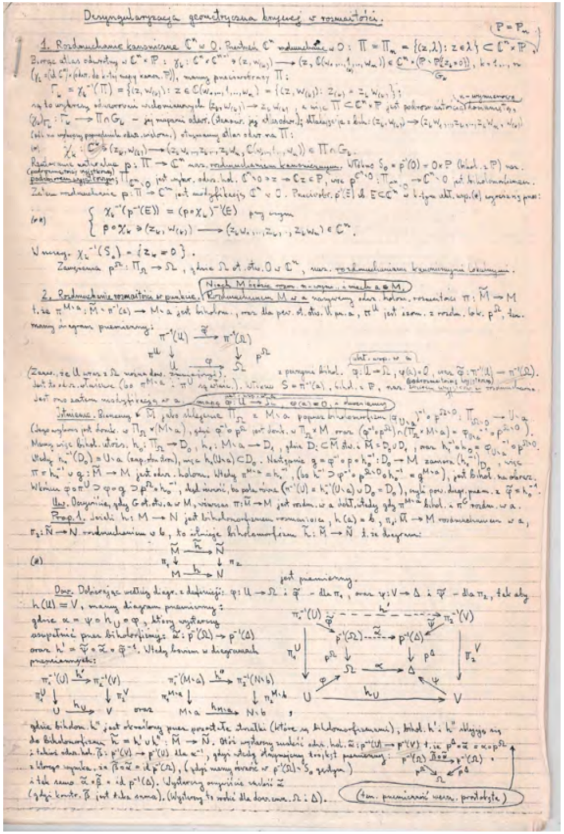
$$\Pi = \Pi_n = \{(z, \lambda) : z \in \lambda\} \subset \mathbb{C}^n \times \mathbb{P}, \quad \mathbb{P} = \mathbb{P}_{n-1}.$$

Taking the inverse atlas for $\mathbb{C}^n \times \mathbb{P}$

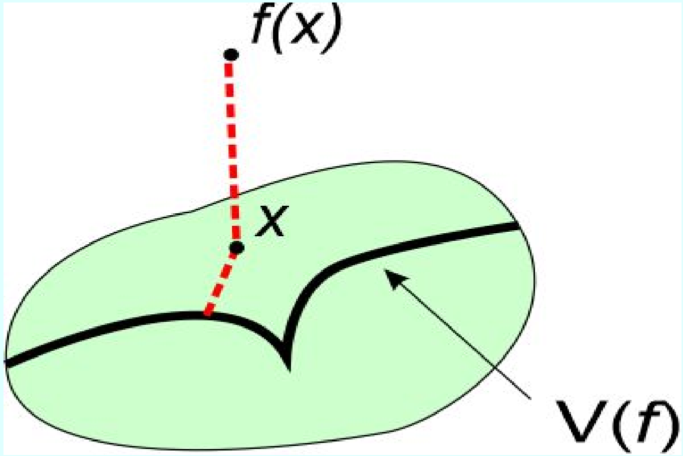
$$\gamma_k : \mathbb{C}^n \times \mathbb{C}^{n-1} \ni (z, w_{(k)}) \mapsto (z, \mathbb{C}(w_1, ..., \underset{(k)}{1}, ..., w_n)) \in \mathbb{C}^n \times \{\mathbb{P} \setminus \mathbb{P}(\{z_k = 0\})\} = G_k, \ k = 1, ..., n,$$

2010 *Mathematics Subject Classification.* Primary 32Sxx, Secondary 14Hxx.
Key words and phrases. Resolution of singularities, curve, blowing-up, coherent analytic sheaf.

*) This article was published (in Polish) in the proceedings of X^{th} Workshop on Theory of Extremal Problems (1989) and has never appeared in translation elsewhere. To honor this



The idea of the Łojasiewicz inequality - to compare the value of an analytic function f at a point x to the distance of x to the zero-set $V(f)$



Locally

$$|f(x)| \geqslant C(\mathrm{dist}(x, V(f)))^{\alpha}$$

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Remark 4. We consider the following variant of the Łojasiewicz inequality.

- $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ — a holomorphic function defined in a neighborhood of $0\in\mathbb{C}^n$ satisfying $f(0)=0$,
- f possesses an isolated critical point at 0, ie., $\nabla f(0)=0$, $\nabla f(z)\neq 0$ for $z\ll 1$.

In this situation we shall say that *f defines (or is) an isolated singularity*.

In this case the maping

$$\nabla f=\left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$$

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We will consider also a slight general setting:

Let $F=(F_1,\ldots,F_m):(\mathbb{C}^n,0)\to(\mathbb{C}^m,0)$, $m\geqslant n$, be a holomorphic mapping in a neighborhood of $0\in\mathbb{C}^n$ possessing an isolated zero at $0\in\mathbb{C}^n$.

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Remark. Any fact concerning a finite mapping F holds also for the gradient mapping $\nabla f=\left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right)$.

Let $\theta > 0$. Consider the inequality

$$C\,|z|^{\theta} \leqslant |F(z)|,$$

where $C > 0$ is a certain constant and $|z| \ll 1$.

The optimal θ in the above inequality (that is the smallest one) is called the *Łojasiewicz exponent* and denoted by $Ł(F)$.

