Gdańsk-Kraków-Łódź-Warszawa Workshop in Singularity Theory
dedicated to the memory of Stanisław Łojasiewicz

Motivic, logarithmic, and topological Milnor fibrations

joint work with Jean-Baptiste Campesato and Goulwen Fichou

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Introduction: the topological Milnor fibration

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function germ.

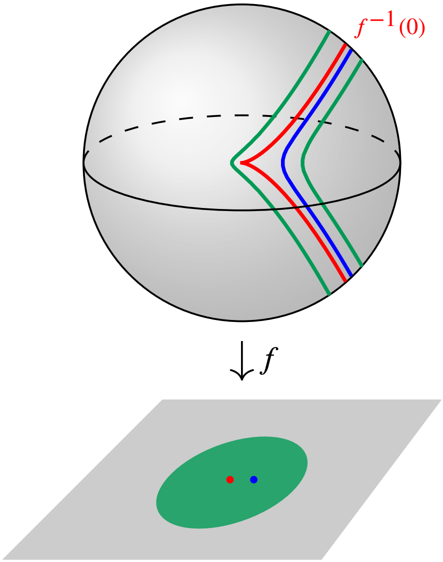
Milnor fibration: Then, for $0 < \eta \ll \varepsilon \ll 1$, the restriction of f

$$f_| : f^{-1}(\partial D_\eta^*) \cap B_\varepsilon \to \partial D_\eta^* = S_\eta^1$$

is a smooth locally trivial fibration. Its fibre

$$F_f := f^{-1}(y) \cap B_\varepsilon$$

is called **the Milnor fibre.**



Remark: Milnor fibre is not an algebraic variety.

Introduction: motivic Milnor fibre (fibration)

Definition: Grothendieck ring of algebraic varieties

We denote by $K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)$ the free abelian group spanned by isomorphism classes $[f : X \to \mathbb{C}^*]$ of complex algebraic varieties over $\mathbb{C}^*$ modulo the relation

$$[f : X \to \mathbb{C}^*] = [f_{|Y} : Y \to \mathbb{C}^*] + [f_{|X\setminus Y} : X \setminus Y \to \mathbb{C}^*] \qquad \text{if } Y \subset X.$$

The ring structure is given by the fibre product

$$[f : X \to \mathbb{C}^*][g : Y \to \mathbb{C}^*] = [X \times_{\mathbb{C}^*} Y \to \mathbb{C}^*].$$

- $0 = [\emptyset \to \mathbb{C}^*]$,
- $1 = [\mathrm{id} : \mathbb{C}^* \to \mathbb{C}^*]$,
- $\mathbb{L} := [\mathrm{pr} : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}^*]$.

Example

$$[\mathrm{pr} : \mathbb{P}^n \times \mathbb{C}^* \to \mathbb{C}^*] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1$$

Introduction: motivic Milnor fibre (fibration)

Definition: motivic zeta function

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a regular function,

$$Z_f(\mathbb{L}^{-s}) := \int_{\mathcal{L}(\mathbb{C}^{n+1},0)} (\mathrm{ac}_f, \mathbb{L}^{-\operatorname{ord}_t f \cdot s}) \in \mathcal{M}[\![\mathbb{L}^{-s}]\!], \qquad \mathcal{M} := K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)[\mathbb{L}^{-1}].$$

$$Z_f(\mathbb{L}^{-s}) = \sum_{k\geq 0} \mu_{\mathrm{mot}} \left(\mathrm{ac}_f : \begin{array}{ccc} \{\gamma \in \mathbb{C}[\![t]\!]^{n+1} : \gamma(0) = 0, \operatorname{ord}_t f(\gamma(t)) = k\} & \to & \mathbb{C}^* \\ \gamma & \mapsto & \mathrm{ac}(f \circ \gamma) \end{array} \right) \mathbb{L}^{-sk}$$

Introduction: motivic Milnor fibre (fibration)

Definition: motivic zeta function

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$$Z_f(\mathbb{L}^{-s}) := \int_{\mathcal{L}(\mathbb{C}^{n+1},0)} (\mathrm{ac}_f, \mathbb{L}^{-\mathrm{ord}_t f \cdot s}) \in \mathcal{M}[\![\mathbb{L}^{-s}]\!], \qquad \mathcal{M} := K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)[\mathbb{L}^{-1}].$$

