# Lipschitz Geometry of Semialgebraic Surface Germs. Overview.

**Lev Birbrair** (UFC, Fortaleza, Brazil, Jagiellonian University, Krakow, Poland)

Banach Center Conference dedicated to the memory of Stanislaw Lojasiewicz.

December 15, 2022.

All sets and maps are definable in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the field of exponents $\mathbb{F}$, e.g., semialgebraic or subanalytic with $\mathbb{F} = \mathbb{Q}$.

A set $X \subset \mathbb{R}^n$ inherits from $\mathbb{R}^n$ two metrics:
the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in $X$ connecting $x$ and $y$.

$X$ is **Normally Embedded** if these two metrics on $X$ are equivalent.

A **surface germ** is a closed two-dimensional germ $X$ at the origin.

**Lipschitz classification problems:**

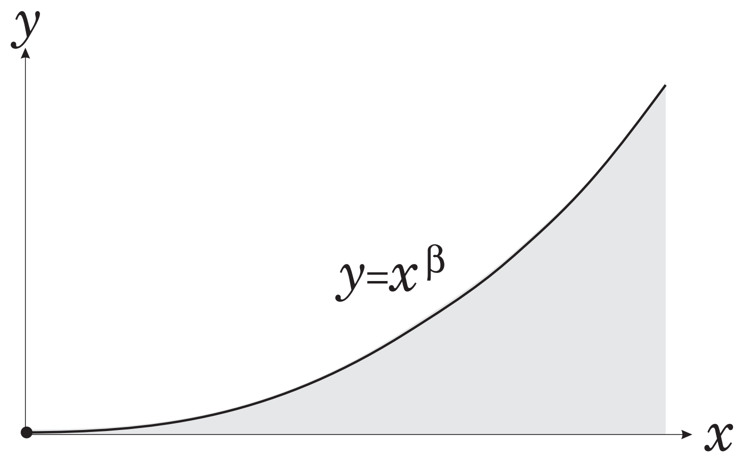
1) **Inner Lipschitz equivalence:** $(X,0) \sim_i (Y,0)$ if there is a homeomorphism $h : (X,0) \to (Y,0)$ bi-Lipschitz with respect to the inner metric.

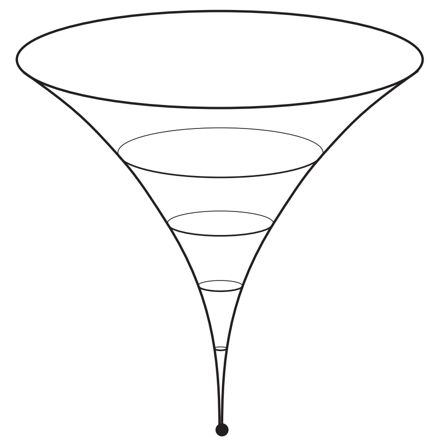
2)**Outer Lipschitz equivalence:** $(X,0) \sim_o (Y,0)$ if there is a homeomorphism $h : (X,0) \to (Y,0)$ bi-Lipschitz with respect to the outer metric.

3) **Ambient Lipschitz equivalence:** $(X,0) \sim_a (Y,0)$ if there is an orientation preserving bi-Lipschitz homeomorphism $H : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ such that $H(X) = Y$.

For $\beta \in \mathbb{F}$, $\beta \geq 1$, the **standard $\beta$-Hölder triangle** is the set
$$T_\beta = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ 0 \leq y \leq x^\beta\}.$$

The **standard $\beta$-horn** is $C_\beta = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$.



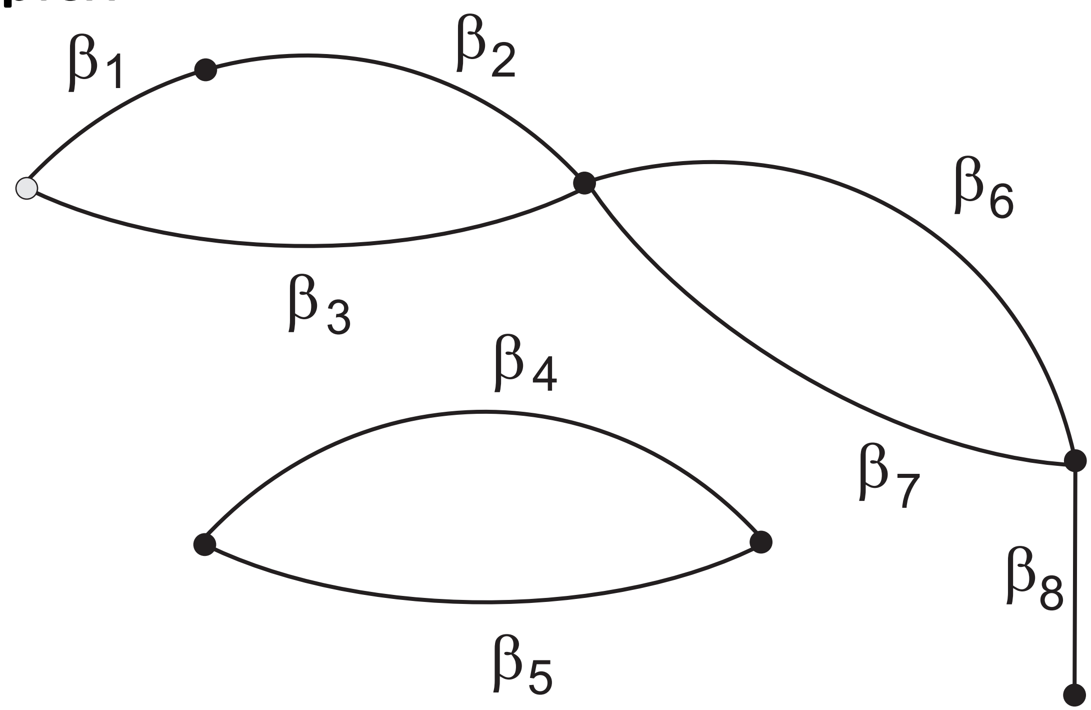
A **$\beta$-Hölder triangle** is a germ inner Lipschitz equivalent to $T_\beta$.

