GDAŃSK-KRAKÓW-ŁÓDŹ-WARSZAWA

WORKSHOP IN SINGULARITY THEORY
- A SPECIAL SESSION DEDICATED TO
THE MEMORY OF STANISŁAW ŁOJASIEWICZ

APPROXIMATION AND HOMOTOPY IN REAL ALGEBRAIC GEOMETRY

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1. Real algebraic varieties

DEFINITION 1.1

By a <u>real algebraic variety</u> we mean a set X in $\mathbb{R}^n$, for some n, that can be written as $X = X_1 \setminus X_2$, where X_1, X_2 are algebraic sets in $\mathbb{R}^n$. In other words, X is a Zariski locally closed set in $\mathbb{R}^n$.

Example 1.2

(i) Unit n-sphere $\mathbb{S}^n$:

$$\mathbb{S}^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \dots + x_n^2 = 1\}.$$

(ii) Real projective n-space $\mathbb{P}^n(\mathbb{R})$:

we identify $\mathbb{P}^n(\mathbb{R})$ with the image of the map

$$\mathbb{P}^n(\mathbb{R}) \longrightarrow \mathbb{R}^{(n+1)^2}, \quad (x_0 : \dots : x_n) \longmapsto \left(\frac{x_i x_j}{x_0^2 + \dots + x_n^2} \right)_{\substack{0 \le i \le n \\ 0 \le j \le n}}.$$

2. Regular functions

Let $\varphi: X \longrightarrow \mathbb{R}$ be a function defined on a real algebraic variety $X \subseteq \mathbb{R}^n$.

DEFINITION 2.1

(i) φ is <u>regular at</u> $x_0 \in X$ if there exist an algebraic set $Z \subseteq \mathbb{R}^n$, with $x_0 \in X \setminus Z$, and two polynomial functions $P, Q: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$Q(x) \neq 0 \text{ and } \varphi(x) = \frac{P(x)}{Q(x)} \text{ for all } x \in X \setminus Z.$$

(ii) φ is <u>regular</u> (or <u>regular on</u> X) if φ is regular at every point in X.

GLOBAL REPRESENTATION

φ is regular on X iff there exist two polynomial functions $P, Q: \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$Q(x) \neq 0 \text{ and } \varphi(x) = \frac{P(x)}{Q(x)} \text{ for all } x \in X.$$

3. Regular maps

Let $f: X \longrightarrow Y$ be a map between real algebraic varieties $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^p$. Then

$$f = (f_1, \dots, f_p) : X \longrightarrow Y \subseteq \mathbb{R}^p.$$

DEFINITION 3.1

The map f is called <u>regular</u> if the components $f_1, \dots, f_p$ are regular functions.

Convention 3.2

Unless explicitly stated otherwise, we consider real algebraic varieties endowed with the Euclidean topology.

Notation 3.3

$\mathcal{R}(X,Y) :=$ the set of all regular maps $X \longrightarrow Y$

$\mathcal{C}(X,Y) :=$ the set of all continuous maps $X \longrightarrow Y$

$$\mathcal{R}(X,Y) \subseteq \mathcal{C}(X,Y)$$

$\mathcal{C}(X,Y)$ is endowed with the compact-open topology

4. General approximation problem

Let X and Y be real algebraic varieties.

PROBLEM 4.1

Which continuous maps $f: X \longrightarrow Y$ can be approximated by regular maps?

Equivalently, which continuous maps $f: X \longrightarrow Y$ belong to the closure $\overline{\mathcal{R}(X,Y)}$ of $\mathcal{R}(X,Y)$ in $\mathcal{C}(X,Y)$?

Example 4.2

(i) $\overline{\mathcal{R}(X,\mathbb{R}^p)} = \mathcal{C}(X,\mathbb{R}^p)$ by the Weierstrass approximation theorem.

(ii) Let $Y = \{(x,y) \in \mathbb{R}^2 : y^2 = x^3 - 1\}$. Note that Y is a real analytic submanifold of $\mathbb{R}^2$ that is diffeomorphic to $\mathbb{R}$. We have

$$\mathcal{R}(\mathbb{R},Y) = \overline{\mathcal{R}(\mathbb{R},Y)} = \text{the constant maps}.$$

5. Maps into spheres

THEOREM 5.1 (Bochnak - K, 2022)

Let X be a real algebraic variety, and $f \in \mathcal{C}(X, \mathbb{S}^p)$. Then the following conditions are equivalent:

(a) $f \in \overline{\mathcal{R}(X, \mathbb{S}^p)}$.