Proposition (Denef–Loeser, 1998)

$$Z_f(\mathbb{L}^{-s}) = \sum_{\emptyset \neq J \subset \{1,\ldots,r\}} [f_J : U_J \to \mathbb{C}^*] \prod_{i \in J} \frac{\mathbb{L}^{-\nu_i - sN_i}}{1 - \mathbb{L}^{-\nu_i - sN_i}} \qquad \text{for some } \nu_i, N_i \in \mathbb{N}_{>0}.$$

Definition: motivic Milnor fibre

$$S_f := -\lim_{s \to -\infty} Z_f(\mathbb{L}^{-s}) = -\sum_{\emptyset \neq J \subset \{1,\ldots,r\}} (-1)^{|J|} [f_J : U_J \to \mathbb{C}^*] \in \mathcal{M}$$

Introduction

Question

Given a regular (e.g. polynomial) function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$,
how the topological and motivic Milnor fibres (fibrations) are related to each other?

Theorem (Denef–Loeser, 1998)

Motivic Milnor fibre and topological Milnor fibration have same numerical invariants :

$$\chi_c(F_f) = \chi_c(S_f)\ ,\ E_{F_f} = E_{S_f}\ .$$

$E_X \in \mathbf{Z}[u, v]$ is the Hodge-Deligne polynomial.

Introduction

Question

Given a regular (e.g. polynomial) function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$,
how the topological and motivic Milnor fibres (fibrations) can be related to each other?

Result

We give two (essentially equivalent) comparisons between topological and motivic Milnor fibres (fibrations):

- Using logarithmic geometry.
- Using line bundles and a real oriented version of the deformation onto normal cone (MacPherson's graph construction)

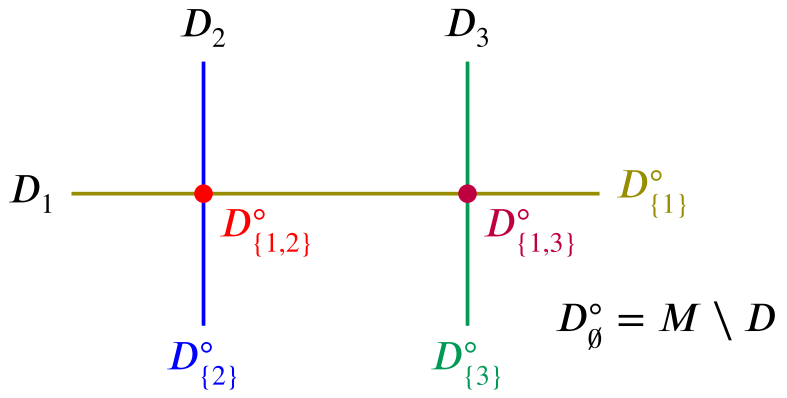
Logarithmic geometry: set-up

Set-up

Let $f : (M, D) \to (\mathbb{C}, 0)$ be a regular function with $D := f^{-1}(0) = \bigcup_{i \in I} D_i$ a divisor with simple normal crossings.

Canonical stratification induced by D

For $J \subset I$, we set $D_J^\circ := \bigcap_{j \in J} D_j \setminus \bigcup_{i \in I \setminus J} D_i$.



A'Campo's model of topological Milnor fibration

Real oriented blowing-up of D in M.

- Real oriented blowing-up of $0 \in \mathbb{C}$ = polar coordinates: $\hat{\mathbb{C}} = \mathbb{R}_{\geq 0} \times S^1 \ni (r, \theta) \to z = r\theta \in \mathbb{C}$.
- Real oriented blowing-up of $D = \{z_1 \cdots z_r = 0\} \subset M = \mathbb{C}^n$ is

$$\hat{M} = \hat{\mathbb{C}}^r \times \mathbb{C}^{n-r} \to M = \mathbb{C}^n$$

  and induces (by functoriality)

