The Łojasiewicz exponent of the gradient of a plane complex curve with respect to its polar curve

Andrzej Lenarcik

Kielce University of Technology

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The Łojasiewicz exponent of the gradient

- For $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ holomorphic with an isolated singularity at zero $0 \in \mathbb{C}^2$, $\text{ł}(f)$ is the smallest $\theta > 0$ in the inequality

$$|\text{grad}\, f(z)| \geqslant C|z|^{\theta}$$

for a positive constant C and small $|z|$.

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  for a positive constant C and small $|z|$.
- An application (Chang & Lu 1973, Teissier 1977)

$$\lfloor \text{ł}(f) \rfloor + 1$$

  equals the minimal possible r such that adding to f monomilas of order strictly greater than r does not change the equisingularity class of f.

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- $ł(f) \geqslant ł(f|A)$
- We say that $ł(f)$ is attained on A if $ł(f) = ł(f|A)$.

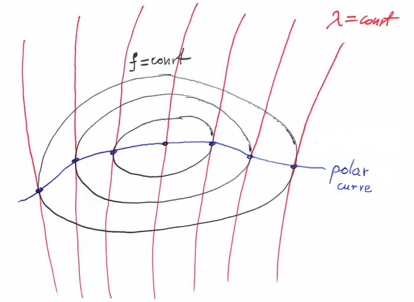
Polar curves

- Every smooth $\lambda : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ defines a polar curve of f

$$\Gamma_{f,\lambda} = \{\mathbf{J}(\lambda, f) = 0\}$$

where

$$\mathbf{J}(\lambda, f) = \frac{\partial \lambda}{\partial X}\frac{\partial f}{\partial Y} - \frac{\partial \lambda}{\partial Y}\frac{\partial f}{\partial X} .$$



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- Bogusławska 1999, Kuo & Parusiński 1998, Płoski 2001: if λ is transversal to f then

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- Proof. Suppose that there exist two $(a:b) \neq (c:d)$ special directions. Then there exist λ tangent to $(a:b)$ and μ tangent to $(c:d)$ such that

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- Contradiction follows form Chądzyński & Krasiński 1988.

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- The special direction exists if and only if there exists exactly one maximum among numbers $\text{\l}_1, \dots, \text{\l}_t$.
- If the special direction exists then for every λ tangent to this direction

$$\text{\l}(f) > \text{\l}(f|\Gamma_{f,\lambda}) \, .$$

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- If f is unitangent $(t(f) = 1)$ then the direction tangent to f is special.
- If f has only smooth and pairwise transversal components then there is no special direction and $\text{ł}(f) = \operatorname{ord} f - 1$.
- More difficult example of series without the special direction: $f^{(1)} = X(X + Y^2)(X^2 + Y^3)$, $f^{(2)} = Y^2 + X^5$, $f = f^{(1)} f^{(2)}$. This example gives an occasion to show how to compute the Łojasiewicz exponent by using the Newton diagram.

Łojasiewicz exponent and the Newton Polygon (set of pairwise nonparallel compact segments of the boundary)

- $f(X,Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta, \quad \operatorname{supp} f = \{(\alpha,\beta) : c_{\alpha\beta} \neq 0\}.$

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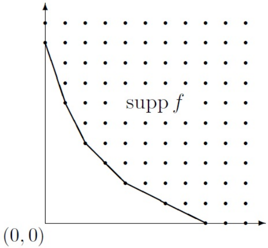

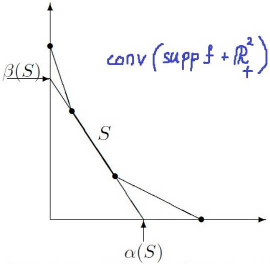

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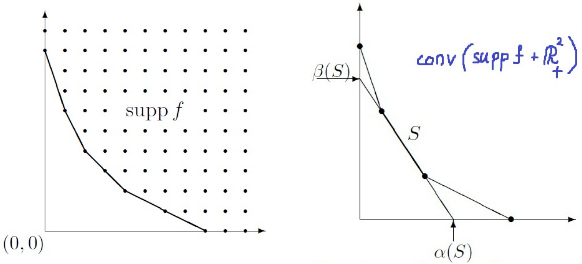
- 

- We say that f in nondegenerate on the segment S of the Newton polygon if the polynomial

$$\operatorname{in}(f,S) = \sum_{(\alpha,\beta)\in S} c_{\alpha\beta} X^\alpha Y^\beta$$

has no multiple factors different from X and Y.

