

# The Łojasiewicz exponent of the gradient of a plane complex curve with respect to its polar curve

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GDAŃSK-KRAKÓW-ŁÓDŹ-WARSZAWA  
WORKSHOP IN SINGULARITY THEORY

A special session dedicated to the memory of  
STANISŁAW ŁOJASIEWICZ

Warszawa, December 12-16, 2022

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- For  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  holomorphic with an isolated singularity at zero  $0 \in \mathbb{C}^2$ ,  $\mathfrak{I}(f)$  is the smallest  $\theta > 0$  in the inequality

$$|\text{grad } f(z)| \geq C|z|^\theta$$

for a positive constant  $C$  and small  $|z|$ .

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$$|\text{grad } f(z)| \geq C|z|^\theta$$

for a positive constant  $C$  and small  $|z|$ .

- An application (Chang & Lu 1973, Teissier 1977)

$$\lfloor \mathfrak{l}(f) \rfloor + 1$$

equals the minimal possible  $r$  such that adding to  $f$  monomials of order strictly greater than  $r$  does not change the equisingularity class of  $f$ .

# The relative Łojasiewicz exponent

- Restricting the inequality

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- $\mathfrak{t}(f) \geq \mathfrak{t}(f|A)$
- We say that  $\mathfrak{t}(f)$  is attained on  $A$  if  $\mathfrak{t}(f) = \mathfrak{t}(f|A)$ .

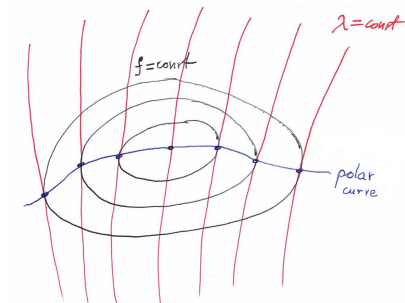
# Polar curves

- Every smooth  $\lambda : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  defines a polar curve of  $f$

$$\Gamma_{f,\lambda} = \{\mathbf{J}(\lambda, f) = 0\}$$

where

$$\mathbf{J}(\lambda, f) = \frac{\partial \lambda}{\partial X} \frac{\partial f}{\partial Y} - \frac{\partial \lambda}{\partial Y} \frac{\partial f}{\partial X}.$$



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- Bogusławska 1999, Kuo & Parusiński 1998, Płoski 2001: if  $\lambda$  is transversal to  $f$  then

$$\mathbf{t}(f) = \mathbf{t}(f|\Gamma_{f,\lambda}) .$$

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- Proof. Suppose that there exist two  $(a : b) \neq (c : d)$  special directions. Then there exist  $\lambda$  tangent to  $(a : b)$  and  $\mu$  tangent to  $(c : d)$  such that

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- Contradiction follows from Chądzyński & Krasieński 1988.

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- The special direction exists if and only if there exists exactly one maximum among numbers  $\mathbf{t}_1, \dots, \mathbf{t}_t$ .
- If the special direction exists then for every  $\lambda$  tangent to this direction

$$\mathbf{t}(f) > \mathbf{t}(f|_{\Gamma_{f,\lambda}}).$$

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- If  $f$  is unitangent ( $t(f) = 1$ ) then the direction tangent to  $f$  is special.
- If  $f$  has only smooth and pairwise transversal components then there is no special direction and  $\mathfrak{t}(f) = \text{ord } f - 1$ .

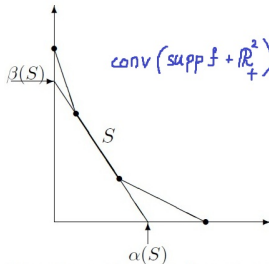
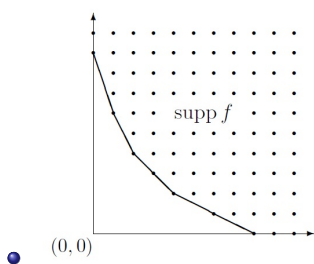
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- More difficult example of series without the special direction:  $f^{(1)} = X(X + Y^2)(X^2 + Y^3)$ ,  $f^{(2)} = Y^2 + X^5$ ,  $f = f^{(1)}f^{(2)}$ .  
This example gives an occasion to show how to compute the Łojasiewicz exponent by using the Newton diagram.

# Łojasiewicz exponent and the Newton Polygon (set of pairwise nonparallel compact segments of the boundary)

- $f(X, Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta$ ,  $\text{supp } f = \{(\alpha, \beta) : c_{\alpha\beta} \neq 0\}$ .