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Example. Let $F := (z_2^3 + z_1^2, z_1 z_2^2)$. Then $Ł(F) = \frac{7}{2} > \operatorname{ord} F = 2$.

For a singularity f we define its *Łojasiewicz exponent* $ł(f)$ as

$$ł(f) := Ł(\nabla f).$$

- There exist effective methods of computation of Ł (eg. using PŁOSKI's characteristic polynomial or Groebner bases) but they are computationally expensive.
- In the 2-dimensional case there exists another useful way.

Namely, for $F=(F_1,F_2)$ we have

$$Ł(F)=\max_{\varphi\in V(F_1\cdot F_2)}\frac{\operatorname{ord} F\circ\varphi}{\operatorname{ord}\varphi},$$

so it is enough to find (finitely many!) parametrizations $\varphi:(\mathbb{C},0)\to(\mathbb{C}^2,0)$ of the curve $F_1\cdot F_2=0$ (**Chądzyński & Krasiński '88**).

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Example. Let, again, $F := (z_1^2 + z_2^3, z_1 z_2^2)$. Then $V(F_1 \cdot F_2) = \{(t^3, -t^2), (0, t), (t, 0)\}$ so $Ł(F) = \max\left\{\frac{7}{2}, \frac{3}{1}, \frac{2}{1}\right\} = \frac{7}{2}$.

- The above result does not extend to dimension $n \geqslant 3$:

> Let $F(z_1, z_2, z_3) := (z_1^2, z_2^3, z_3^3 - z_1 z_2)$. Then $V(F_1, F_2) = \{(0, 0, t)\}$, $V(F_2, F_3) = \{(t, 0, 0)\}$, $V(F_1, F_3) = \{(0, t, 0)\}$. Hence, $\max_{\varphi \in \cup_{i<j} V(F_i, F_j)} \frac{\operatorname{ord} F \circ \varphi}{\operatorname{ord} \varphi} = \max\left\{\frac{3}{1}, \frac{2}{1}, \frac{3}{1}\right\} = 3$, while $Ł(F) = \frac{18}{5}$.
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- A certain positive result of this kind: for $F = (F_1, \ldots, F_m)$: $(\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ we have (**Chądzyński & Krasiński '98**)

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(This result is not very useful in practice because one must test infinitely many parametrizations φ.)

Conjecture I — ł(f) on „coordinate polar curves"

Focusing on isolated singularities, we pose

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be an isolated singularity. Then

$$ł(f)=\max_{\varphi\in\cup_{i=1}^{n} V\left(\frac{\partial f}{\partial z_1},\ldots,\widehat{\frac{\partial f}{\partial z_i}},\ldots,\frac{\partial f}{\partial z_n}\right)}\left(\frac{\operatorname{ord}\nabla f\circ\varphi}{\operatorname{ord}\varphi}\right).$$

CONJECTURE I

- We are not aware of any counterexample to this.
- Below, we will give *many examples* in its favor. A fast one:

Example. For $f(x,y,z):=x^2z+yz^2+y^3z+(2y+3)x^2y^4$ we have $ł(f)=6$ and this exponent is achieved on the parametrization $\varphi(t):=(-t^3\cdot\sqrt{1-6t^4+4t^6},-t^2,-3\cdot t^8+2\cdot t^{10})\in V\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial z}\right)$.

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Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic function. We shall say:

- f is *quasihomogeneous of type* $(d;l_1,\ldots,l_n)$, shortly: $f\in \mathrm{QH}(d;l_1,\ldots,l_n)$, if $d,l_1,\ldots,l_n\in\mathbb{Q}_{>0}$, $l_1/d,\ldots,l_n/d\in(0,1/2]$ and for all monomials $z^a=z_1^{a_1}\cdot\ldots\cdot z_n^{a_n}$ appearing in f with a non-zero coefficient we have $a_1l_1+\ldots+a_nl_n=d$

The numbers $l_1,\ldots,l_n$ are *weights*. The number d is the *weighted degree* of the polynomial f.

- f is *semiquasihomogeneous of type* $(d;l_1,\ldots,l_n)$, shortly: $f\in\mathrm{SQH}(d;l_1,\ldots,l_n)$, if $f=f_d+f_{d+1}+\ldots$, where $f_i\in\mathrm{QH}(i;l_1,\ldots,l_n)$ and f_d is an isolated singularity

Example

Let $f:=(xz+y^5)+x^3$. Then $f\in\mathrm{SQH}\left(1;\frac{1}{2},\frac{1}{5},\frac{1}{2}\right)$.

The following theorem holds (**Krasiński, Oleksik & Płoski ’09** for $n \leqslant 3$; **Brzostowski ’14** and **Abderrahmane ’15** for general n):

Let $f \in \mathrm{QH}(1; l_1, \ldots, l_n)$ *be an isolated singularity.*

Put $l_{\min} := \min\{l_1, \ldots, l_n\}$. *Then*

$$ł(f) = \frac{1}{l_{\min}} - 1.$$

Theorem

- The above theorem holds also for SQH functions.
- $ł(f)$ is achieved on coordinate polar curves, confirming ł CONJECTURE in this case.