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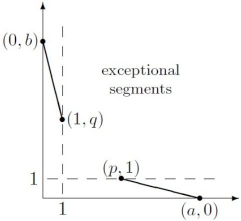
- Theorem (A.L. 1998) f has an isolated singularity, the Newton polygon has at least one nonexceptional segment, f is nondegenerate in the Kouchnirenko sense. Then

$$\text{ł}(f) = \max_{S}\{\alpha(S), \beta(S)\} - 1$$

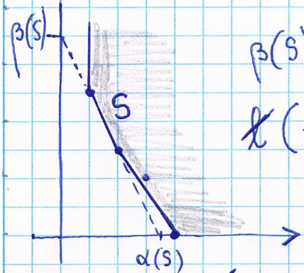
where S runs over the nonexceptional segments of the Newton polygon



Computations

$(4,0) \quad (2,3) \quad (3,2) \quad (1,5)$

$$f^{(1)} = X(X+Y^2)(X^2+Y^3) = X^4 + X^2Y^3 + X^3Y^2 + XY^5$$

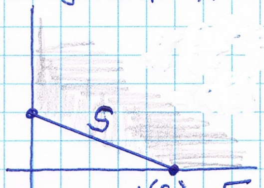


$$\beta(S) = 7$$

$$\mathcal{L}(f^{(1)}) = \beta(S) - 1 = 6$$

$$\mathrm{ord}\, f^{(1)} = 4$$

$$f^{(2)} = Y^2 + X^5$$



$$\alpha(S) = 5$$

$$\mathrm{ord}\, f = 4 + 2 = 6$$

$$\mathcal{L}_1 = 6 + 6 - 4 = 8; \quad \mathcal{L}_2 = 4 + 6 - 2 = 8$$

$$\mathcal{L}(f^{(2)}) = \alpha(S) - 1 = 4$$

$$\forall \lambda \quad \mathcal{L}(f) = \mathcal{L}(f|\Gamma_{f,\lambda}) = 8$$

$$\mathrm{ord}\, f^{(2)} = 2$$

Intersection multiplicity of local curves f and g, semigroup of branch

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- $\Gamma(h) = \mathbb{N}\,\bar{\beta}_0 + \cdots + \mathbb{N}\,\bar{\beta}_{\mathbf{g}}$, $\bar{\beta}_0 < \cdots < \bar{\beta}_{\mathbf{g}}$ the minimal sequence of generators, called the branch characteristics (singularity invariant).

$\mathcal{J}(\lambda, f) = g_1 \cdots g_s$ factorization into branches

Thm a. λ is not a branch of f

$$\text{ł}(f|\Gamma_{f,\lambda}) = \max_j \frac{(f, g_j)_0}{\operatorname{ord} g_j} - \frac{(\lambda, g_j)_0}{\operatorname{ord} g_j}$$

we may write

$$\frac{\left(\frac{f}{\lambda}, g_j\right)_0}{\operatorname{ord} g_j}$$

b. in general

$$\text{ł}(f|\Gamma_{f,\lambda}) = \max_g \frac{\left(\frac{f}{\lambda}, g\right)_0}{\operatorname{ord} g}$$

