Tame non-Archimedean geometry: Łojasiewicz inequalities and applications

GKŁW Workshop in Singularity Theory

a special session dedicated to the memory of S.Łojasiewicz

December 12-16, 2022, Banach Center, Warsaw, Poland

Krzysztof Jan Nowak

Institute of Mathematics, Jagiellonian University

E-mail: nowak@im.uj.edu.pl

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Tame geometry in Hensel minimal fields

I deal with tame geometry in Hensel minimal, non-trivially valued fields K from my recent papers [N6, N7], with special attention focused on the Łojasiewicz inequalities. The axiomatic theory of Hensel minimality was introduced in the recent article [CHR] by Cluckers–Halupczok–Rideau (with several variants, l-h-minimality, $l \in \mathbb{N} \cup \{\omega\}$, the stronger, the larger is the number l). Yet at least in the equicharacteristic zero case, already 1-h-minimality provides, likewise o-minimality does, powerful geometric tools.

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Łojasiewicz inequalities and some other results of mine (as the closedness theorem) require an additional condition that every definable subset in the imaginary sort RV be already definable in the pure valued field language. This condition ensures that the residue field is orthogonal to the value group, and is satisfied by many of the classical tame structures on Henselian fields (including Henselian fields with analytic structure, V-minimal fields and polynomially bounded o-minimal structures with a convex subring).

Main results

The main presented results are the theorem on existence of limit, curve selection, the closedness theorem and several non-Archimedean versions of the Łojasiewicz inequalities. Relying on them, I establish an embedding theorem for regular definable spaces, the definable ultranormality and ultraparacompactness of definable Hausdorff LC-spaces, the theorems on extending continuous definable functions and existence of definable retractions, and a non-Archimedean, non-locally compact version of Kirszbraun's theorem on extending definable Lipschitz maps.

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The closedness theorem and ultraparacompactness of definable LC-spaces are ingredients of my definable ultrametric version of Bierstone–Milman's desingularization algorithm which provides resolution of singularities and transformation to normal crossings by blowing up. Actually, my research into non-Archimedean geometry was a continuation of my joint paper [K-N] on real and p-adic hereditarily rational functions, which in turn had been inspired by some discussions with Wojciech Kucharz on the subject.

Basic tools

The basic tools and results involved in my research into non-Archimedean geometry are among others the following:

1. cell decomposition with centers (due to Cluckers–Halupczok–Rideau [CHR]);
2. the Jacobian property and definable spherical completeness;
3. relative quantifier elimination for ordered abelian groups in a many sorted language (due to Cluckers–Halupczok);
4. orthogonality of the value group and residue field in the leading term structure RV;
5. my closedness theorem;
6. my non-Archimedean version of curve selection;
7. my non-Archimedean versions of the Łojasiewicz inequalities;
8. the theory of risometries (developed by Halupczok);
9. the concept of a skeleton of a parametrized Lipschitz open cell introduced recently by myself.

Some background

Soon after o-minimality had become a fundamental concept in real algebraic geometry (realizing the postulate of both tame topology and tame model theory), numerous attempts were made to find similar approaches in algebraic geometry of valued fields. This led to axiomatically based notions such as C-minimality (Haskel, Macpherson, Steinhorn, 1994), P-minimality (Haskel, Macpherson, 1997), V-minimality (Hrushovski, Kazhdan, 2006), b-minimality (Cluckers, Loeser, 2007), tame structures (Cluckers, Comte, Loeser, 2015), and eventually Hensel minimality ([CHR], 2019).

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The last concept seems to enjoy most natural and desirable properties. It is tame with respect to the leading term structure RV and provides, likewise o-minimality, powerful geometric tools as, for instance, cell decomposition, a good dimension theory or the Jacobian property (an analogue of the o-minimal monotonicity theorem).

Existence of the limit

Theorem

Let $f : E \to K$ be a 0-definable function on a subset E of K. Suppose that 0 is an accumulation point of E. Then there is a subset F of E, definable over algebraic closure of $\emptyset$, with accumulation point 0, and a point $w \in \mathbb{P}^1(K)$ such that $\lim_{x\to 0} f|F(x) = w$, and the set

$$\{(v(x), v(f(x))) : \; x \in F \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\})$$

is contained either in an affine line with rational slope

$$\{(k,l) \in \Gamma \times \Gamma : \;\; q \cdot l = p \cdot k + \beta\}$$

with $p, q \in \mathbb{Z}$, $q > 0$, $\beta \in \Gamma$, or in $\Gamma \times \{\infty\}$.