(b) f is homotopic to a regular map.

This is a special case of our main theorem that will be discussed later.

COROLLARY 5.2

If a continuous map $f: X \longrightarrow \mathbb{S}^p$ is null homotopic, then $f \in \overline{\mathcal{R}(X, \mathbb{S}^p)}$.

Example 5.3 (Bochnak-K, 1988)

$\overline{\mathcal{R}(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{S}^2)}$ = the null homotopic maps

6. Maps between spheres

CONJECTURE 6.1

$\overline{\mathcal{R}(\mathbb{S}^n, \mathbb{S}^p)} = \mathcal{C}(\mathbb{S}^n, \mathbb{S}^p)$.

This conjecture has been studied since at least 1980s or even 1960s (in the 1960s the main related question was whether every continuous map $\mathbb{S}^n \to \mathbb{S}^p$ is homotopic to a regular map). It is still largly open. In our paper, we have the following result.

THEOREM 6.2 (Bochnak-K, 2022)

$\overline{\mathcal{R}(\mathbb{S}^n, \mathbb{S}^p)} = \mathcal{C}(\mathbb{S}^n, \mathbb{S}^p)$ in each of the following cases:

(i) $p \in \{1, 2, 4\}$,

(ii) $n - p \leq 3$

(iii) $4 \leq n - p \leq 5$ with possible exception for the pairs $(9,5)$, $(7,3)$, $(11,6)$, $(10,5)$, $(8,3)$.

7. Linear real algebraic groups

$\mathrm{Mat}_n(\mathbb{R})$:= the set of all real $n\times n$ matrices

$\mathrm{Mat}_n(\mathbb{R}) = \mathbb{R}^{n^2}$

$GL_n(\mathbb{R}) = \{A\in \mathrm{Mat}_n(\mathbb{R}) : \det(A)\neq 0\}$ is a real algebraic variety

DEFINITION 7.1

A <u>linear real algebraic group</u> G is a subgroup of $GL_n(\mathbb{R})$, for some n, such that $G = GL_n(\mathbb{R})\cap Z$ for some algebraic set $Z\subseteq \mathrm{Mat}_n(\mathbb{R})$.

Example 7.2

$$SO(n)\subseteq O(n)\subseteq GL_n(\mathbb{R}).$$

8. Homogeneous spaces

Let G be a linear real algebraic group, and Y a real algebraic variety.

DEFINITION 8.1

Y is a <u>G-variety</u> if G acts on Y, and the map

$$G \times Y \longrightarrow Y, \quad (a, y) \longmapsto a \cdot y$$

is regular.

DEFINITION 8.2

Y is a <u>homogeneous space</u> for G if Y is a G-variety and G acts transitively on Y.

Example 8.3

(i) G is a homogeneous space for G.

(ii) $\mathbb{S}^n$ is a homogeneous space for $O(n+1)$.

(iii) $\mathbb{P}^n(\mathbb{R})$ is a homogeneous space for $O(n+1)$.

9. Maps into homogeneous spaces

THEOREM 9.1 (Bochnak-K, 2022, main thm.)

Let X, Y be real algebraic varieties. Assume that Y is a homogeneous space for some linear real algebraic group. Then, for a continuous map $f: X \to Y$, the following conditions are equivalent:

(a) $f \in \overline{\mathcal{R}(X,Y)}$.

(b) f is homotopic to a regular map.

I will give a brief outline of the proof of this theorem. But first, some preparation is necessary.

10. Dominating sprays

Let Y be a nonsingular real algebraic variety.

DEFINITION 10.1

(i) A <u>spray</u> for Y is a regular map $s: Y \times \mathbb{R}^n \to Y$ such that $s(y,0) = y$ for all $y \in Y$.

(ii) A spray $s: Y \times \mathbb{R}^n \to Y$ is <u>dominating</u> if for every point $y \in Y$, the map

$$s(y,\cdot): \mathbb{R}^n \to Y, \quad v \longmapsto s(y,v)$$

is a submersion at $0 \in \mathbb{R}^n$.