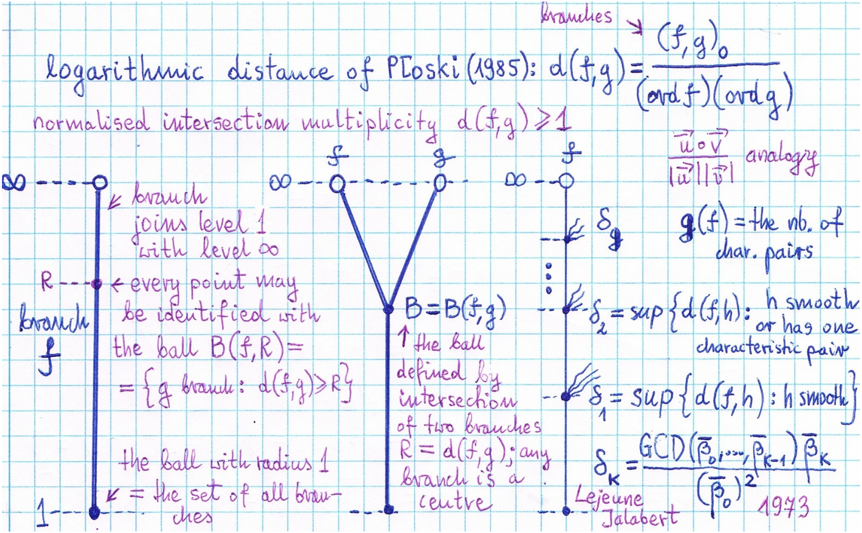
any branch of $\mathcal{J}(\lambda, f)$ → g

The case $f = \lambda \tilde{f}$

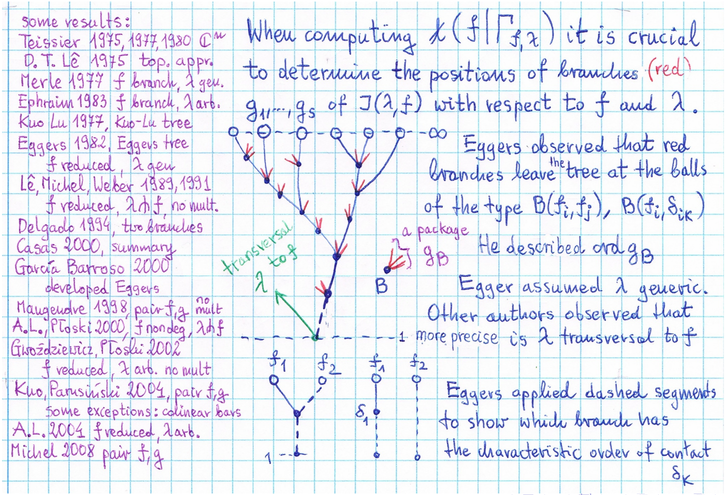
$$\mathcal{J}(\lambda, f) = \lambda \mathcal{J}(\lambda, \tilde{f})$$

$$\frac{f}{\lambda} = \tilde{f}$$

Version of Eggers tree: A.L. 2011, 2013

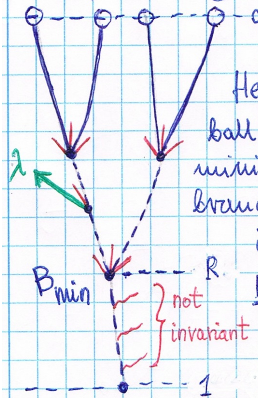


Positions of branches of $\mathbf{J}(\lambda, f)$ with respect to f. Spirit of Kuo Lu Lemma 1977



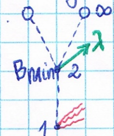
The case when λ is tangent to f

Kuo-Parus 2004 observed that when the tree (Kuo-Lu) of the pair f, g has what they call collinear points or bars, the way the roots of $\mathcal{J}(f,g)$ leave the tree is not an invariant of the tree.



Here we have similar: we define B_{min} as the ball among $B(f_i, f_j)$, $B(f_i, \delta_{ik})$, $B(f_i, \lambda)$ with the minimal radius R. When $R > 1$ then the way how branches of $\mathcal{J}(\lambda, f)$ leave the tree of f below B_{min} is not equisingularity invariant of the pair f, λ.

Ex $f = Y^4 - X^2$,
$\lambda = X$
$\ell(f|\Gamma_{f,\lambda}) = 1$



$f' = Y^4 - X^2 + XY^2$
$\lambda' = X$
$\ell(f'|\Gamma_{f',\lambda'}) = \frac{3}{2}$