Curve selection

Theorem

Consider an $\mathcal{L}$-definable subset A of K^n with an accumulation point $a_0 \in K^n$, i.e. a_0 lies in the closure of $A \setminus \{a_0\}$. Then there exists a continuous function $a : E \to K^n$, which is definable (with parameters) in the language $\mathcal{L}$ augmented by an angular component map, such that 0 is an accumulation point of $E \subset K$, and

$$a(E \setminus \{0\}) \subset A \setminus \{a_0\}, \quad \lim_{t \to 0} a(t) = a_0.$$

We then say that $a(t)$ is a definable curve in A and write $a(t) \longrightarrow a_0$.

It seems that the only non-Archimedean versions of curve selection known before are the ones over an algebraically closed ground field K, provided by Huber [Hub] for (valuative) semianalytic subsets, and Lipshitz–Robinson [L-R] for (valuative) subanalytic sets.

Closedness theorem

Theorem

Given a definable subset D of K^n, the canonical projection

$$\pi : D \times \mathcal{O}_K^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $A \subset D \times \mathcal{O}_K^m$ is a closed definable subset, so is its image $\pi(A) \subset D$.

It has numerous applications in geometry and topology of valued fields. In particular, it allows one to use resolution of singularities in much the same way as over the locally compact fields.

It seems that the only results in this direction known before are the ones by Moret-Bailly [MB] for a proper morphism of K-schemes of finite type, and by Huber [Hub] for a quasi-compact morphism of rigid analytic varieties, over an algebraically closed valued field K.

Non-Archimedean classical Łojasiewicz inequality

Theorem

Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geqslant v((g_1(x), \ldots, g_m(x))), \quad x \in A.$$

Equivalently, there is a $C \in |K|$ such that

$$|f(x)|^s \leqslant C \cdot \max\{|g_1(x)|, \ldots, |g_m(x)|\}, \quad x \in A.$$

It seems that the only non-Archimedean (and non-locally compact) versions of Łojasiewicz inequalities known before are the ones for (valuative) subanalytic functions over an algebraically closed valued field K, provided by Lipshitz [Lip] and Lipshitz–Robinson [L-R].

Corollary

Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.

$$s \cdot v(f(x) - f(z)) + \beta \geqslant v(x - z), \quad x, z \in A.$$

Equivalently, there is a $C \in |K|$ such that

$$|f(x) - f(z)|^s \leqslant C \cdot |x - z|, \quad x, z \in A.$$

Corollary

Every continuous $\mathcal{L}$-definable function $f : A \to K$ on a closed bounded subset $A \subset K^n$ is uniformly continuous.

Non-Archimedean Łojasiewicz inequalities on LC-subsets

Theorem

Let $f, g : A \to K$ be two continuous $\mathcal{L}$-definable functions on a locally closed subset A of K^n. If

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Theorem

Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a locally closed subset A of K^n and $g : \mathcal{D}(f) := \{x \in A : f(x) \neq 0\} \to K$ a continuous $\mathcal{L}$-definable function. Then $f^s \cdot g$ extends, for $s \gg 0$, by zero through the set $\mathcal{Z}(f) := \{x \in A : f(x) = 0\}$ to a (unique) continuous $\mathcal{L}$-definable function on A.

Ideas behind the proof of the Łojasiewicz inequalities

Several problems of non-Archimedean geometry, as e.g. separation of definable subsets or Łojasiewicz inequalities, come down to certain problems of piecewise linear geometry over $\mathbb{Q}$. This is by elimination of valued field quantifiers, whereby we are reduced to consider some sets definable in a generalized Presburger language in the value group sort Γ. Those sets consist of the valuations of the points of the definable sets in the valued field sort K.

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Archimedean ordered abelian groups Γ admit quantifier elimination in the Presburger language $(<, +, -, 0, 1, \equiv_n, \; n > 1)$. However, general ordered abelian groups admit only relative (with respect to some auxiliary, linearly ordered sorts) quantifier elimination in a generalized Presburger language. Fortunately, we can often get rid of the congruence fragment of the Presburger language, analyzing only underlying sets determined by linear equations and inequalities (with rational coefficients). In this manner, we are reduced to study the piecewise linear shadows of definable sets in K^n, being objects with a polyhedral structure.