(iii) Y is a <u>malleable</u> variety if it admits a dominating spray (for some n).

Example 10.2 (trivial)

The map

$$s: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad (y,v) \longmapsto y+v$$

is a dominating spray for $\mathbb{R}^n$.

11. Maps into malleable varieties

THEORE 11.1 (Bochnak-K, 2022)

Let X, Y be real algebraic varieties. Assume that the variety Y is malleable. Then, for a continuous map $f: X \longrightarrow Y$, the following conditions are equivalent:

(a) $f \in \overline{\mathcal{R}(X,Y)}$.

(b) f is homotopic to a regular map.

<u>Sketch of proof.</u>

(a)$\Longrightarrow$(b) is a general fact. It is known that for some compact set $K \subseteq X$ the inclusion map $K \hookrightarrow X$ is a homotopy equivalence. If $h \in \mathcal{R}(X,Y)$ is close to f, then $h|_K$ is close to $f|_K$ in $\mathcal{C}(K,Y)$, so $f|_K$ is homotopic to $h|_K$. It follows that f is homotopic to h.

11. Maps into malleable varieties (continued)

(b)$\Rightarrow$(a). For simplicity, we assume that X is nonsingular and compact. Let $F: X\times[0,1]\longrightarrow Y$ be a homotopy such that F_0 is a regular map and $F_1 = f$. Here, for $t\in[0,1]$, we define $F_t : X\longrightarrow Y$ by $F_t(x) = F(x,t)$.

Then there exists a dominating spray $s: Y\times\mathbb{R}^n\longrightarrow Y$ for Y and a continuous map $\eta: X\times[0,1]\longrightarrow\mathbb{R}^n$ such that

$$s(F(x,0),\eta(x,t)) = F(x,t) \text{ for all } (x,t)\in X\times[0,1]$$

(the proof of this assertion is rather long and technical). In particular,

$$s(F_0(x),\eta(x,1)) = F(x,1) = f(x) \text{ for all } x\in X.$$

If $\beta: X\longrightarrow\mathbb{R}^n$ is a regular map close to the map $X\longrightarrow\mathbb{R}^n$, $x\longmapsto\eta(x,1)$, then $g: X\longrightarrow Y$ defined by

$$g(x) = s(F_0(x),\beta(x)) \text{ for all } x\in X$$

is a regular map close to f. $\square$

12. Homogeneous spaces are malleable varieties

PROPOSITION 12.1 (Bochnak-K, 2022)

Let Y be a homogeneous space for a linear real algebraic group G. Then Y is a malleable variety.

<u>Proof.</u> Let $n := \dim G$ and let G_0 be the irreducible component of G that contains the neutral element e of G. By Chevalley's theorem (1954), G_0 is a unirational variety. This means that there exist a Zariski open set $U \subseteq \mathbb{R}^n$ and a regular map $\varphi : U \longrightarrow G_0$ with $\varphi(U)$ Zariski dense in G_0. Thus, for some point $u \in U$, the derivative

$$d_u\varphi : T_u U = \mathbb{R}^n \longrightarrow T_{\varphi(u)} G_0 = T_{\varphi(u)} G$$

is an isomorphism. Using translations (in $\mathbb{R}^n$ and in G), we may assume that $u = 0 \in U$ and $\varphi(u) = e$, so

$$d_0\varphi : \mathbb{R}^n \longrightarrow T_e G$$

is an isomorphism. Define

$$h: \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad h(v) = \frac{cv}{1+\|v\|^2},$$

where $c > 0$. Clearly, the derivative

$$d_0 h: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is an isomorphism. Moreover,

$$h(\mathbb{R}^n) \subseteq U$$

if c is sufficiently small. The composite

$$g := \varphi \circ h : \mathbb{R}^n \longrightarrow G$$

is a regular map, and the derivative

$$d_0 g : \mathbb{R}^n \longrightarrow T_e G$$

is an isomorphism.

It follows that the regular map

$$s: Y \times \mathbb{R}^n \longrightarrow Y, \quad s(y,v) = g(v)\cdot y$$

is a dominating spray for Y. $\square$

The use of dominating sprays was inspired by Gromov's work in complex geometry, further developed by Forstnerič and others.

Thank you for your attention.