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Conjecture II — ƚ(f) as an invariant

ƚ(f) is a topological invariant of a singularity

Conjecture II

$\text{ł}(f)$ is a topological invariant of a singularity

CONJECTURE II

Precisely:

If f and g are two singularities (isolated) $\mathscr{R}$-topologically equivalent i.e. $g = f \circ \Phi$, where $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a homeomorphism, then $\text{ł}(f) = \text{ł}(g)$.

An intuitive reason to believe this: ł *is* connected with the topology of the singularity; for, the number $\lfloor ł(f) \rfloor + 1$ is the degree of C^0-$\mathscr{R}$-determinacy of the germ f in $\mathscr{O}_n$ (**Chang & Lu ’73**, **Teissier ’77**, **Bochnak & Kucharz ’79**).

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Thus, if $\operatorname{ord}(f-g) > \lfloor \text{ł}(f) \rfloor + 1$, then g is $\mathscr{R}$-topologically equivalent to f; moreover, for such f and g we have $\text{ł}(f) = \text{ł}(g)$ by Płoski's lemma so the conjecture holds in this important case.

CONJECTURE II — $\text{ł}(f)$ as an invariant

Results confirming this conjecture:

- dimension 2 (**Teissier '77**); in this case, one can provide a formula for $\text{ł}(f)$ in terms of so-called characteristic sequences of the branches of the curve $\{f=0\}$ and their intersection multiplicities (**Płoski '01**)
- SQH singularities of 3 variables (**Krasiński, Oleksik & Płoski '09**)
- If $f, g\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ are two isolated hypersurface singularities that are $\mathscr{R}-\mathscr{L}$-bi-Lipschitz equivalent, that is $g=\Psi\circ f\circ\Phi$, where $\Phi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$, $\Psi\colon(\mathbb{C},0)\to(\mathbb{C},0)$ are bi-Lipschitz homeomorphisms, then $\text{ł}(f)=\text{ł}(g)$ (**Bivià-Ausina & Fukui '17**).

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A weaker form of CONJECTURE II is:

(Teissier '77) If (f_s) is a topologically trivial (holomorphic) deformation of a singularity f_0, then $\text{ł}(f_s) = \text{ł}(f_0)$ for small $s \in \mathbb{C}$.

CONJECTURE II'

- $\text{ł}(f_s) = \text{ł}(f_0)$ for an SQH function f_0 **(S. Brzostowski. '14)**.

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> **Conjecture II'**

In this direction, we have, for a μ-constant deformation (f_s) of the germ f_0:

- $\text{ł}(f_s) \geqslant \text{ł}(f_0)$ (**Teissier '77**, **Płoski '10**); this is so-called „lower semicontinuity" of the Łojasiewicz exponent,
- $\text{ł}(f_s) = \text{ł}(f_0)$ for an SQH function f_0 (**S. Brzostowski. '14**).

Conjecture II' — a weaker version

A weaker form of Conjecture II is:

> **(Teissier '77)** If (f_s) is a topologically trivial (holomorphic) deformation of a singularity f_0, then $\text{Ł}(f_s) = \text{Ł}(f_0)$ for small $s \in \mathbb{C}$.
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CONJECTURE II' — a weaker version

- If for a holomorphic family $\{f_s(z)\}$ of isolated hypersurface singularities we assume a somewhat stronger triviality than the topological one, namely that it satisfies Teissier's condition (c), then $\text{ł}(f_s) = \text{ł}(f_0)$ (**Teissier '77**).

- If for a holomorphic family $\{f_s(z)\}$ of isolated hypersurface singularities we assume a somewhat stronger triviality than the topological one, namely that it satisfies Teissier's condition (c), then ł$(f_s) =$ ł(f_0) (**Teissier '77**).

Here, *condition (c)* means that for the family $\{f_s\}$ we have:

$$\frac{\partial f_s(z)}{\partial s} \in \overline{(z_1, \ldots, z_n) \cdot (\nabla_z f_s(z))},$$

where, as before, the bar "$^{-}$" designates integral closure of an ideal (in the ring $\mathcal{O}_{n+1} \simeq \mathbb{C}\{s, z_1, \ldots, z_n\}$). Relaxing the above requirement on $\{f_s\}$ to $\frac{\partial f_s(z)}{\partial s} \in \overline{\nabla_z f_s(z)}$ we get a condition equivalent to μ-constancy of the family (cf. **Greuel '86**).

The exponent in a family of mappings

One can also ask: *and what about families of mappings with constant intersection multiplicity?* There is the following answer (**Płoski '10** for full intersections; **Rodak, Różycki & Spodzieja '16** in general):

Theorem

If (F_s) *is a deformation of a map-germ* $F:(\mathbb{C}^n,0)\to(\mathbb{C}^m,0)$ *having constant intersection multiplicity for small* s, *then* $Ł(F_s)\geqslant Ł(F_0)$.

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In general, however, the above inequality may be strict:

Example. (Mc Neal & Némethi '05) Let $F_s(z_1,z_2):=(s z_1+z_1^2+z_2^2, z_1^2-z_2^5):(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$. Then $Ł(F_s)\underset{s\neq 0}{=}4>2=Ł(F_0)$, although $e(F_s)=e(F_0)$.

