GDAŃSK-KRAKÓW-ŁÓDŹ-WARSZAWA WORKSHOP IN SINGULARITY THEOREY

A special session dedicated to the memory of STANISŁAW ŁOJASIEWICZ

A view of point for the 2-dimensional Jacobian conjecture

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Real and Complex Jacobian Conjecture

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"Let $F = (F_1, \ldots, F_n) : \mathbb{K}^n \to \mathbb{K}^n$, where $n \geq 1$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, be a *polynomial map*. If

$$\det JF(p) \neq 0, \quad \forall p \in \mathbb{K}^n,$$

then F is an automorphism." (*Ott-Heinrich Keller, 1939*)

- $n = 1$: the conjecture is true for both $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.
- $n = 2$:
  1. The conjecture is false for $\mathbb{K} = \mathbb{R}$ (Pinchuk, 1994)
  2. The conjecture is still open for $\mathbb{K} = \mathbb{C}$.

An equivalent statement of the Jacobian conjecture

Let $F : \mathbb{K}^n \to \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a polynomial mapping. If $\det JF(p) \neq 0, \forall p \in \mathbb{K}^n$, then F is proper.

- The map F is proper if and only if its asymptotic set S_F is empty.

$$S_F := \{a \in \mathbb{K}^n : \exists \{\xi_k\} \subset \mathbb{K}^n, |\xi_k| \to \infty, F(\xi_k) \to a\}.$$

Motivation: *Study a subset (suficient large) of non-proper mappings and investigate the Condition of the Jacobian conjecture for this subset.*

Structure of the talk

Part 1. "Asymptotic set" approach (*2-dimensional Real and Complex cases*).

Part 2. Newton polygon approach and the 2-dimensional complex Jacobian conjecture.

Part 3. Intersection homology approach and a singular variety associated to a Pinchuk's map.

Part 1:
“Asymptotic set” approach

Part 1: "Asymptotic set" approach

Consider a polynomial mapping $F : \mathbb{K}^2 \to \mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Purpose:

1. Define a so-called "pertinent variables" in such a way that any mapping written under these variables (called *"pertinent mappings"*) is non-proper.
2. Clasify the pertinent mappings into two types: "dependent" type and "independent" type, such that *for the "dependent" type, the pertinent mappings do not satisfy the condition of the Jacobian conjecture.*

Pertinent variables

Definition

Let $u : \mathbb{K}^2 \to \mathbb{K}$ be a polynomial function. We say that the function u is a *pertinent variable* if there is a sequence $\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset \mathbb{K}^2$ such that $\lim_{k\to\infty} \|(x_k, y_k)\| = \infty$ and $\lim_{k\to\infty} u(x_k, y_k) = c \in \mathbb{K}$.

Definition

A polynomial mapping $F : \mathbb{K}^2 \to \mathbb{K}^2$ is called *pertinent mapping* with respect to pertinent variables $u_0, u_1, \ldots, u_n$ if there exists a polynomial mapping $H : \mathbb{K}^{n+1} \to \mathbb{K}^2$ such that

$$F(x, y) = H(u_0(x, y), u_1(x, y), \ldots, u_n(x, y)).$$

Remark

A pertinent mapping is a non-proper mapping.

Example: Pinchuk map is a pertinent map satisfying the condition of the Jacobian conjecture for the *real* case.

Counter-example to the Real Jacobian Conjecture (Pinchuk, 1994).

Construction of the smallest degree Pinchuk's map: given $(x, y) \in \mathbb{R}^2$, denote

$$t = xy - 1, \quad h = t(xt + 1), \quad f = (xt + 1)^2(t^2 + y).$$

The smallest degree (25) of Pinchuk's maps is the one $\mathcal{P} = (p, q) : \mathbb{R}^2 \to \mathbb{R}^2$, where

$p = f + h$ and

$q = -t^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - \frac{75}{4}h^4.$

Then $\det(J\mathcal{P}(x, y)) > 0, \forall (x, y) \in \mathbb{R}^2$ but $\mathcal{P}$ is not injective.

Affirmation: The variables t, h, f are pertinent.

Classify pertinent variables

Definition

Let $u_0, u_1, \ldots, u_n : \mathbb{K}^2 \to \mathbb{K}$ be pertinent variables. We say that they are *dependent pertinent variables* if there exists $z \in \mathbb{K}^2$ such that the two vectors

$$\left(\frac{\partial u_0}{\partial x}(z), \frac{\partial u_1}{\partial x}(z), \ldots, \frac{\partial u_n}{\partial x}(z)\right),$$

$$\left(\frac{\partial u_0}{\partial y}(z), \frac{\partial u_1}{\partial y}(z), \ldots, \frac{\partial u_n}{\partial y}(z)\right)$$

are *linearly dependent* in $\mathbb{K}^{n+1}$. Otherwise, the pertinent variables $u_0, u_1, \ldots, u_n$ are *independent pertinent variables*.

Example:

Proposition

The Pinchuk pertinent variables:

$$t = xy - 1, \quad h = t(xt+1), \quad f = (xt+1)^2(t^2+y)$$

are independent in both real and complex cases.

Notice that the Pinchuk map is not a counter-example in the complex case.

Proposition

If $u_0, u_1, \ldots, u_n$ are dependent then any pertinent maping $F(u_0, u_1, \ldots, u_n)$ does not satisfy the condition of the Jacobian conjecture.