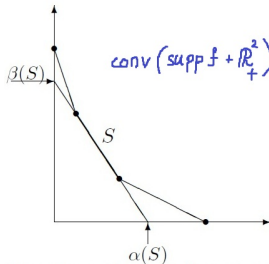
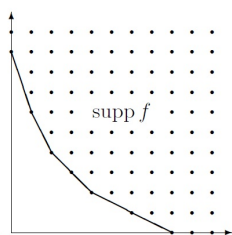
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- $(0,0)$
- We say that  $f$  is nondegenerate on the segment  $S$  of the Newton polygon if the polynomial

$$\text{in}(f, S) = \sum_{(\alpha, \beta) \in S} c_{\alpha\beta} X^\alpha Y^\beta$$

has no multiple factors different from  $X$  and  $Y$ .

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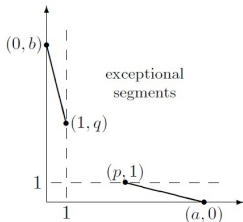
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- DEF  $f$  is nondegenerate in the Kouchnirenko sense if  $f$  is nondegenerate on every segment of the Newton polygon.
- Theorem (A.L. 1998)  $f$  has an isolated singularity, the Newton polygon has at least one nonexceptional segment,  $f$  is nondegenerate in the Kouchnirenko sense. Then

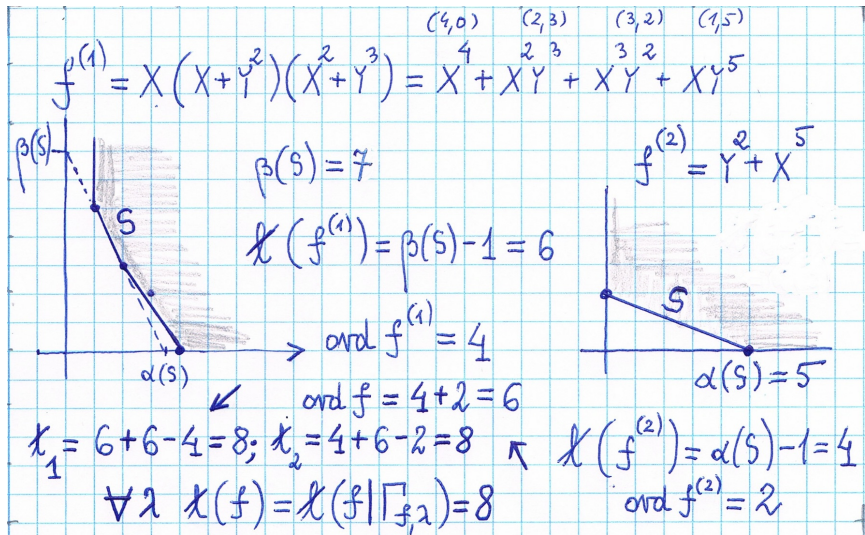
$$\sharp(f) = \max_S \{\alpha(S), \beta(S)\} - 1$$

where  $S$  runs over the nonexceptional segments of the Newton polygon





# Computations



# Intersection multiplicity of local curves $f$ and $g$ , semigroup of branch

- Algebra:  $(f, g)_0$  is the  $\mathbb{C}$  codimension of the ideal generated by  $f$  and  $g$  in the ring  $\mathbb{C}\{X, Y\}$

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- Semigroup  $\Gamma$  of a branch  $h$  (irreducible curve)

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- $\Gamma(h) = \mathbb{N}\bar{\beta}_0 + \cdots + \mathbb{N}\bar{\beta}_g$ ,  $\bar{\beta}_0 < \cdots < \bar{\beta}_g$  the minimal sequence of generators, called the branch characteristics (singularity invariant).

# Description of $\mathfrak{t}(f|\Gamma_{f,\lambda})$

$\mathfrak{J}(\lambda, f) = g_1 \dots g_s$  factorization into branches

Thm a.  $\lambda$  is not a branch of  $f$

$$\mathfrak{X}(f|\Gamma_{f,\lambda}) = \max_j \frac{(f, g_j)_0}{\text{ord } g_j} - \frac{(\lambda, g_j)_0}{\text{ord } g_j}$$

we may write

$$\frac{\left(\frac{f}{\lambda}, g_j\right)_0}{\text{ord } g_j}$$

b. in general

$$\mathfrak{X}(f|\Gamma_{f,\lambda}) = \max \frac{\left(\frac{f}{\lambda}, g\right)_0}{\text{ord } g}$$

any branch of  $\mathfrak{J}(\lambda, f) \rightarrow g$

The case  $f = \lambda \tilde{f}$

$$\mathfrak{J}(\lambda, f) = \lambda \mathfrak{J}(\lambda, \tilde{f})$$

$$\frac{f}{\lambda} = \tilde{f}$$

# Version of Eggers tree: A.L. 2011, 2013

logarithmic distance of Płoski (1985):  $d(f, g) = \frac{(f, g)_0}{(\text{ord } f)(\text{ord } g)}$

normalised intersection multiplicity  $d(f, g) \geq 1$

branches  $\rightarrow (f, g)_0$

$\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$  analogy

$g(f) =$  the nb. of char. pairs

$\delta_g$

$\delta_2 = \sup \{ d(f, h) : h \text{ smooth or has one characteristic pair} \}$