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The Łojasiewicz exponent is "an interesting invariant" of an isolated singularity although we don't know if it is a topological invariant.

We pose

> The Łojasiewicz exponent can be read off the Newton diagram for singularities non-degenerate in the sense of Kushnirenko.
>
> CONJECTURE III

- There exist some partial results in this direction for $n \geq 3$ (**Abderrahmane '06, Fukui '91, Oka '18**).
- The solution to CONJECTURE III must in particular involve an accurate definition of an „*exceptional*"=*irrelevant* face of a Newton diagram.

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Conjecture III – a starting point

Brzostowski '19 proved "first-half" of the **Conjecture III.**

If $f, g \colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *are Kushnirenko non-degenerate isolated singularities with **the same Newton diagrams,** then* $\text{ł}(f) = \text{ł}(g)$.

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- Loosely speaking, this theorem follows from the fact that, after some preparations, we can join f and g by a piecewise linear curve (in the space of coefficients) along which, locally, we have Teissier's condition (c) satisfied.
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Conjecture III in dimension 3

We have the following theorem

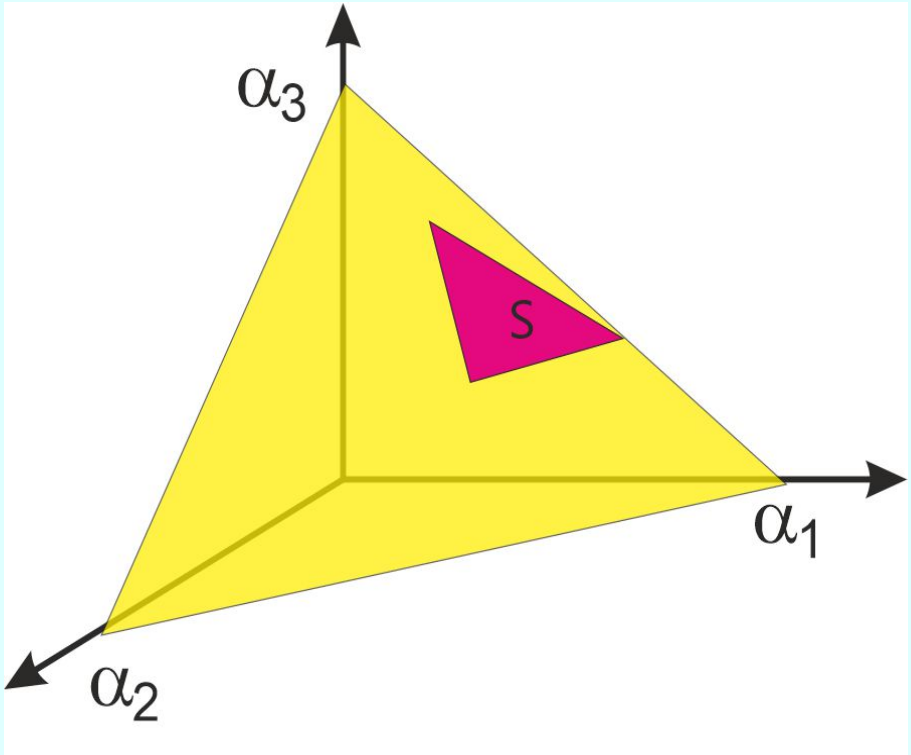
If $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ *is a Kushnirenko non-degenerate isolated* ***surface*** *singularity, then*

$$ł(f) = \max_{S - \text{ relevant facets}} m(S) - 1,$$

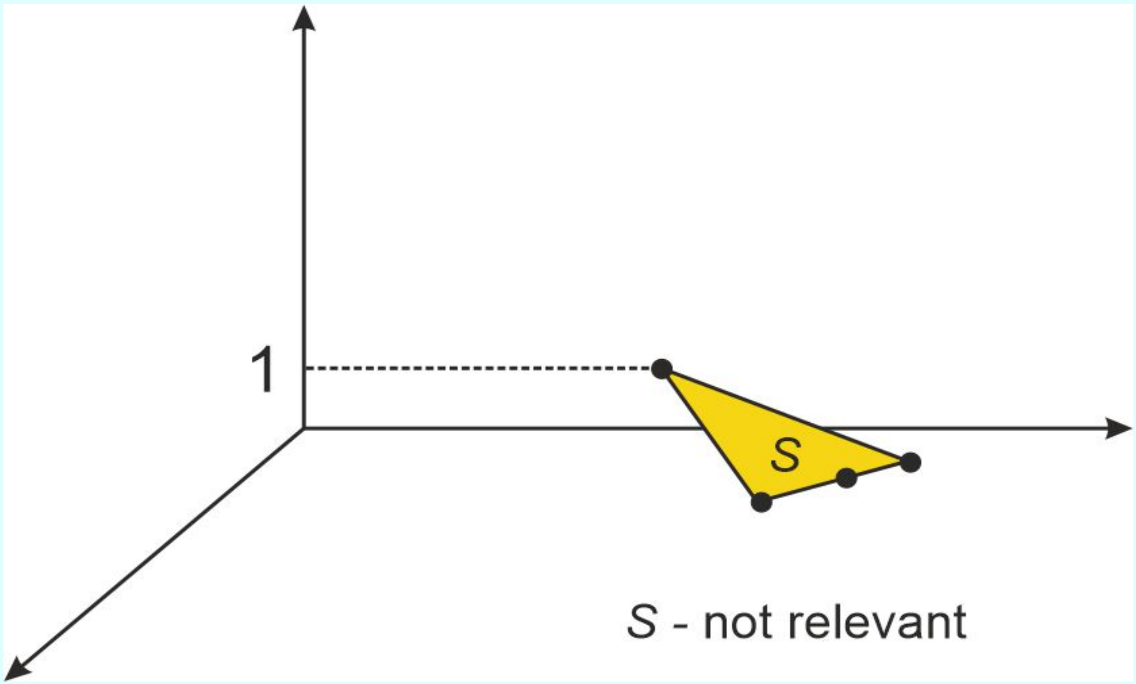
provided the set of relevant facets is non-empty.