Application to regulous functions

In our joint paper [K-N], we applied the Łojasiewicz inequality (second version) to establish, in the real and p-adic cases, the extension theorem that a continuous rational function on an affine variety extends to a continuous rational function on the ambient affine space if and only if its restriction to each subvariety remains rational (an intrinsic condition). In the proof, we were working to a large extent on a variety obtained by blowing up an ideal on the algebraic variety under study. The passage back to the initial variety was possible over locally compact fields, because projections with projective fibers are closed (in fact, proper) maps.

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Afterwards, in order to carry over this extension theorem of regulous (continuous hereditarily rational) functions to any Henselian non-trivially valued field (cf. [N1, N3]), I established the closedness theorem that every such a projection is a definably closed map.

Embedding theorem

Definable spaces X are defined, after van den Dries, by gluing finitely many affine definable sets (i.e. definable subsets of affine spaces K^n). The embedding theorem for definable spaces in o-minimal structures due to van den Dries, which originates from Robson in the semialgebraic case, can be carried over to our non-Archimedean settings, as stated below.

Theorem

Every regular definable space X is affine, i.e. X can be embedded into an affine space K^N.

One of the ingredients of the proof is the fact that every closed definable subset of an affine space is the zero set of a continuous definable function. This, in turn, relies on the łojasiewicz inequalities (similarly as for o-minimal structures) and on model-theoretic compactness arguments.

Definable LC-spaces

By *definable LC-spaces* we mean those definable spaces which are defined by gluing finitely many definable, locally closed subsets of affine spaces K^n. Such spaces include, in particular, definable topological manifolds obtained by gluing definable open subsets of K^n.

Proposition

Every definable Hausdorff LC-space X is regular.

Theorem

Every definable Hausdorff LC-space X is definably ultranormal and ultraparacompact.

Extension theorem and definable retractions

Theorem

Let A be a closed 0-definable subset of a definable Hausdorff LC-space X. Then
1) every continuous 0-definable function $f : A \longrightarrow K$ can be extended to a continuous 0-definable function $F : X \longrightarrow K$;
2) there exists a 0-definable retraction $r : X \longrightarrow A$.

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In the purely topological case, a non-Archimedean version of the Tietze–Urysohn theorem (on extending continuous functions from a closed subset of an ultranormal spaces into a locally compact field with non-Archimedean absolute value) was given by Elis (1967). The existence of a continuous retraction onto a closed subset of an ultranormal metrizable space was established by Dancis (1993).

Kirszbraun's extension theorem

Theorem

Let K be an arbitrary Hensel minimal field of equicharacteristic zero, and $f : A \to K^m$ be a 0-definable 1-Lipschitz map on a (possibly non-closed) subset $A \subset K^n$ of dimension k.

I. Suppose the value group vK has no minimal element among the elements > 1. Then, for any $\epsilon \in |K|$, $\epsilon > 1$, f extends to a 0-definable ϵ-Lipschitz map $F : K^n \to K^m$.

II. Suppose the value group vK has the minimal element ϵ among the elements > 1. Then f extends to a 0-definable ϵ^k-Lipschitz map $F : K^n \to K^m$.

To my best knowledge, the only definable, non-Archimedean version of Kirszbraun's extension theorem was achieved by Cluckers–Martin [C-M] (2018) in the p-adic, thus locally compact case (for Lipschitz maps definable in p-adic fields also with an analytic structure).

Valuation preliminaries

We begin with basic notions from valuation theory. By (K, v) we mean a field K endowed with a valuation v. Let

$$\Gamma = vK, \ \mathcal{O}_K, \ \mathcal{M}_K \ \text{ and } \ \widetilde{K} = Kv$$

denote the value group, valuation ring, its maximal ideal and residue field, respectively. Let $r : \mathcal{O}_K \to Kv$ be the residue map. In this paper, we shall consider the equicharacteristic zero case, i.e. the characteristic of the fields K and Kv are assumed to be zero. For elements $a \in K$, the value is denoted by va and the residue by av or $r(a)$ when $a \in \mathcal{O}_K$. Then

$$\mathcal{O}_K = \{a \in K : \ va \geqslant 0\}, \ \ \mathcal{M}_K = \{a \in K : \ va > 0\}.$$

For a ring R, let $R^{\times}$ denote the multiplicative group of units of R. Obviously, $1 + \mathcal{M}_K$ is a subgroup of the multiplicative group $K^{\times}$.