A **$\beta$-horn** is a germ inner Lipschitz equivalent to $C_\beta$.

**Inner Lipschitz classification** of surface germs: (LB, 1999).
Canonical decomposition of a surface germ $X$ into $\beta_i$-Hölder triangles with singular boundary arcs and $\beta_j$-horns.
Complete invariant of the inner Lipschitz equivalence class of $X$.

The invariant is called **Canonical Hölder Complex** (for semialgebraic or sublanalytic sets) or **Valuation Complex** for definable sets in o-minimal structures.

**Hölder Complex**



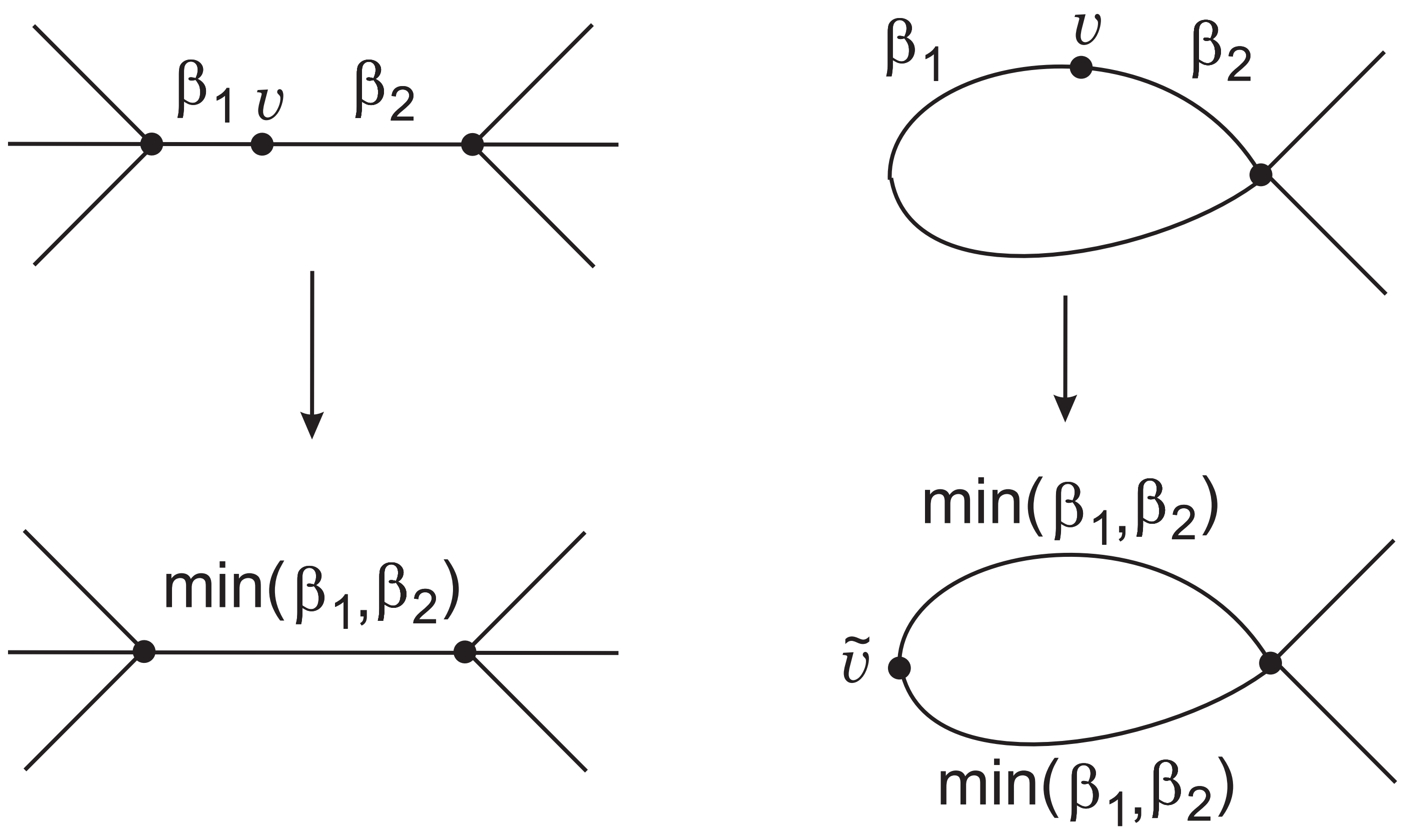

Consider a graph Γ with no loops. Let $E_\Gamma$ be the set of edges of Γ. Let $\beta : E_\Gamma \to \mathbb{Q} \cap [1, \infty)$ be a function. A pair $(\Gamma, \beta)$ is called **Formal Hölder Complex**.

**Geometric Hölder Complex** corresponding to $(\Gamma, \beta)$ is a partition of a surface germ $X$ into $\beta_i$-Hölder triangles. The edges of the graph $\Gamma$ correspond to the Hölder triangles.

**Theorem.** (LB 99)For any germ $X$ of a semialgebraic (or definable) surface there exists a Hölder Complex $(\Gamma, \beta)$ and a semialgebraic partition, Geometric Hölder Complex, corresponding to $(\Gamma, \beta)$.