Brzostowski, Krasiński, & Oleksik '20

- There is also a *direct formula* for ł if the set of relevant facets happens to be empty.



$$m(S) = \max(\alpha_1, \alpha_2, \alpha_3)$$


1
S
S - not relevant

- In **Arnold's, Gusein-Zade's & Varchenko's** book there is given a full classification of singularities with Milnor numbers $\mu \leqslant 16$ with respect to stable C^∞-$\mathscr{R}$-equivalence.
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- The numerical results are in favor of the conjectures:

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Testing the hypotheses — Arnold's zoo

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- The numerical results are in favor of the conjectures:

Conjecture I.The exponents are always achieved on some coordinate polar curves (in the coordinate system in which the singularity has so-called **normal form**) →

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Conjecture IIIThe value of the exponents can be read off the Newton diagram (for $n=3$) as predicted above

Modality 0

	Class	Formula	μ	ł	Zeroes of	Parametrization	Why
1	A_k	x^{k+1}	k	k	–	$[x=t]$	1 var.
2	D_k	x^2y+y^{k-1}	k	$k-2$	$\frac{\partial}{\partial x}$	$[x=0,y=t]$	2 var.
3	E_6	y^4+x^3	6	3	$\frac{\partial}{\partial x}$	$[x=y=t]$	2 var.
4	E_7	xy^3+x^3	7	$\frac{7}{2}$	$\frac{\partial}{\partial x}$	$[x=3t^3,y=-3t^2]$	2 var.
5	E_8	y^5+x^3	8	4	$\frac{\partial}{\partial x}$	$[x=0,y=t]$	2 var.

Modality 1

	Class	*Formula*	μ	ł	*Zeroes of*	*Parametrization*	*Why*
6	P_8	$axyz+x^3+y^3+z^3$	8	2	$\frac{\partial}{\partial x},\frac{\partial}{\partial y}$	$[x=0,y=0,z=t]$	qh
7	X_9	$ax^2y^2+x^4+y^4$	9	3	$\frac{\partial}{\partial x}$	$[x=0,y=t]$	2 var.
8	J_{10}	$y^6+ax^2y^2+x^3$	10	5	$\frac{\partial}{\partial x}$	$[x=0,y=t]$	2 var.
9*	$T_{p,q,r}$	$x^p+y^q+z^r+axyz$	$p+q+r-1$	$r-1$	$\frac{\partial}{\partial x},\frac{\partial}{\partial y}$	$[x=0,y=0,z=t]$	ndg.
10	E_{12}	$axy^5+y^7+x^3$	12	6	$\frac{\partial}{\partial x}$	$[x=9a^3t^5,y=-3at^2]$	2 var.
11	E_{13}	$ay^8+xy^5+x^3$	13	$\frac{13}{2}$	$\frac{\partial}{\partial x}$	$[x=-9t^5,y=-3t^2]$	2 var.
12	E_{14}	$axy^6+y^8+x^3$	14	7	$\frac{\partial}{\partial x}$	$[x=\sqrt{\frac{-a}{3}}t^3,y=t]$	2 var.
13	Z_{11}	$axy^4+y^5+x^3y$	11	4	$\frac{\partial}{\partial x}$	$[x=3a^2t^3,y=-3at^2]$	2 var.
14	Z_{12}	$ax^2y^3+xy^4+x^3y$	12	$\frac{9}{2}$	$\frac{\partial}{\partial x}$	$[x=-3\frac{t^3}{2at+1},y=-3\frac{t^2}{2at+1}]$	2 var.

9*. For $\max(p,q,r)=r$. Here $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$.

Modality 1 (cont.)