Leading term structure

Let

$$rv : K^\times \to G(K) := K^\times/(1 + \mathcal{M}_K)$$

be the canonical group epimorphism; this leading term map rv corresponds in a sense to the sign function in real geometry. Since $vK \cong K^\times/\mathcal{O}_K^\times$, we get the canonical group epimorphism $\bar{v} : G(K) \to vK$ and the following exact sequence

$$1 \to \widetilde{K}^\times \to G(K) \to vK \to 0. \qquad (*)$$

We put $v(0) = \infty$ and $\bar{v}(0) = \infty$.

We shall consider the following 2-sorted pure valued field language $\mathcal{L}_{hen}$ (with imaginary auxiliary sort RV) on Henselian fields (K, v) of equicharacteristic zero, which goes back to Basarab [B] and yields (even resplendent) quantifier elimination of valued field quantifiers for the theory of Henselian fields.

A 2-sorted language of valued fields

Main sort: a valued field with the language of rings $(K, 0, 1, +, -, \cdot)$ or with the language $\mathcal{L}_{vf}$ of valued fields $(K, 0, 1, +, -, \cdot, \mathcal{O}_K)$.

Auxiliary sort: $RV(K) := G(K) \cup \{0\}$ with the language specified as follows: (multiplicative) language of groups $(1, \cdot)$ and one unary predicate $\mathcal{P}$ so that $\mathcal{P}_K(\xi)$ iff $\bar{v}(\xi) \geqslant 0$; here we put $\xi \cdot 0 = 0$ for all $\xi \in RV(K)$. The predicate

$$\mathcal{R}(\xi) \iff [\xi = 0 \ \vee \ (\xi \neq 0 \wedge \mathcal{P}(\xi) \wedge \mathcal{P}(1/\xi))]$$

will be construed as the residue field $Kv = \widetilde{K}$ with the language of rings $(0, 1, +, \cdot)$; obviously, $\mathcal{R}_K(\xi)$ iff $\bar{v}(\xi) = 0$. The sort RV binds together the residue field and value group.

One connecting map: $rv : K \to RV(K), \quad rv(0) = 0.$

The language of rings in the leading term structure RV

The valuation ring can be defined by putting $\mathcal{O}_K = rv^{-1}(\mathcal{P}_K)$. The residue map $r : \mathcal{O}_K \to Kv$ will be identified with the map

$$r(x) = \begin{cases} rv(x) & \text{if } x \in \mathcal{O}_K^{\times}, \\ 0 & \text{if } x \in \mathcal{M}_K. \end{cases}$$

Remark

Addition in the residue field $\mathcal{R}_K \cup \{0\}$ is the restriction of the following algebraic operation on $RV(K)$:

$$rv(x) + rv(y) = \begin{cases} rv(x+y) & \textit{if } v(x+y) = \min\{v(x), v(y)\}, \\ 0 & \textit{otherwise} \end{cases}$$

for all $x, y \in K^{\times}$; clearly, we put $\xi + 0 = \xi$ for every $\xi \in RV(K)$.

Angular component map

Remark

The standard language for the sort RV, whose vocabulary has just been introduced, is of course equivalent to the language of rings $(0, 1, +, \cdot)$ from the previous remark. In particular, $\bar{v}(\xi) > 0$ iff $1 + \xi = 1$. This language of rings for RV will be denoted by $\mathcal{L}_{rv}$.

It is well known that exact sequence $(*)$ splits whenever the residue field Kv is $\aleph_1$-saturated. In this case, there is a section $\theta : G(K) \to \widetilde{K}^\times$ of the monomorphism $\iota : \widetilde{K}^\times \to G(K)$ and the map

$$(\theta, \bar{v}) : G(K) \to \widetilde{K}^\times \times vK$$

is an isomorphism. Generally, the existence of such a section θ is equivalent to that of an angular component map $\overline{ac} = \theta \circ rv$.

Orthogonality property

Remark

It is easy to check that the language $\mathcal{L}_{rv}$ with the section θ is equivalent with the language which consists of two maps

$$\theta : RV(K) \to Kv,\ \theta(0) = 0,\ \textit{and}\ \bar{v} : RV(K) \to vK \cup \{\infty\},\ \bar{v}(0) = \infty,$$

of the language of rings $(0, 1, +, -, \cdot)$ on the residue field Kv, and the language of ordered groups $(0, +, -, <)$ on the value group vK.

Hence the residue field is orthogonal to the value group, i.e. every definable subset $C \subset (Kv)^p \times (vK)^q$ is a finite union of Cartesian products

$$C = \bigcup_{i=1}^{k} X_i \times Y_i \qquad (\dagger)$$

for some definable subsets $X_i \subset (Kv)^p$ and $Y_i \subset (vK)^q$.

