Łojasiewicz' Gradient Inequality for analytic (o-minimal) maps

Thm (Łojasiewicz 60's) $f: \mathcal{U} \to \mathbb{R}$ analytic $0 \in \mathcal{U} \subset \mathbb{R}^n$, $f(0) = 0$, $\nabla f(0) = 0$, then $\exists$ <u>$\varsigma < 1$</u>, $c > 0$ such that

$$|\nabla f| \geq c\,|f|^{\varsigma} \quad \text{in a nghb of } 0.$$

Equivalent formulation:

$\exists\ \Psi : (-\varepsilon, 0) \cup (0, \varepsilon) \to \mathbb{R}$ suban. continuous increasing such that

$$|\nabla(\Psi \circ f)| \geq c > 0 \quad (c = 1)$$

for $x \in \mathcal{U}$ close to $0 \in \mathbb{R}^n$.

(Take $\Psi(t) = t\,|t|^{-\varsigma}$)

<u>Corollary</u>. Let S be C^1 curve in $\mathcal{U}$ s.t

a) $\measuredangle(T_x S, \operatorname{Ker} d_x f) \geq \alpha > 0$,

b) $f|_S$ is injective.

Then: length $S \leq C(\alpha) \cdot \text{length}(\Psi \circ f)(S) < +\infty$

In particular trajectories of ∇f have finite length.

We consider $f: \mathcal{U} \to \mathbb{R}^k$ analytic, $\mathcal{U} \subset \mathbb{R}^n$, $\underline{k \leq n}$.
$\operatorname{rank} d_x f = k$ (generically).

<u>Distance to singular maps.</u>

X, Y normed spaces, $L(X,Y) = \{A : X \to Y$, lin. and cont$\}$

$\Sigma = \{A : A$ - nonsurjective$\}$.

$$\nu(A) \overset{\text{def}}{=} \operatorname{dist}(A, \Sigma)$$

<u>Prop1.</u> $\nu(A) = \sup\{r \geq 0 : B_Y(r) \subset A(B_X(1))\}$.

Prop2. 1) $Y = \mathbb{R} \Rightarrow \nu(A) = \|A\|$.

2) $A : X \to Y$ isom. $\Rightarrow \nu(A) = \dfrac{1}{\|A^{-1}\|}$.

3) $X = \mathbb{R}^n$, $Y = \mathbb{R}^k$ with Euclid. norms

a) $\nu(A) = \nu(A|(\ker A)^{\perp})$

b) $\nu(A) = \min\{|\mu| : \mu^2$ - eigenvalue of $AA^*\}$

4) let $k(A) = \max\{\nu(A|Z) : Z$ lin. s. of X, $\dim Z = k\}$

$g(A) =$ —— | —— Z coordin. k-plane$\}$.

a) $\alpha g(A) \leq \nu(A) \leq \beta g(A$ $\quad \alpha = \alpha(k,n)$, $\beta = \beta(k,n)$

b) for Euclid. norm $k(A) = \nu(A)$

<u>Question</u> $k(A) = \nu(A)$ for any norms?
$\leq$ holds always.

Theorem 1 $f : \mathcal{U} \to \mathbb{R}^k$ analytic, nondegenerate $\mathcal{U} \subset \mathbb{R}^n$, $K \subset \mathcal{U}$ compact. Then there exists $\Psi : W \to \mathbb{R}^k$ suban, analytic, bounded and injective such that

$$\nu\left(d_x(\Psi \circ f)\right) \geq c > 0 \qquad (c = 1)$$

for any $x \in f^{-1}(W) \cap K$.

Moreover $W = V \setminus (K_f \cup \Gamma)$, where

K_f = critical values of f on K,

Γ - suban, closed, nowhere dense in $\mathbb{R}^k$

Corollary, f as above, S a k-dim C^1 subman. of $\mathcal{U} \subset \mathbb{R}^n$.