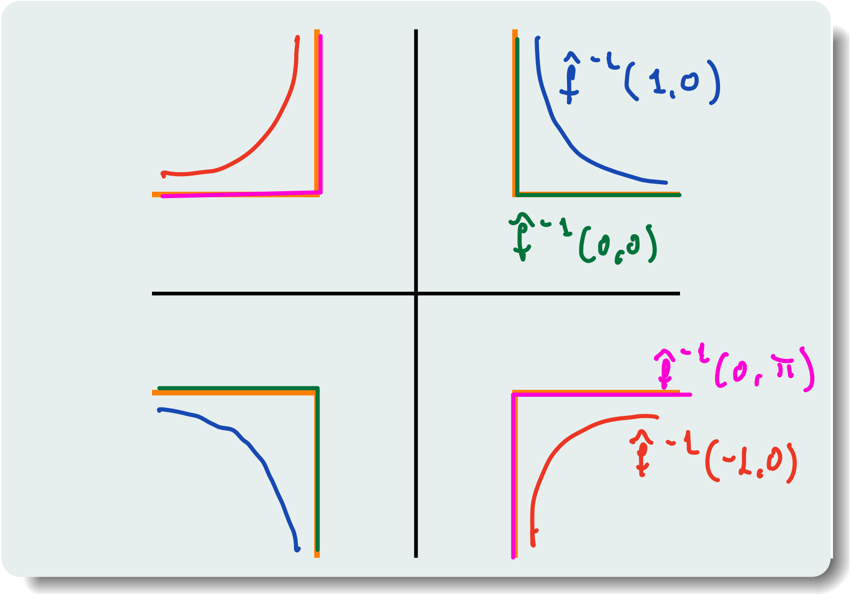
$$\hat{f} : (\hat{M}, \hat{D}) \to (\hat{\mathbb{C}}, \hat{0}) = (\hat{\mathbb{C}}, S^1).$$

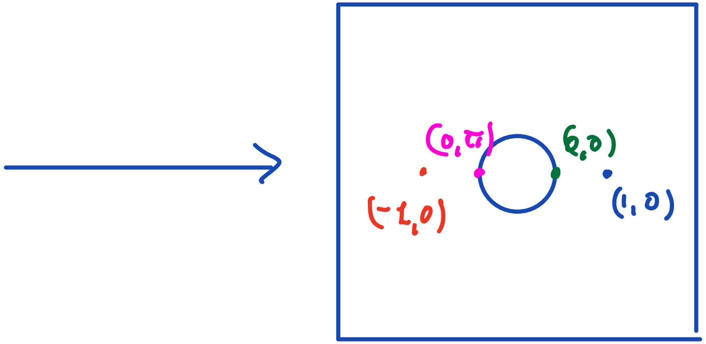
- $\hat{f}$ is locally topologically trivial and over S^1 gives A'Campo's model: $\hat{f}_{|\hat{D}} : \hat{D} \to S^1$.
- In local coordinates at a point of D_J°:

$$f(z) = u(z) \prod_{i \in J} z_i^{N_i} \qquad \hat{f}(z) = (|u(z)| \prod_{i \in J} r_i^{N_i}, \arg u(z) \prod_{i \in J} \theta_i^{N_i}).$$

Example: $f(z_1, z_2) = z_1 z_2$

$$\hat f : (\hat{\mathbb{C}}^2, \hat D) \to (\hat{\mathbb{C}}, \hat 0) = (\hat{\mathbb{C}}, S^1)$$



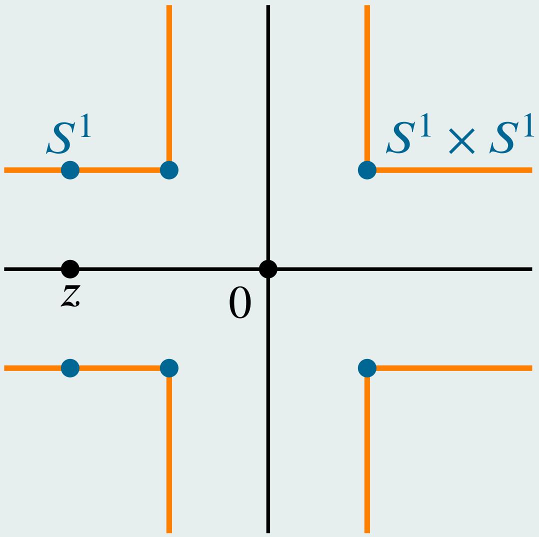


Logarithmic geometry. Topological space

Sheaf of monoids: $\mathcal{M}(U) := \{ f \in \mathcal{O}_M(U) \ : \ f_{|U\cap(M\setminus D)} \text{ is invertible} \} \supset \mathcal{O}^*_M(U)$.