	Class	*Formula*	μ	ł	*Zeroes of*	*Parametrization*	*Why*
15	Z_{13}	$axy^5+y^6+x^3y$	13	5	$\frac{\partial}{\partial x}$	$[x=\sqrt{\frac{-a}{3}}t^2,y=t]$	2 var.
16	W_{12}	$ax^2y^3+y^5+x^4$	12	4	$\frac{\partial}{\partial x}$	$[x=0,y=t]$	2 var.
17	W_{13}	$ay^6+xy^4+x^4$	13	$\frac{13}{3}$	$\frac{\partial}{\partial x}$	$[x=-4t^4,y=4t^3]$	2 var.
18	Q_{10}	$axy^3+y^4+x^3+yz^2$	10	3	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=3\sqrt{a}t^3,y=-3t^2,z=0]$	sqh
19	Q_{11}	$az^5+xz^3+x^3+y^2z$	11	$\frac{7}{2}$	$\frac{\partial}{\partial x},\frac{\partial}{\partial y}$	$[x=3t^3,y=0,z=-3t^2]$	sqh
20	Q_{12}	$axy^4+y^5+x^3+yz^2$	12	4	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=-\sqrt{\frac{-a}{3}}t^2,y=t,z=0]$	sqh
21	S_{11}	$ax^3z+x^4+xz^2+y^2z$	11	3	$\frac{\partial}{\partial y},\frac{\partial}{\partial z}$	$[x=2t,y=0,z=-2at^2]$	sqh
22	S_{12}	$az^5+xz^3+x^2y+y^2z$	12	$\frac{10}{3}$	$\frac{\partial}{\partial x},\frac{\partial}{\partial y}$	$[x=16t^4,y=16t^5,z=-8t^3]$	sqh
23	U_{12}	$axyz^2+z^4+x^3+y^3$	12	3	$\frac{\partial}{\partial x},\frac{\partial}{\partial y}$	$[x=0,y=0,z=t]$	sqh

Modality 2

	Class	Formula	μ	ł	Zeroes of	Parametrization	Why
24	$J_{3,0}$	$cxy^7+y^9+bx^2y^3+x^3$	16	8	$\frac{\partial}{\partial x}$	$[x=-128\frac{b^7t^3}{(ct+3)^7}, y=4\frac{b^2t}{(ct+3)^2}]$ for $b\neq0$ $[x=27\sqrt{c}t^7, y=-3t^2]$ for $b=0$	2 var.
25	$J_{3,p}$	$x^3+x^2y^3+\boldsymbol{a}y^{9+p}$	$16+p$	$8+p$	$\frac{\partial}{\partial x}$	$[x=0, y=t]$	2 var.
26	$Z_{1,0}$	$cxy^6+y^7+dx^2y^3+x^3y$	15	6	$\frac{\partial}{\partial x}$	$[x=3t^2(9ct-2d)^3, y=-3t(9ct-2d)]$ for $c\neq0\vee d\neq0$ $[x=0, y=t]$ for $c=0\wedge d=0$	2 var.
27	$Z_{1,p}$	$x^3y+x^2y^3+\boldsymbol{a}y^{7+p}$	$15+p$	$6+p$	$\frac{\partial}{\partial x}$	$[x=0, y=t]$	2 var.
28	$W_{1,0}$	$x^4+\boldsymbol{a}x^2y^3+y^6$	15	5	$\frac{\partial}{\partial x}$	$[x=0, y=t]$	2 var.
29	$W_{1,p}$	$x^4+x^2y^3+\boldsymbol{a}y^{6+p}$	$15+p$	$5+p$	$\frac{\partial}{\partial x}$	$[x=0, y=t]$	2 var.
30	$W^{\#}_{1,2q-1}$	$(y^3+x^2)^2+\boldsymbol{a}xy^{4+q}$	$14+2q$	$\frac{9}{2}+q$	$\frac{\partial}{\partial x}$	$\left[x=t^3+\frac{1}{8}a_0t^{2q+2}+\dots, y=-t^2\right]$	2 var.
31	$W^{\#}_{1,2q}$	$(y^3+x^2)^2+\boldsymbol{a}x^2y^{3+q}$	$15+2q$	$5+q$	$\frac{\partial}{\partial x}$	$\left[x=t^3+\frac{1}{4}a_0(-1)^qt^{2q+3}+\dots, y=-t^2\right]$	2 var.

Here $\boldsymbol{a} := (a_0 + a_1 y)$, $p, q > 0$.