**Remark.** A Hölder complex corresponding to $X$ may be not unique. That is why an additional **simplification procedure** must be applied.

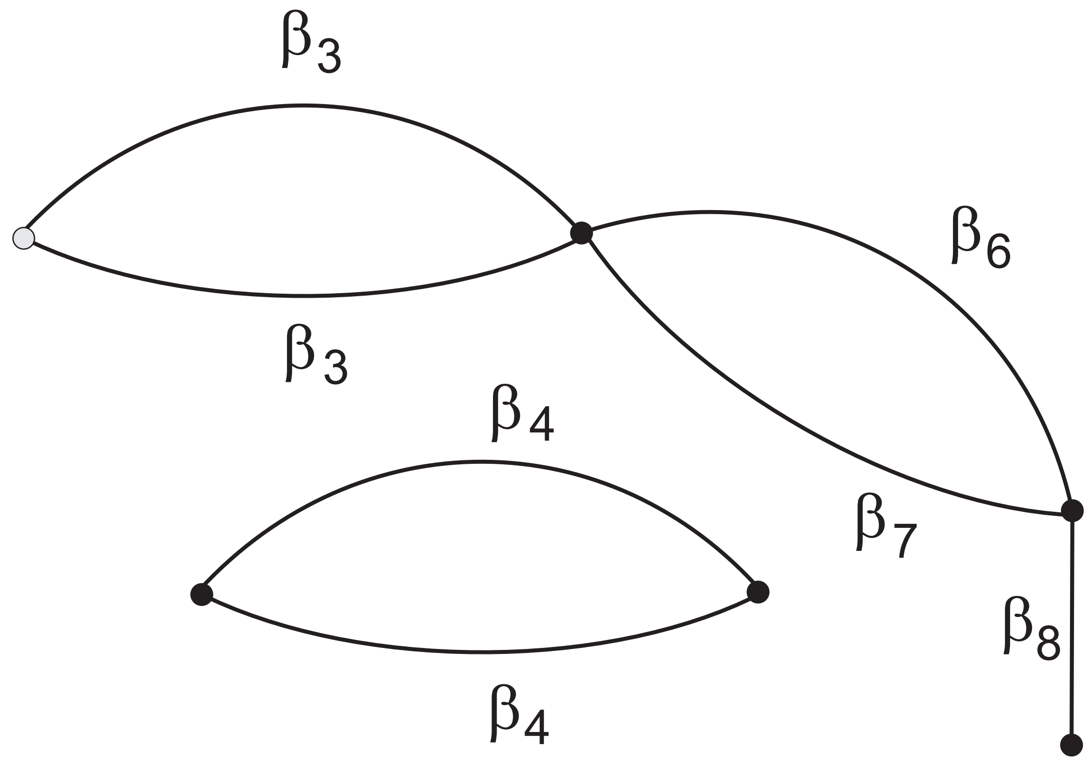
**Simplification.**




Notice that simplification does not change inner Lipschitz geometry.

**Definition** The resulting Hölder Complex after simplification is called a **Canonical Hölder Complex** corresponding to $X$.

Canonical Hölder Complex for the previous example:



**Theorem** The Canonical Hölder Complex corresponding to $X$ is unique up to combinatorial equivalence. It is a complete bi-Lipschitz invariant with respect to the inner metric.

**Remark** (LB, M. Sobolevsky) For any formal Hölder Complex $(\Gamma, \beta)$ there exists a germ of a surface $X$, such that $X$ is a Geometric Hölder Complex corresponding to $(\Gamma, \beta)$.

## Outer Lipschitz Equivalence

**Normal Embedding Theorem** (LB, Mostowski)

Let $X \subset \mathbb{R}^n$ be a compact semialgebraic set. Then there exists another semialgebraic set $\tilde{X} \subset \mathbb{R}^m$, such that

1. $\tilde{X}$ is bi-Lipschitz equivalent to $X$ with respect to the inner metric.

2. $\tilde{X}$ is Normally embedded in $\mathbb{R}^m$.

**Finiteness theorems:** Mostowski 85, Parusinski 94, Valette 05. Any definable family has finitely many outer Lipschitz equivalence classes.

**Theorem** (Pham-Tessier, Fernandes, Neumann-Pichon). Two germs of irreducible complex curves in $\mathbb{C}^2$ are **ambient** Lipschitz equivalent if and only if they are **outer** Lipschitz equivalent (have the same Puiseux exponents).

A special case of Pancake Decomposition of Kurdyka for surface germs can be stated as follows:

**Theorem** Let $(X, 0)$ be a surface germ. Then there exists a decomposition of $(X, 0)$ into the germs $(X_i, 0)$ such that