$$(M, D)^{\log} := \left\{ (x, \varphi) \ : \ x \in M,\ \varphi \in \mathrm{Hom}_{\mathrm{mon}}(\mathcal{M}_x, S^1),\ \forall g \in \mathcal{O}^*_x,\ \varphi(g) = \frac{g(x)}{|g(x)|} \right\}$$

Example: $D = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$



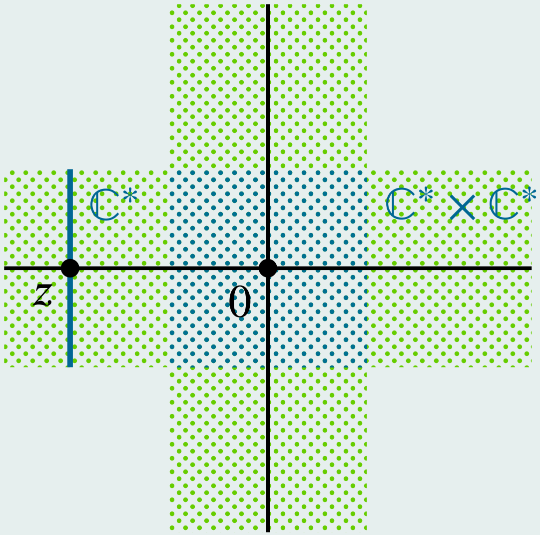
- $(M, D)^{\log}_{|M\setminus D} = M \setminus D$.
- $(M, D)^{\log}_{|\{z\}} = S^1$ for $z \in D \setminus \{0\}$.
  (just decide $\varphi(z_2)$)
- $(M, D)^{\log}_{|\{0\}} = S^1 \times S^1$.
  (decide $\varphi(z_1)$, $\varphi(z_2)$)

Logarithmic geometry. Algebraic space

Sheaf of monoids: $\mathcal{M}(U) := \left\{ f \in \mathcal{O}_M(U) \ : \ f_{|U\cap(M\setminus D)} \text{ is invertible} \right\} \supset \mathcal{O}^*_M(U)$.

$$(M, D)^{\text{alog}} := \left\{ (x, \Phi) \ : \ x \in M, \ \Phi \in \operatorname{Hom}_{\text{mon}}(\mathcal{M}_x, \mathbb{C}^*), \ \forall g \in \mathcal{O}^*_x, \ \Phi(g) = g(x) \right\}$$

Example: $D = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$



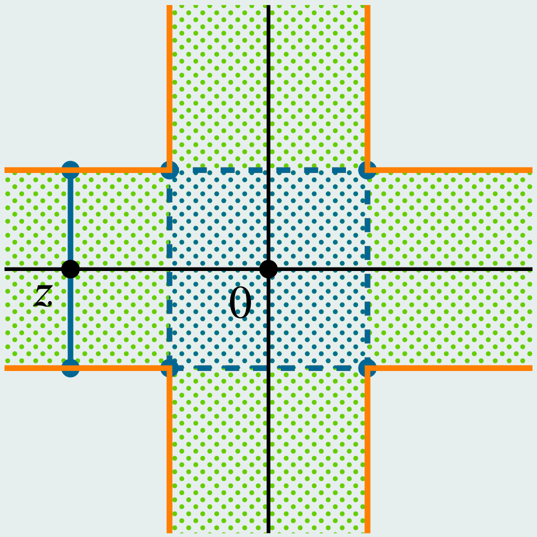
- $(M, D)^{\text{alog}}_{|M\setminus D} = M \setminus D$.
- $(M, D)^{\text{alog}}_{|\{z\}} = \mathbb{C}^*$ for $z \in D \setminus \{0\}$.
  (decide $\Phi(z_2)$)
- $(M, D)^{\text{alog}}_{|\{0\}} = \mathbb{C}^* \times \mathbb{C}^*$.
  (decide $\Phi(z_1)$, $\Phi(z_2)$)

Logarithmic geometry. Complete space

Sheaf of monoids: $\mathcal{M}(U) := \left\{ f \in \mathcal{O}_M(U) \ : \ f_{|U\cap(M\setminus D)} \text{ is invertible} \right\} \supset \mathcal{O}^*_M(U)$.

$$(M,D)^{\mathrm{clog}} := \left\{ (x,\varphi,\psi) \ : \ (x,\varphi) \in (M,D)^{\log},\ \psi \in \mathrm{Hom}_{\mathrm{mon}}\left(\mathcal{M}_x, (0,+\infty]\right), \forall g \in \mathcal{O}^*_x,\ \psi(g) = |g(x)| \right\}$$