Assume that

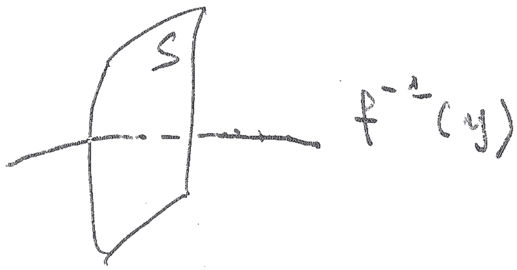
1) $\bar{S} \subset \mathcal{U}$ is compact, $\bar{S} \subset \mathcal{U} \setminus \{\text{crit. points of } f\}$,

2) f injective on S,

3) $\exists_{\alpha > 0}$ $\measuredangle(T_x S, \operatorname{Ker}(d_x f)) \geq \alpha$ for $\forall_{x \in S}$.

Then $\operatorname{vol}_k(S) < +\infty$, more precisely

$$\operatorname{vol}_k(S) \leq C(\alpha, c) \operatorname{vol}_k(\Psi(f(S))).$$



Towards a nicer statement

<u>Prop</u> (K., W. PAWŁUCKI). Let $\psi : W \to \mathbb{R}$, $W \subset \mathbb{R}^k$ be a C^1 suban, bounded function. Then

$$|\nabla \psi(y)| \leq \frac{C}{\operatorname{dist}(y, \Gamma_1)^{\varrho}}$$

where Γ_1 is suban., closed, nowhere dense in $\mathbb{R}$

In general $\varrho = 1$.
If ψ extends continuously on $\overline{W}$ (compact) then $\varrho < 1$.

Applying this to components of $\Psi = (\psi_1, \dots, \psi_n)$ we obtain a subset $\Gamma_1 \subset \mathbb{R}^k$ such that

$$\|d_y \Psi\| \leq \frac{C}{\operatorname{dist}(y, \Gamma_1)^{\varrho}}.$$

<u>Theorem 2 ($\Leftrightarrow$ Thm 1)</u> $f : U \to \mathbb{R}^k$ an. nondegen. $U \subset \mathbb{R}^n$, $K \subset U$ compact, then

$$\nu(d_x f) \geq C \operatorname{dist}(f(x), \Gamma_1), \; x \in K,$$

where $\Gamma_1 \subset \mathbb{R}^k$ is suban, closed, nowhere dense in $\mathbb{R}^k$.

Example $f(x,y) = (x, xy)$

$$A = \begin{pmatrix} 1 & 0 \\ y & x \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{y}{x} & \frac{1}{x} \end{pmatrix}$$

$$\nu(A) = \frac{1}{\|A^{-1}\|} = |x| = \mathrm{dist}(f(x), \Gamma_1)$$

$$\Gamma_1 = \{0\} \subset \mathbb{R}^2.$$

So in general one cannot have $\varsigma < 1$ in Thm 2.

But if the function Ψ extends continuously on $\bar{W}$, then we have

$$(*) \qquad \nu(d_x f) \geq C\, \mathrm{dist}(f(x), \Gamma_1)^{\varsigma}$$

where $\varsigma < 1$. In particular it holds if $k = 1$.

<u>Theorem 3.</u> $f: \mathcal{U} \to \mathbb{R}^k$ an. nondegen. $\mathcal{U} \subset \mathbb{R}^n$, $K \subset \mathcal{U}$ compact. Assume that f is finite on the critical locus of f (on K).

Then
$$\nu(d_x f) \geq C\, \mathrm{dist}(f(x), \Gamma_1)^{\varsigma}, \quad x \in K,$$

with $\varsigma < 1$, $\Gamma_1 \subset \mathbb{R}^k$ suban, closed, nowhere dense in $\mathbb{R}^k$

Remark (conjecture)

In $\mathbb{C}$-case $\Gamma_1 = K_f$ = critical values of f

but $\varsigma = ?$

Some conjectures (and speculations)

Is the validity of $(*)$ $(\rho < 1)$ related to:

- f is "sans eclatement"
- f triangulable
- f admits A_f (or W_f) stratification.