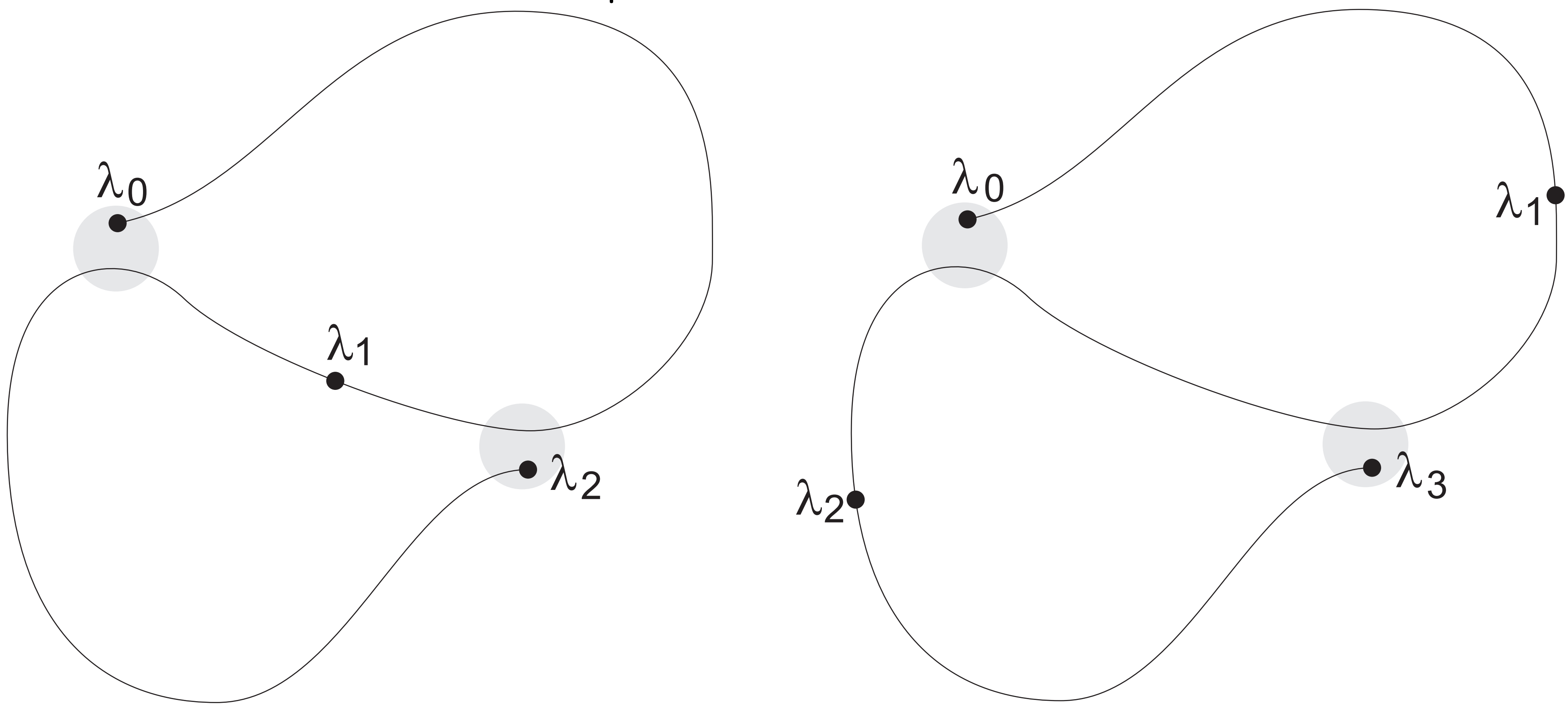
1. Each $(X_i, 0)$ is a Nornally Enbedded $\beta_i$-Hölder triangle.

2. For $i \neq j$, the intersection $(X_i, 0) \cap (X_j, 0)$ is either the origin or a common boundary arc of $(X_i, 0)$ and $(X_j, 0)$.

A pancake decomposition is called **minimal** if it is not a refinement of another pancake decomposition.

**Natural question:** is it true that any two minimal pancake decompositions of a surface germ are combinatorially equivalent as Hölder Complexes?

The answer is **No**. Example of Gabrielov and Sousa:



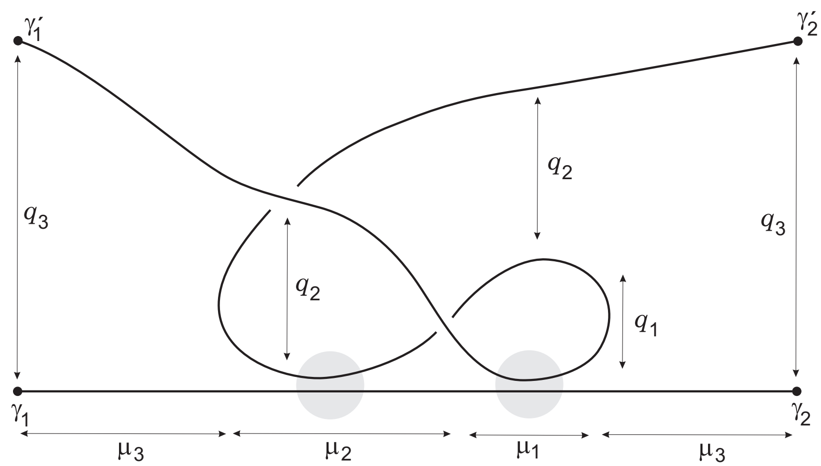

Two combinatorially non-equivalent minimal pancake decompositions of a snake. Black dots indicate the boundary arcs of pancakes.

## Pair of Normally Embedded Triangles.

The Simplest case - a single Hölder triangle.

We consider the next simplest case: a **pair** $(T, T')$ of Normally Embedded $\beta$-Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$.

The outer Lipschitz Geometry of such pairs is surprisingly nontrivial.



In order to formulate the results we need some terminology.

An **arc** $\gamma$ in a surface germ $X$ is a germ of a map $\gamma : [0, \epsilon) \to X$ such that $|\gamma(t)| = t$. We write $\gamma \subset X$ identifying $\gamma$ with its image in $X$.

The **Valette link** $V(X)$ is the space of all arcs in $X$.

The **tangency order** $tord(\gamma, \gamma')$ of two arcs $\gamma$ and $\gamma'$ is the exponent $\kappa \in \mathbb{F} \cup \{\infty\}$ in $|\gamma(t) - \gamma'(t)| = ct^{\kappa}+$ (higher terms), where $c \neq 0$. We define $tord(\gamma, Y) = \sup_{\lambda \in V(Y)} tord(\gamma, \lambda)$ for a surface germ $Y$.

The tangency order defines **non-archimedean metric** on $V(X)$.