Modality 2 (cont.); here $\boldsymbol{a} := (a_0 + a_1 y)$, $p, q > 0$

	Class	Formula	μ	ł	Zer.	Parametrization	Why
32	$Q_{2,0}$	$x^3+yz^2+\boldsymbol{a}x^2y^2+xy^4$	14	5	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=-\frac{1}{8}\frac{t^2\left(t+4a_1\sqrt{a_0^2-3}\right)^2}{(t+2a_0a_1+2a_1\sqrt{a_0^2-3})^3a_1^3},y=\frac{1}{4}\frac{t\left(t+4a_1\sqrt{a_0^2-3}\right)}{(t+2a_0a_1+2a_1\sqrt{a_0^2-3})a_1^2},$ $z=0]$ for $a_1\neq 0$ $[x=(-a_0+\sqrt{a_0^2-3})t^2,y=(-a_0+\sqrt{a_0^2-3})t,z=0]$ for $a_1=0$	sqh
33	$Q_{2,p}$	$x^3+yz^2+x^2y^2+\boldsymbol{a}y^{6+p}$	$14+p$	$5+p$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=0,y=t,z=0]$	ndg.
34	$S_{1,0}$	$x^2z+yz^2+y^5+\boldsymbol{a}y^3z$	14	4	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[x=0,y=t,z=-\frac{1}{2}a_1t^3-\frac{1}{2}a_0t^2\right]$	sqh
35	$S_{1,p}$	$x^2z+yz^2+x^2y^2+\boldsymbol{a}y^{5+p}$	$14+p$	$4+p$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=0,y=t,z=0]$	ndg.
36	$S^\#_{1,2q-1}$	$x^2z+yz^2+y^3z+\boldsymbol{a}xy^{3+q}$	$13+2q$	$\frac{7}{2}+q$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[x=\frac{1}{2}a_0(-1)^qt^{2q+2}+t^3,y=-t^2,z=\frac{1}{2}a_0(-1)^qt^{2q+3}\right]$ for $1<q$ $\left[x=-\frac{3}{8}a_0^2t^5-\frac{1}{2}a_0t^4+t^3,y=-t^2,z=-\frac{1}{4}a_0^2t^6-\frac{1}{2}a_0t^5\right]$ for $q=1$	ndg.
37	$S^\#_{1,2q}$	$x^2z+yz^2+y^3z+\boldsymbol{a}x^2y^{2+q}$	$14+2q$	$4+q$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=-t^3\sqrt{1+2a_0(-1)^{1+q}t^{2q}+2a_1(-1)^qt^{2q+2}},y=-t^2,$ $z=(-1)^q(-a_0t^{2q+4}+a_1t^{2q+6})]$	ndg.
38	$U_{1,0}$	$x^3+xz^2+xy^3+\boldsymbol{a}y^3z$	14	$\frac{7}{2}$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[y=(\frac{-1}{6}+\frac{1}{6}\eta)t^2,x=(\frac{-1}{12}a_0^2+\frac{1}{18}-\frac{1}{18}\eta)t^3,z=\frac{-1}{12}a_0(-1+\eta)t^3]$ for $\eta=\sqrt{-3a_0^2+1}$ and $a_0^2\neq 1/3$	sqh
39	$U_{1,2q-1}$	$x^3+xz^2+xy^3+\boldsymbol{a}y^{1+q}z^2$	$13+2q$	$3+q$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=(-1)^q(a_0t^{2q+2}-a_1t^{2q+4}),y=-t^2,$ $z=-t^3\sqrt{1-3a_0^2t^{4q-2}+6a_0a_1t^{4q}-3a_1^2t^{4q+2}}]$	ndg.
40	$U_{1,2q}$	$x^3+xz^2+xy^3+\boldsymbol{a}y^{3+q}z$	$14+2q$	$\frac{7}{2}+q$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[x=\frac{1}{2}a_0(-1)^qt^{2q+3},y=-t^2,z=-\frac{3}{8}a_0^2t^{4q+3}+t^3\right]$	ndg.

Modality 2 (cont.); here $\boldsymbol{a} := (a_0 + a_1 y)$

	Class	Formula	μ	ł	Zeroes of	Parametrization	Why
41	E_{18}	$x^3+y^{10}+\boldsymbol{a}xy^7$	18	9	$\frac{\partial}{\partial x}$	$\left[x=-\frac{a_0^4t^7}{(a_1t^2+3)^4}, y=-\frac{a_0t^2}{a_1t^2+3}\right]$ for $a_0\neq 0$ $\left[x=\frac{1}{3}\sqrt{-3a_1}t^4, y=t\right]$ for $a_0=0$	2var.
42	E_{19}	$x^3+xy^7+\boldsymbol{a}y^{11}$	19	$\frac{19}{2}$	$\frac{\partial}{\partial x}$	$\left[x=\frac{1}{81}t^7, y=-\frac{1}{3}t^2\right]$	2var.
43	E_{20}	$x^3+y^{11}+\boldsymbol{a}xy^8$	20	10	$\frac{\partial}{\partial x}$	$\left[x=\frac{1}{3}\sqrt{-3a_1t-3a_0}t^4, y=t\right]$	2var.
44	Z_{17}	$x^3y+y^8+\boldsymbol{a}xy^6$	17	7	$\frac{\partial}{\partial x}$	$\left[x=\frac{1}{3}\sqrt{-3a_1t^2+3a_0}t^5, y=-t^2\right]$	2var.
45	Z_{18}	$x^3y+xy^6+\boldsymbol{a}y^9$	18	$\frac{15}{2}$	$\frac{\partial}{\partial x}$	$[x=9t^5, y=-3t^2]$	2var.
46	Z_{19}	$x^3y+y^9+\boldsymbol{a}xy^7$	19	8	$\frac{\partial}{\partial x}$	$\left[x=-\frac{1}{3}\sqrt{3a_1t^2-3a_0}t^6, y=-t^2\right]$	2var.
47	W_{17}	$x^4+xy^5+\boldsymbol{a}y^7$	17	$\frac{17}{3}$	$\frac{\partial}{\partial x}$	$[x=-2t^5, y=2t^3]$	2var.
48	W_{18}	$x^4+y^7+\boldsymbol{a}x^2y^4$	18	6	$\frac{\partial}{\partial x}$	$[x=0, y=t]$	2var.