Example: $D = \{z_1 z_2 = 0\} \subset \mathbb{C}^2$



- $(M,D)^{\mathrm{clog}}_{|M\setminus D} = M \setminus D$.
- $(M,D)^{\mathrm{clog}}_{|\{z\}} = (0,+\infty] \times S^1 \simeq \mathbb{C}^* \sqcup S^1$
  $= (M,D)^{\mathrm{alog}}_{|\{z\}} \sqcup (M,D)^{\log}_{|\{x\}}$.
- $(M,D)^{\mathrm{clog}}_{|\{0\}} = \left((0,+\infty] \times S^1\right)^2$
  $\simeq (\mathbb{C}^* \times \mathbb{C}^*) \sqcup (S^1 \times S^1) \sqcup (\mathbb{C}^* \times S^1) \sqcup (S^1 \times \mathbb{C}^*)$

Comparison of logarithmic spaces I

$$(M,D)^{\log} := \left\{(x,\varphi) \ : \ x \in M,\ \varphi \in \mathrm{Hom}_{\mathrm{mon}}(\mathcal{M}_x, S^1),\ \forall g \in \mathcal{O}_x^*,\ \varphi(g) = \frac{g(x)}{|g(x)|}\right\}$$

$$(M,D)^{\mathrm{alog}} := \left\{(x,\Phi) \ : \ x \in M,\ \Phi \in \mathrm{Hom}_{\mathrm{mon}}(\mathcal{M}_x, \mathbb{C}^*),\ \forall g \in \mathcal{O}_x^*,\ \Phi(g) = g(x)\right\}$$

$$(M,D)^{\mathrm{clog}} := \left\{(x,\varphi,\psi) : (x,\varphi) \in (M,D)^{\log},\ \psi \in \mathrm{Hom}_{\mathrm{mon}}\left(\mathcal{M}_x, (0,+\infty]\right), \forall g \in \mathcal{O}_x^*,\ \psi(g) = |g(x)|\right\}$$

$$\begin{array}{ccccc} (M,D)^{\mathrm{alog}} & \hookrightarrow & (M,D)^{\mathrm{clog}} & \hookleftarrow & (M,D)^{\log} \\ & {\scriptstyle /\mathbb{R}^J_{>0}} \searrow & \downarrow {\scriptstyle \mathrm{pr}} & \swarrow\!\!= & \\ & & (M,D)^{\log} & & \end{array}$$

Comparison of logarithmic fibrations II

$$
\begin{array}{ccccc}
(M,D)^{\mathrm{alog}} & \hookrightarrow & (M,D)^{\mathrm{clog}} & \xrightarrow{\mathrm{pr}} & (M,D)^{\mathrm{log}} \\
\downarrow{\scriptstyle f^{\mathrm{alog}}} & & \downarrow{\scriptstyle f^{\mathrm{clog}}} & & \downarrow{\scriptstyle f^{\mathrm{log}}} \\
(\mathbb{C},0)^{\mathrm{alog}} & \hookrightarrow & (\mathbb{C},0)^{\mathrm{clog}} & \xrightarrow{\mathrm{pr}} & (\mathbb{C},0)^{\mathrm{log}}
\end{array}
$$

Comparison of logarithmic fibrations II

Applying functoriality to $f : (M, D) \to (\mathbb{C}, 0)$, we get

$$\begin{array}{ccccc}
(M,D)^{\mathrm{alog}}{}_{|D} & \hookrightarrow & (M,D)^{\mathrm{clog}}{}_{|D} & \xrightarrow{\mathrm{pr}} & (M,D)^{\log}{}_{|D} \\
\downarrow f^{\mathrm{alog}} & & \downarrow f^{\mathrm{clog}} & & \downarrow f^{\log} \\
(\mathbb{C},0)^{\mathrm{alog}}{}_{|0} & \hookrightarrow & (\mathbb{C},0)^{\mathrm{clog}}{}_{|0} & \xrightarrow{\mathrm{pr}} & (\mathbb{C},0)^{\log}{}_{|0} \\
\| & & \| & & \| \\
\mathbb{C}^* & \hookrightarrow & \mathbb{C}^* \sqcup S^1 \simeq (0,+\infty] \times S^1 & \xrightarrow{\mathrm{pr}} & S^1
\end{array}$$

Milnor fibrations and logarithmic geometry

Topological Milnor fibration

$f^{\log}_{|D} : (M, D)^{\log}_{|D} \to S^1$ coincides with A'Campo's model for the topological Milnor fibration.