For a surface germ $X$, a subset $Z \subset V(X)$ is a **zone** if, for any arcs $\gamma \neq \gamma'$ in $Z$, there is a Hölder triangle $T = T(\gamma, \gamma')$ such that $V(T) \subset Z$.

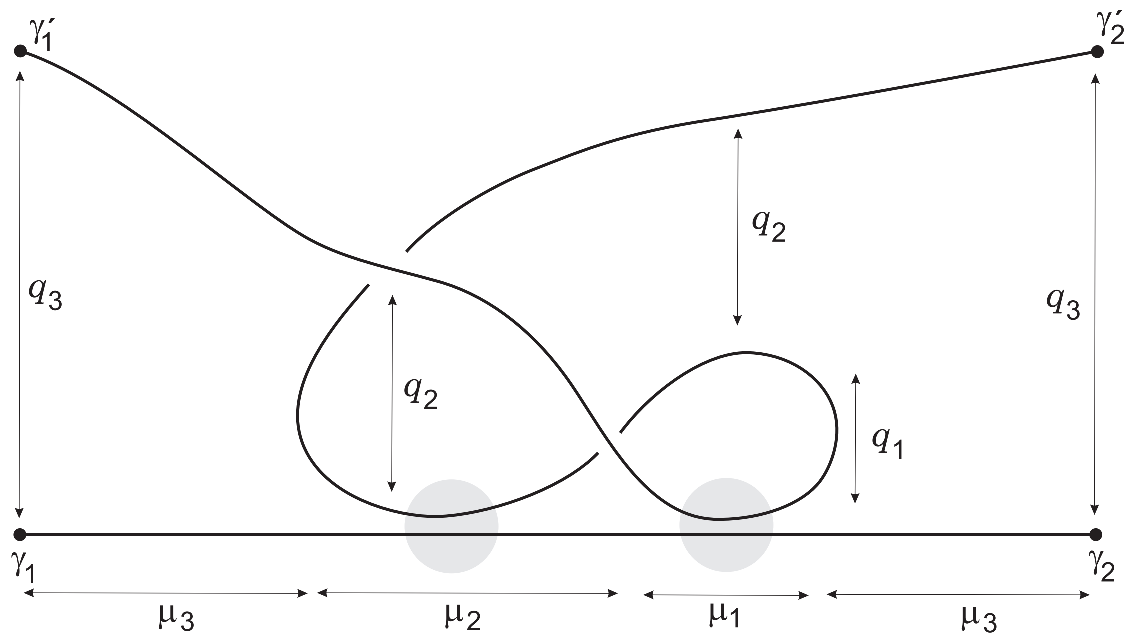
Let $Q(T) \subset \mathbb{F} \cup \{\infty\}$ be the set of exponents $q = ord_\gamma f$ for all $\gamma \subset T$. The set $Q(T)$ is a closed interval in $\mathbb{F} \cup \{\infty\}$.

$T$ is **elementary** if $Z_q = \{\gamma \subset T,\ ord_\gamma f = q\}$ is a zone for any $q \in Q(T)$.

Given two Hölder triangles $T$ and $T'$, a pair of arcs $\gamma \subset T$ and $\gamma' \subset T'$, is **normal** if $tord(\gamma, T') = tord(\gamma, \gamma') = tord(\gamma', T)$. A pair $(T, T')$ of LNE Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ is **normal** if both pairs $(\gamma_1, \gamma_1')$ and $(\gamma_2, \gamma_2')$ of their boundary arcs are normal.

For example, if $T'$ is a graph of a Lipschitz function $f$ on $T$, then any pair of arcs $(\gamma, \gamma')$, where $\gamma \subset T$ and $\gamma' \subset T'$ is the graph of $f|_\gamma$, is normal, and the pair $(T, T')$ of LNE Hölder triangles is normal.

**Theorem**(LB Andrei Gabrielov). Let $(T, T')$ be a normal pair of LNE Hölder triangles, such that $T$ is elementary with respect to $f(x) = dist(x, T')$. Then the pair $(T, T')$ is outer bi-Lipschitz equivalent to the pair $(T, \Gamma)$, where $\Gamma$ is the graph of $f$.




**Example:** The link of a pair of LNE $\beta$-Hölder triangles. Shaded discs indicate zones with the tangency order higher than $\beta$.

**Ambient Lipschitz Geometry.**

**Unique Embedding Theorem** (LB, A.Fernades, Z.Jelonek) If $X_1$ and $X_2$ are two semialgebraic sets of dimension $k$, embedded to $\mathbb{R}^m$, where $m > 2k + 1$, and if $X_1$ and $X_2$ are outer bi-Lipschitz equivalent, then they are ambient bi-Lipschitz equivalent.