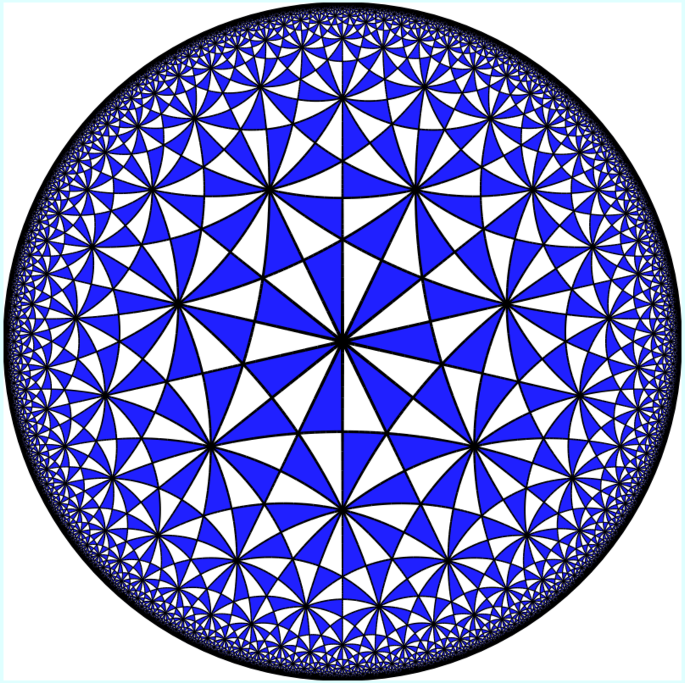
Modality 2 (cont.); here $\boldsymbol{a} := (a_0 + a_1 y)$

	Class	Formula	μ	ł	Zeroes of	Parametrization	Why
49	Q_{16}	$x^3+yz^2+y^7+\boldsymbol{a}xy^5$	16	6	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=-\frac{a_0^3t^5}{(a_1t^2+3)^3},y=-\frac{a_0t^2}{a_1t^2+3},z=0]$ for $a_0\neq0$ $[y=t,x=\sqrt{-\frac{1}{3}a_1t^3},z=0]$ for $a_0=0$	sqh
50	Q_{17}	$x^3+yz^2+xy^5+\boldsymbol{a}y^8$	17	$\frac{13}{2}$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[y=-\frac{1}{3}t^2,x=-\frac{1}{27}t^5,z=0\right]$	sqh
51	Q_{18}	$x^3+yz^2+y^8+\boldsymbol{a}xy^6$	18	7	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[x=\frac{1}{3}\sqrt{-3a_0-3a_1t}\,t^3,y=t,z=0\right]$	sqh
52	S_{16}	$x^2z+yz^2+xy^4+\boldsymbol{a}y^6$	16	$\frac{14}{3}$	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[x=t^5,y=t^3,z=-\frac{1}{2}t^7\right]$	sqh
53	S_{17}	$x^2z+yz^2+y^6+\boldsymbol{a}y^4z$	17	5	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$\left[x=0,y=t,z=-\frac{1}{2}a_0t^3-\frac{1}{2}a_1t^4\right]$	sqh
54	U_{16}	$x^3+xz^2+y^5+\boldsymbol{a}x^2y^2$	16	4	$\frac{\partial}{\partial x},\frac{\partial}{\partial z}$	$[x=0,y=t,z=0]$	sqh

Modality $k-1$

	Class	Formula	μ	ł	Zer.	Parametrization	Why
55	$J_{k,0}$	$x^3+bx^2y^k+y^{3k}+\left(\sum_{j=0}^{k-3}c_jy^j\right)xy^{2k+1}$	$6k-2$	$3k-1$	$\frac{\partial}{\partial x}$	$\left[x=\frac{1}{3}\left(-b+\sqrt{b^2-3\sum_{j=0}^{k-3}c_jt^{j+1}}\right)t^k, y=t\right]$	2var.
56	$J_{k,i}$	$x^3+x^2y^k+\left(\sum_{j=0}^{k-2}a_jy^j\right)y^{3k+i}$	$6k+i-2$	$3k+i-1$	$\frac{\partial}{\partial x}$	$[x=0, y=t]$	2var.
57	E_{6k}	$x^3+y^{3k+1}+\left(\sum_{j=0}^{k-2}a_jy^j\right)xy^{2k+1}$	$6k$	$3k$	$\frac{\partial}{\partial x}$	$\left[x=\frac{\sqrt{3}}{3}\sqrt{-\sum_{j=0}^{k-2}a_jt^{2j}}\,t^{2k+1}, y=t^2\right]$	2var.
58	E_{6k+1}	$x^3+xy^{2k+1}+\left(\sum_{j=0}^{k-2}a_jy^j\right)y^{3k+2}$	$6k+1$	$3k+\frac{1}{2}$	$\frac{\partial}{\partial x}$	$\left[x=\frac{1}{3}i\sqrt{3}t^{2k+1}, y=t^2\right]$	2var.
59	E_{6k+2}	$x^3+y^{3k+2}+\left(\sum_{j=0}^{k-2}a_jy^j\right)xy^{2k+2}$	$6k+2$	$3k+1$	$\frac{\partial}{\partial x}$	$\left[x=\frac{\sqrt{3}}{3}\sqrt{-\sum_{j=0}^{k-2}a_jt^{2j}}\,t^{2k+2}, y=t^2\right]$	2var.

Thanks for your attention!



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