The language under consideration

We shall work with a language $\mathcal{L}$ which is an expansion of the language $\mathcal{L}_{vf}$ of valued fields, possibly with some auxiliary imaginary sorts. The words 0-definable and A-definable shall mean $\mathcal{L}$-definable and $\mathcal{L}_A$-definable in a fixed language $\mathcal{L}$; "definable" will refer to definable in $\mathcal{L}$ with arbitrary parameters.

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For presentation of Hensel minimality, it is more convenient to adopt multiplicative notation for valuations $|\cdot|$. For any $\lambda \in \Gamma_K$, $\lambda \leqslant 1$, put

$$\mathcal{M}_\lambda := \{x \in K : |x| < \lambda\}, \quad RV_\lambda^\times := K^\times/(1 + \mathcal{M}_\lambda),$$

$$rv_\lambda : K \to RV_\lambda := RV_\lambda^\times \cup \{0\}, \quad 0 \mapsto 0.$$

We say that a finite set $C \subset K$ λ-prepares a subset $X \subset K$ if whether some $x \in K$ lies in X depends only on the tuple $(rv_\lambda(x - c))_{c \in C}$.

The notion of Hensel minimality

A first approximation to the notion of Hensel minimality is: Any A-definable subset $X \subset K$ (for $A \subset K$) can be 1-prepared by a finite A-definable set $C \subset K$. This, however, is not yet strong enough: a strong control of parameters from RV_λ is needed and, consequently, different variants, l-h-minimality with $l \in \mathbb{N}\{\omega\}$, are considered.

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Let T be a complete $\mathcal{L}$-theory of valued fields of equicharacteristic 0. We say that T is l-h-minimal if every model K of T has the following property: For every $\lambda \in \Gamma_K$, $0 < \lambda \leqslant 1$, for every set $A \subset K$ and for every set $A' \subset RV_\lambda$ of cardinality $|A'| \leqslant l$, every $(A \cup RV \cup A')$-definable subset $X \subset K$ can be λ-prepared by a finite A-definable set $C \subset K$.

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Below we recall the following three results of Hensel minimality from the paper [CHR], which are crucial for our approach:

Valuative Jacobian Property

Proposition

Let $f : K \to K$ be a 0-definable function. Then there exists a finite 0-definable set $C \subset K$ such that for every ball B 1-next to C, either f is constant on B, or there exists a $\mu_B \in vK$ such that
(1) for every open ball $B' \subset B$, $f(B')$ is an open ball of radius $\mu_B + \mathrm{rad}(B')$;
(2) for every $x_1, x_2 \in B$, we have
$v(f(x_1) - f(x_2)) = \mu_B + v(x_1 - x_2)$.

Domain and Range Preparation

Proposition

Let $f : K \to K$ be a 0-definable function and let $C_0 \subset K$ be a finite, 0-definable set. Then there exist finite, 0-definable sets $C, D \subset K$ with $C_0 \subset C$ such that $f(C) \subset D$ and for every ball B 1-next to C, the image $f(B)$ is either a singleton in D or a ball 1-next to D.

Cells

For $m \leqslant n$, denote by $\pi_{\leqslant m}$ or $\pi_{<m+1}$ the projection $K^n \to K^m$ onto the first m coordinates; put $x_{\leqslant m} = \pi_{\leqslant m}(x)$. Let $C \subset K^n$ be a non-empty 0-definable set, $j_i \in \{0, 1\}$ and

$$c_i : \pi_{<m}(X) \to K$$

be 0-definable functions $i = 1, \ldots, n$. Then C is called a 0-definable cell with center tuple $c = (c_i)_{i=1}^n$ and of cell-type $j = (j_i)_{i=1}^n$ if it is of the form:

$$C = \{x \in K^n : (rv(x_i - c_i(x_{<i})))_{i=1}^n \in R\},$$

for a (necessarily 0-definable) set

$$R \subset \prod_{i=1}^n j_i \cdot G(K),$$

where $0 \cdot G(K) = 0 \subset RV(K)$ and $1 \cdot G(K) = G(K) \subset RV(K)$. One can similarly define A-definable cells.