**Definition.** The **tangent cone** of $X$ at 0 is

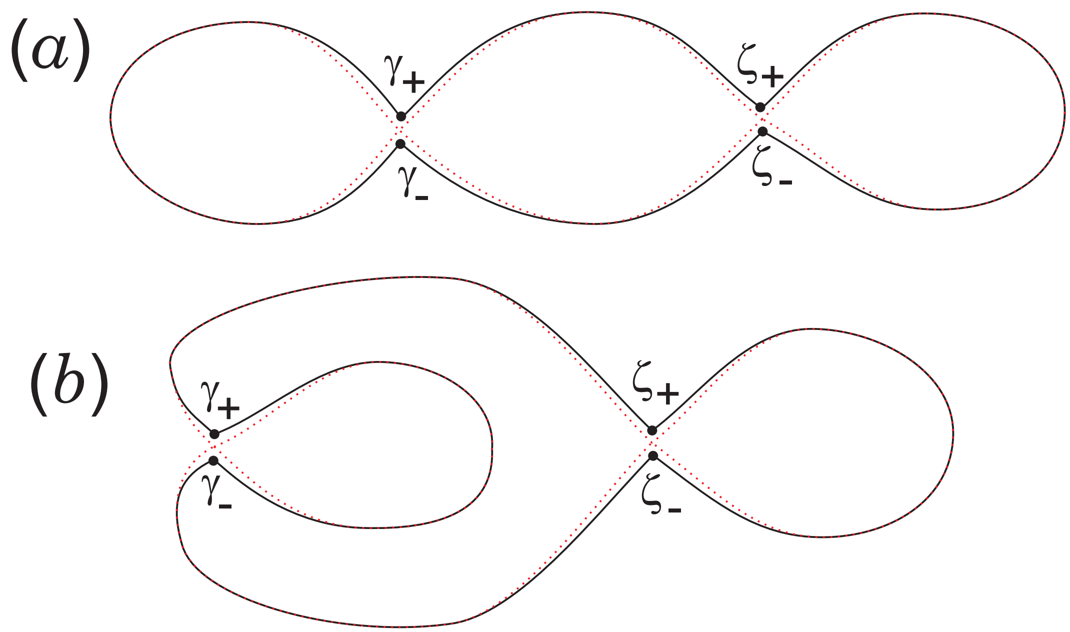
$$C_0 X = Cone\left(\lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big(X \cap \{|x| = \epsilon\}\Big)\right).$$

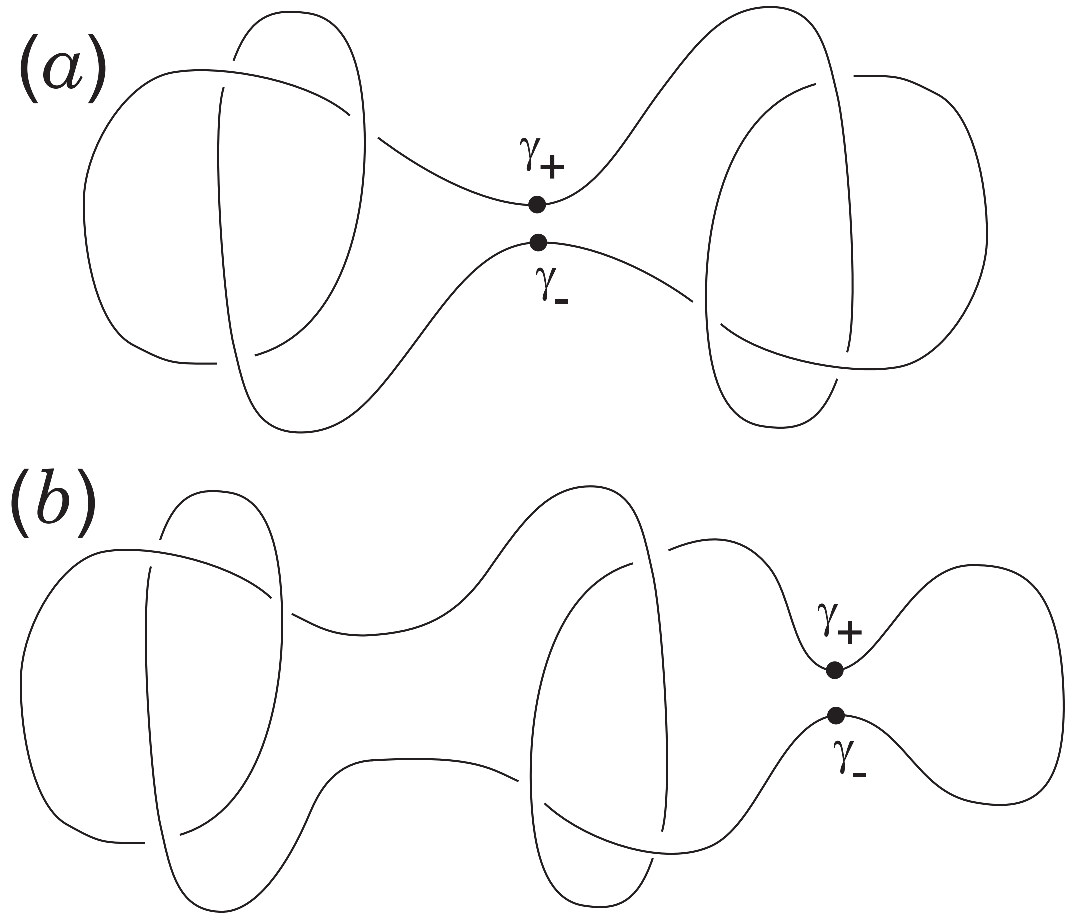
**Theorem (Sampaio).** Let $(X, 0)$ and $(Y, 0)$ be two outer (resp., ambient) Lipschitz equivalent germs of semialgebraic sets. Then their tangent cones are outer (resp., ambient) Lipschitz equivalent.

In other words, tangent cones are outer and ambient bi-Lipschitz invariants.

Sampaio's theorem is useful for the following
**Problem:** Does **outer** Lipschitz equivalence imply **ambient** Lipschitz equivalence? **Answer:** No!

There are examples of isotopic and outer Lipschitz equivalent, but **not** ambient Lipschitz equivalent, surface germs in $\mathbb{R}^3$ and $\mathbb{R}^4$ (LB, Gabrielov).