Motivic Milnor fibre (comparing to the Denef-Loeser formula)

$$S_f = -\sum_{\emptyset \neq J \subset I} (-1)^{|J|} \left[f^{\mathrm{alog}}_{|D^\circ_J} : (M, D)^{\mathrm{alog}}_{D^\circ_J} \to \mathbb{C}^* \right]$$

Interpretation of the sign:
division by powers of $\mathbb{R}_{>0}$.

$$\begin{array}{ccc} (M, D)^{\mathrm{alog}}_{|D^\circ_J} & \xrightarrow{/(\mathbb{R}_{>0})^J} & (M, D)^{\log}_{|D^\circ_J} \\ \downarrow {\scriptstyle f^{\mathrm{alog}}_{|D^\circ_J}} & & \downarrow {\scriptstyle f^{\log}_{|D^\circ_J}} \\ \mathbb{C}^* & \xrightarrow[/\mathbb{R}_{>0}]{} & S^1 \end{array}$$

Applications

Topological Milnor fibration determines motivic Milnor fibre

The motivic Milnor fibre $\mathcal{S}_f$ is determined by the stratified topological Milnor fibration $f^{\log}_{|D} : (M, D)^{\log}_{|D} \to S^1$.

Theorem

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be regular. Then

$$\mathcal{S}_f := -\sum_{\emptyset \neq J \subset I} (-1)^{|J|} \left[f^{\mathrm{alog}}_{|D_J^\circ} : (M, D)^{\mathrm{alog}}_{D_J^\circ} \to \mathbb{C}^* \right] \in K_0\left(\mathrm{Var}_{\mathbb{C}}^{\mathbb{C}^*}\right)$$

(No need to make $\mathbb{L}$ invertible.)

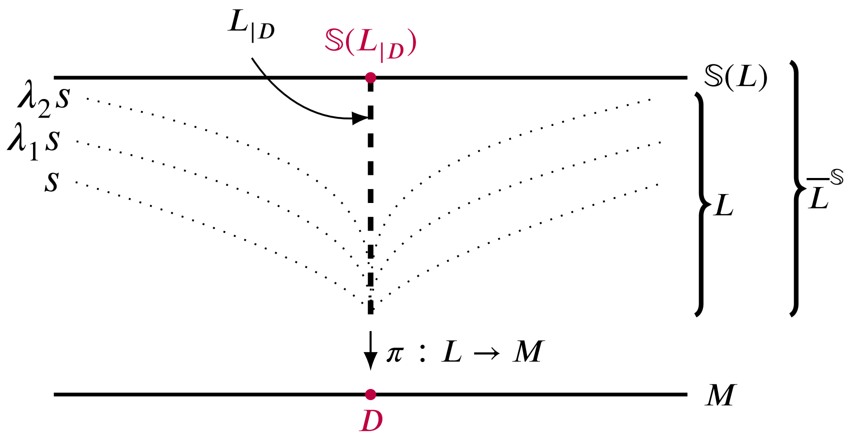
Proof by computation of the effect of the blowing-up on $(M, D)^{\mathrm{clog}}$.

Geometric construction of $(M, D)^{\mathrm{clog}}$

Case of a single smooth hypersurface $D \subset M$.

Fix $L \to M$ a line bundle together with a section $s : M \to L$ such that $D = s^{-1}(0)$.
Denote by $\overline{L}^{\mathbb{S}}$ the (real oriented) compactification of L by the sphere bundle $\mathbb{S}(L)$.