Parametrized cells

In the absence of the condition that algebraic closure and definable closure coincide in $T = \mathrm{Th}(K)$ (i.e. the algebraic closure $\mathrm{acl}(A)$ equals the definable closure $\mathrm{dcl}(A)$ for any Henselian field $K' \equiv K$ and every $A \subset K'$), a concept of parameterized cells must come into play. Let us mention that one can ensure the above condition via an expansion of the language for the sort RV.

Parametrized cells

In the absence of the condition that algebraic closure and definable closure coincide in $T = \mathrm{Th}(K)$ (i.e. the algebraic closure $\mathrm{acl}(A)$ equals the definable closure $\mathrm{dcl}(A)$ for any Henselian field $K' \equiv K$ and every $A \subset K'$), a concept of parameterized cells must come into play. Let us mention that one can ensure the above condition via an expansion of the language for the sort RV.

Consider a 0-definable function $\sigma : C \to RV(K)^k$. Then (C, σ) is called a 0-definable parameterized (by σ) cell if each set $\sigma^{-1}(\xi)$, $\xi \in \sigma(C)$, is a ξ-definable cell with some center tuple c_ξ depending definably on ξ and of cell-type independent of ξ.

Remark

If the language $\mathcal{L}$ has an angular component map, then one can take σ from the above definition to be residue field valued (instead of RV-valued).

Classical and parameterized cell decompositions

Theorem

Suppose that algebraic closure and definable closure coincide in a 1-h-minimal theory $T = \mathrm{Th}(K)$. For every 0-definable set $X \subset K^n$, there exists a finite decomposition of X into 0-definable cells C_k. Furthermore, there exists a finite decomposition of X into 0-definable subsets C_k such that each cell C_k is, after some permutation of the variables, a 0-definable cell of type $(1, \ldots, 1, 0, \ldots, 0)$ with 1-Lipschitz continuous centers $c_1, \ldots, c_n$. Such cells shall be called 1-Lipschitz cells.

Theorem

For every 0-definable set $X \subset K^n$, there exists a finite decomposition of X into 0-definable parametrized cells (C_k, σ_k). Moreover, given finitely many 0-definable functions $f_j : X \to K$, one can require that the restriction of every function f_j to each cell $\sigma_k^{-1}(\xi)$ be continuous.

References

- [B] S.A. Basarab, *Relative elimination of quantifiers for Henselian valued fields*, Ann. Pure Appl. Logic **53** (1991), 51–74.
- [CHR] R. Cluckers, E. Halupczok, S. Rideau, *Hensel minimality I*, Forum Math., Pi, **10** (2022), e11.
- [C-M] R. Cluckers, F. Martin, *A definable p-adic analogue of Kirszbraun's theorem on extension of Lipschitz maps*, J. Inst. Math. Jussieu **17** (2018), 39–57.
- [Hub] R. Huber, *On semianalytic subsets of rigid analytic varieties*, Geometriae Dedicata **58** (1995), 291-311.
- [K-N] J. Kollár, K. Nowak, *Continuous rational functions on real and p-adic varieties*, Math. Zeitschrift **279** (2015), 85–97.
- [Lip] L. Lipshitz, *Rigid subanalytic sets*, Amer. J. Math. **115** (1) (1993), 77-108.

[L-R] L. Lipshitz, Z. Robinson, *Uniform properties of rigid subanalytic sets*, Trans. Amer. Math. Soc. 357 (11) (2005), 4349–4377.

[MB] L. Moret-Bailly, *An extension of Greenberg's theorem to general valuation rings*, Manuscripta Math. **139** (2012), 153–166.

[N1] K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. **23** (2017), 455–495.

[N2] K.J. Nowak, *Definable transformation to normal crossings over Henselian fields with separated analytic structure*, Symmetry **11** (7) (2019), 934.

[N3] K.J. Nowak, *A closedness theorem and applications in geometry of rational points over Henselian valued fields*, J. Singul. **21** (2020), 212–233.

[N4] K.J. Nowak, *A closedness theorem over Henselian fields with analytic structure and its applications*. In: Algebra, Logic and Number Theory, Banach Center Publ. **121** (2020), 141–149.

[N5] K.J. Nowak, *Definable retractions and a non-Archimedean Tietze–Urysohn theorem over Henselian valued fields*, arXiv:1808.09782 [math.AG].

[N6] K.J. Nowak, *Tame topology and desingularization in Hensel minimal structures*, arXiv:2103.01836 [math.AG].

[N7] K.J. Nowak, *Extension of Lipschitz maps definable in Hensel minimal structures*, arXiv:2204.05900 [math.LO].