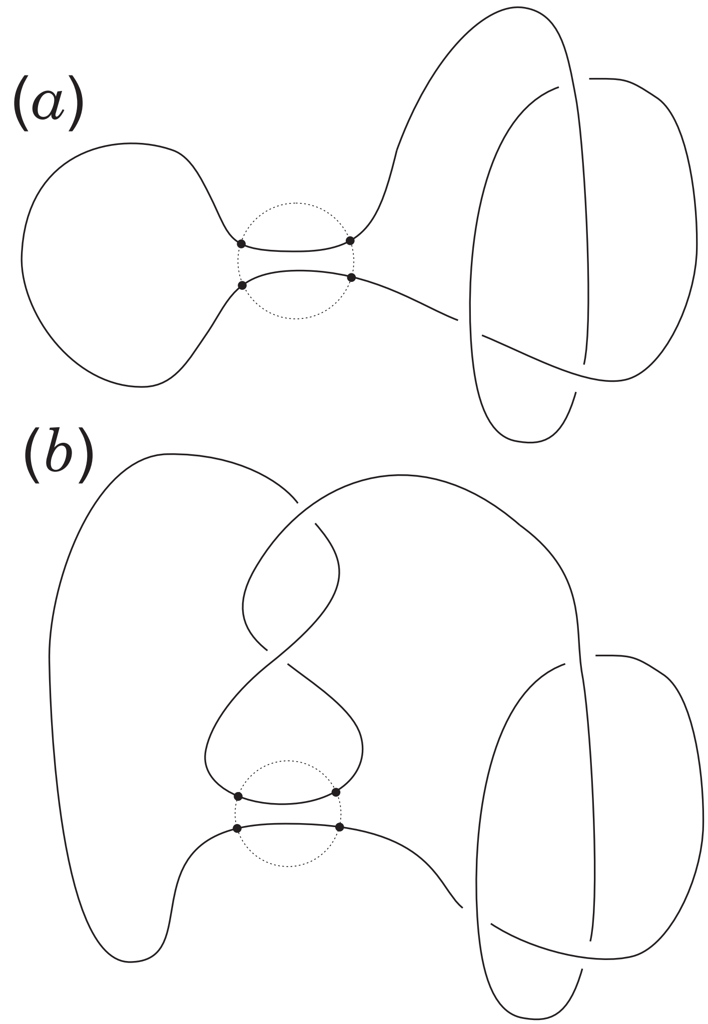

Links of two isotopic and outer Lipschitz equivalent, but not ambient Lipschitz equivalent, surfaces in $\mathbb{R}^4$.

**Theorem**(LB,Gabrielov). For any isotopy type of a real semialgebraic surface germ $(X, 0)$ in $\mathbb{R}^4$ there are infinitely many semialgebraic surface germs $(X_i, 0)$ such that

1) All $X_i$ are isotopic to $X$;

2) All $X_i$ are outer Lipschitz equivalent;

3) $X_i$ and $X_j$ are not ambient Lipschitz equivalent if $i \neq j$.

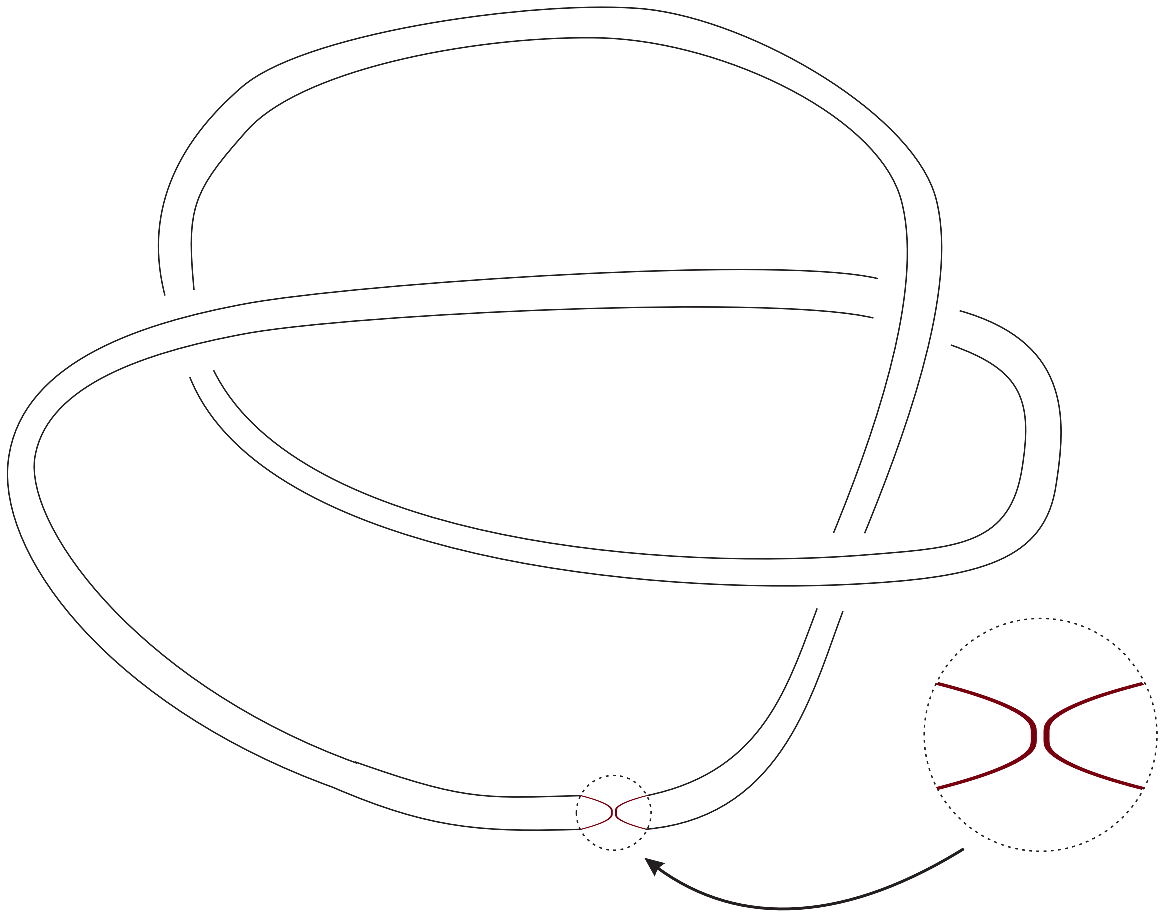

(a)
(b)

**Universality Theorem** (LB , Brandenbursky, Gabrielov). For each knot $K \subset S^3$ there exists a semialgebraic surface $X_K \subset \mathbb{R}^4$ such that

1) The link of $X_K$ at 0 is a trivial knot in $S^3$.