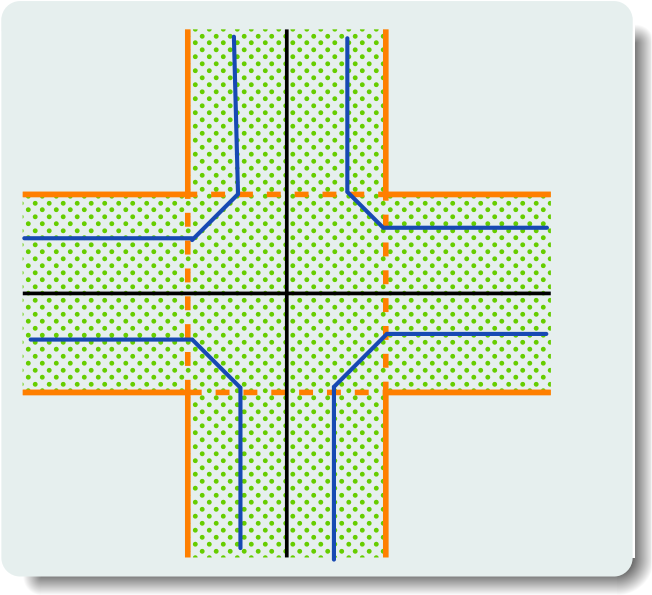
Consider (real oriented) MacPherson graph construction:



$$\lambda_i \to +\infty \quad \rightsquigarrow \quad \overline{L}^{\mathbb{S}} \bigcup_{\mathbb{S}(L_{|D})} (M, D)^{\mathrm{log}}$$

Then $(M, D)^{\mathrm{clog}} = \overline{L^*}^{\mathbb{S}} \bigcup_{\mathbb{S}(L_{|D})} (M, D)^{\mathrm{log}} = L^*_{|D} \bigsqcup (M, D)^{\mathrm{log}}$.

Geometric construction of $(M, D)^{\text{clog}}$



- General case $D = \bigcup_{i\in I} D_i$: $\quad (M, D)^{\text{clog}} = \prod_{i\in I} {}_M (M, D_i)^{\text{clog}}$.
- Formulas over D_J°:
  $f_J^{alog}\left(x, (v_i)_{i\in J}\right) = u_J(x) \prod_{i\in J} v_i^{N_i}$,
  $f_J^{log}\left(x, (\theta_i)_{i\in J}\right) = \arg(u_J(x)) \prod_{i\in J} \theta_i^{N_i}$.
- Define $\xi : (M, D)^{\text{clog}}_{|D} \to \mathbb{R}^I$ by $\xi = (\xi_i)_{i\in I}$, where over D_J°:

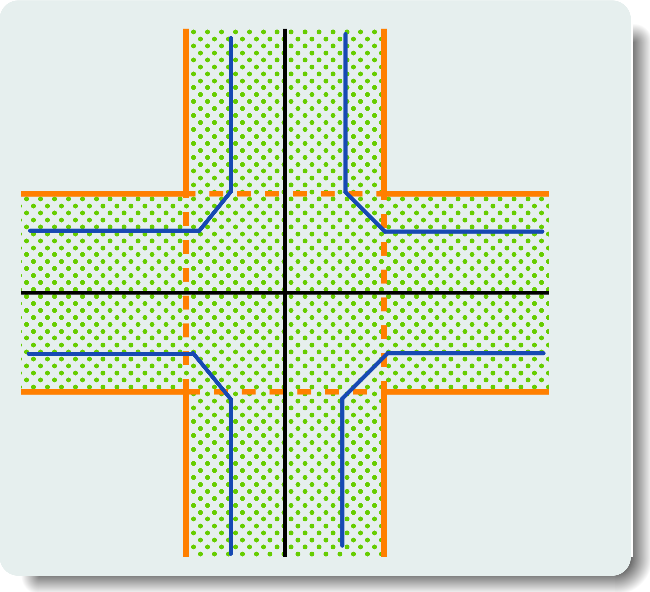
$$\xi_i(x, v) = \begin{cases} (|v_i|N_i)^{-1} & \text{if } i \in J \\ 0 & \text{if } i \notin J. \end{cases}$$

- A' Campo's second model is given by

$$\xi^{-1}(\Delta) \to S^1,$$

where $\Delta = \left\{\xi \in \mathbb{R}^I \ : \ \xi \geq 0, \sum \xi_i = 1\right\}$.

A'Campo's second construction



- A' Campo's second model is given by

$$\xi^{-1}(\Delta) \to S^1,$$

where $\Delta = \left\{\xi \in \mathbb{R}^I \ : \ \xi \geq 0, \sum \xi_i = 1\right\}.$

- We can define a continuous monodromy on $\xi^{-1}(\Delta)$ giving over each D_J°:

$$h_{\lambda,J}(x,v) = \left(x, \left(\exp\left(\lambda \xi_i(x,v) 2\pi\sqrt{-1}\right) v_i\right)_{i\in J}\right).$$