$\delta_1 = \sup \{ d(f, h) : h \text{ smooth} \}$

$\delta_k = \frac{\text{GCD}(\bar{\beta}_0, \dots, \bar{\beta}_{k-1}) \bar{\beta}_k}{(\bar{\beta}_0)^2}$  Lejeune-Jalabert 1973

$B = B(f, g)$   
 $\uparrow$  the ball defined by intersection of two branches  
 $R = d(f, g)$ ; any branch is a centre

branch joins level 1 with level  $\infty$

every point may be identified with the ball  $B(f, R) = \{ g \text{ branch} : d(f, g) \geq R \}$

the ball with radius 1 = the set of all branches

branch  $f$

$\infty$

$1$



# Positions of branches of $J(\lambda, f)$ with respect to $f$ . Spirit of Kuo Lu Lemma 1977

some results:

- Teissier 1975, 1977, 1980  $\mathbb{C}^m$
- D. T. Lê 1975 top. app.
- Merle 1977  $f$  branch,  $\lambda$  gen.
- Ephraïm 1983  $f$  branch,  $\lambda$  arb.
- Kuo Lu 1977, Kuo-Lu tree
- Eggers 1982, Eggers tree
- $f$  reduced,  $\lambda$  gen
- Lê, Michel, Weber 1989, 1991
- $f$  reduced,  $\lambda$  h.f, no mult.
- Delgado 1994, two branches
- Casas 2000, summary
- García Barroso 2000
- developed Eggers
- Maugendre 1998, pair  $f, g$  no mult
- A.L., Płoski 2000,  $f$  nondeg,  $\lambda$  h.f
- Gwoździevicz, Płoski 2002
- $f$  reduced,  $\lambda$  arb. no mult
- Kuo, Pamiński 2004, pair  $f, g$
- some exceptions: colinear bars
- A.L. 2004  $f$  reduced,  $\lambda$  arb.
- Michel 2008 pair  $f, g$

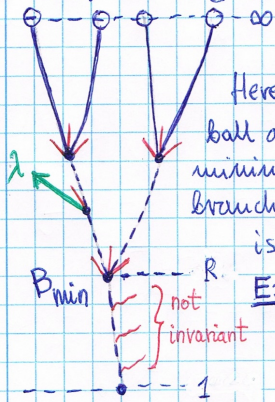
When computing  $\kappa(f|\Gamma_f, \lambda)$  it is crucial to determine the positions of branches (red)  $g_1, \dots, g_s$  of  $J(\lambda, f)$  with respect to  $f$  and  $\lambda$ .

Eggers observed that red branches leave the tree at the balls of the type  $B(f_i, f_j)$ ,  $B(f_i, \delta_{ik})$ . He described ord  $g_B$ . Eggers assumed  $\lambda$  generic. Other authors observed that more precise is  $\lambda$  transversal to  $f$ .

Eggers applied dashed segments to show which branch has the characteristic order of contact  $\delta_k$ .

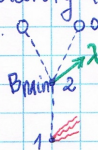
# The case when $\lambda$ is tangent to $f$

Kuo-Parus 2004 observed that when the tree (Kuo-Lu) of the pair  $f, g$  has what they call collinear points or bars, the way the roots of  $J(f, g)$  leave the tree is not an invariant of the tree.



Here we have similar: we define  $B_{\min}$  as the ball among  $B(f_i, f_j)$ ,  $B(f_i, \delta_{ik})$ ,  $B(f_i, \lambda)$  with the minimal radius  $R$ . When  $R > 1$  then the way how branches of  $J(\lambda, f)$  leave the tree of  $f$  below  $B_{\min}$  is not equisingularity invariant of the pair  $f, \lambda$ .

Ex  $f = Y^4 - X^2$   
 $\lambda = X$   
 $\chi(f | \Gamma_{f, \lambda}) = 1$



$f' = Y^4 - X^2 + XY^2$   
 $\lambda' = X$   
 $\chi(f' | \Gamma_{f', \lambda'}) = \frac{3}{2}$