2) All surfaces $X_K$ are outer Lipschitz equivalent.

3) $X_{K_1}$ is ambient bi-Lipschitz equivalent to $X_{K_2}$, only if the knots $K_1$ and $K_2$ are isotopic.



Link of the surface $X_K$ for the trefoil knot $K$.

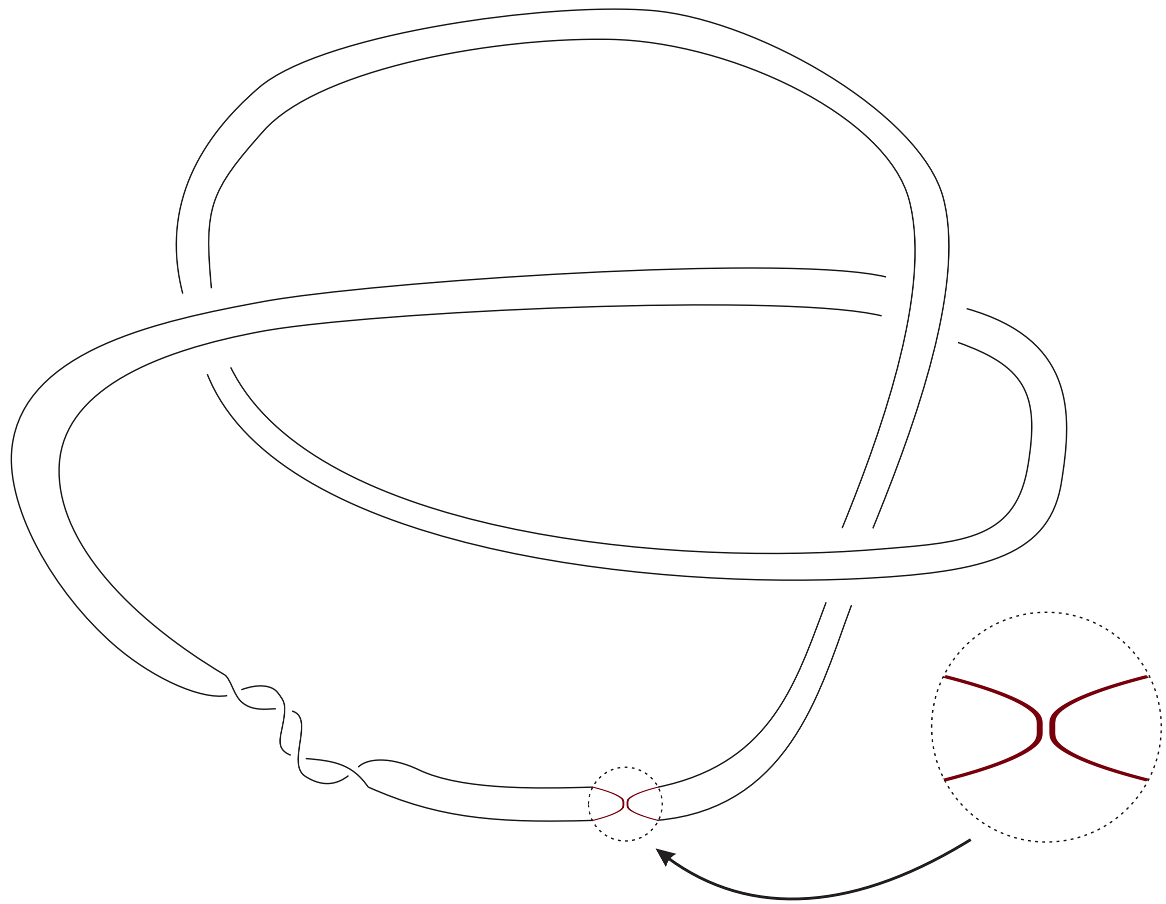
Combining the above constructions, we get the following

**Theorem** (LB, Brandenbursky, Gabrielov). For each knot $K \subset S^3$ there is an infinite sequence of semialgebraic surfaces $X_{K,i} \subset \mathbb{R}^4$ such that

1) The link of each $X_{K,i}$ at 0 is a trivial knot in $S^3$.

2) All surfaces $X_{K,i}$ are outer Lipschitz equivalent.

3) $X_{K,i}$ and $X_{L,j}$ are ambient Lipschitz equivalent only if the knots $K$ and $L$ are isotopic and $i = j$.



Surface $X_{K,1}$ (with one twist) for the trefoil knot $K$.

**Universality Theorem 2**(LB, Brandenbursky, Gabrielov). For any two knots $K$ and $L$, there exists a germ of a semialgebraic surface $X_{KL}$ such that:

1. The link of $X_{KL}$ at zero is isotopic to $L$.

2. For any knots $K$ and $L$, all surface germs $X_{KL}$ are outer bi-Lipschitz equivalent.

3. The tangent link of $X_{KL}$ is isotopic to $K$.