Theorem

Consider the set of pertinent variables from $\mathbb{K}^2$ to $\mathbb{K}$:

$$u_0 = y, u_1 = x - bx^r y^s, \ldots, u_n = x - bx^{nr} y^{ns}, \quad (b \in \mathbb{K},\ r, s > 0)$$

Then the variables $u_0, u_1, \ldots, u_n$ are pertinent and

1. If $r = 1$, then the pertinent variables $u_0, u_1, \ldots, u_n$ are dependent.
2. If $r > 1$, then for a fixed natural number $n \geq 2$, the pertinent variables $u_0, u_1, \ldots, u_n$ are independent.

Denote by $\mathcal{C}$ the class of pertinent mappings $F(u_0, u_1, \ldots, u_n)$ and write:

$$\mathcal{C} = \bigcup_{r=1}^{\infty} \mathcal{C}_r.$$

- If $\mathcal{C}_1$ is a class where Jacobian conjectrue is true;
- If there exists a counter-example to Jacobian conjecture, this one belongs to some class $\mathcal{C}_r$ with $r \geq 2$.

Part 2: Newton polygon and 2-dimensional complex Jacobian conjecture

Newton polygon

Definition

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of two variables x and y, *i.e.*, f can be written as a finite sum:

$$f = \sum_{i,j} a_{ij} x^i y^j.$$

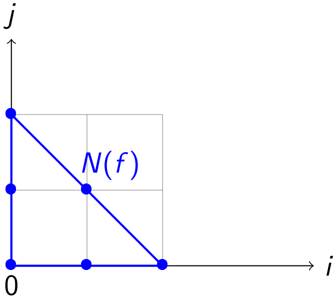
The *support* of f, denoted by $\mathrm{Supp} f$, is defined by:

$$\mathrm{Supp} f = \{(i,j) : a_{ij} \neq 0\}.$$

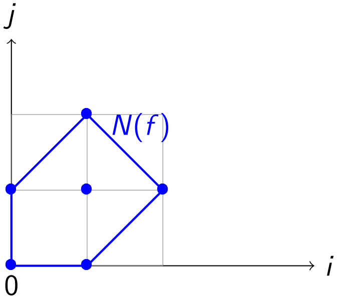
The Newton polygon of f, denoted by $N(f)$, is the convex hull of the origin and $\mathrm{Supp} f$:

$$N(f) = \mathrm{Conv}\left(\{(0,0)\} \cup \{(i,j) : a_{ij} \neq 0\}\right).$$

Example 1: $f(x, y) = x + x^2 + xy + y + y^2$.



Example 2: $f(x, y) = x + x^2y + xy + xy^2 + y$.



The Newton polygon approach was studied by Abhyankar in the book
S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry*, Tata Institute of Fundamental Research, 1977
to study the 2-dimensional Jacobian conjecture in the *complex case*.

Some principal results by Newton polygon approach

Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping.

1. **(Abhyankar, 1977).** If $\deg(f)$ divide $\deg(g)$ or $\deg(g)$ divide $\deg(f)$ then the conjecture is true.
2. **(Appelgate, Onishi, Nagata, 1985).** Let $d := \gcd(\deg(f), \deg(g))$, the greatest common divisor of degrees of f and g. Then If $d \le 8$ or d is a prime number then the conjecture is true.
3. **(Nagata, 1985).** If $\deg(f)$ or $\deg(g)$ is a product of at most two prime numbers, then the conjecture is true.

With a highly non-trivial proof, Moh proved:

Theorem (Moh, 1983)

Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. The 2-dimensional complex Jacobian conjecture is true for $\deg(F) \leq 100$.

Question: What happens if $\deg(F) \geq 101$?

Theorem

The 2-dimensional complex Jacobian conjecture is true for $\deg(F) \leq 104$.

Idea of Proof: Using the principal results obtained by Newton polygon approach and the following result of Żołądek obtained by the so-called Newton-Puiseux chart

Theorem (Żołądek, 2008)

The Jacobian conjecture satisfies for maps with $\gcd(\deg(f), \deg(g)) \leq 16$ and for maps with $\gcd(\deg(f), \deg(g))$ equals to 2 times a prime.

For $\deg(F) = 105$, we have:

Proposition

If the conjecture satisfies for the cases where $(\deg(f), \deg(g))$ is

$$(42, 105), \quad (63, 105), \quad (70, 105), \quad (84, 105)$$

then the conjecture satisfies for polynomial maps of degree 105.

Part 3:
Intersection Homology Approach

Intersection homology approach

A. Valette and G. Valette, *Geometry of polynomial mappings at infinity via intersection homology*, Ann. I. Fourier vol. 64, fascicule 5 (2014), 2147-2163.

Theorem (Anna Valette and Guillaume Valette, 2010).

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping satisfying the condition of the Jacobian conjecture. Then there exist singular varieties V_F (called *Valette varieties*) such that F is a diffeomorphism iff the intersection homology $IH_2^{\overline{p}}(V_F, \mathbb{R}) = 0$, for *any* perversity $\overline{p}$.

- **A generalization of Valettes's Result:**

Theorem (–, Anna Valette and Guillaume Valette, 2013)

The above theorem is still true for the general case $F : \mathbb{C}^n \to \mathbb{C}^n$, where $n \geq 2$, with the additional condition: "The parts at infinity of the fibres of F are complete intersections."

Nguyen, T. B. T., A. Valette and G. Valette, *On a singular variety associated to a polynomial mapping*, Journal of Singularities volume 7 (2013), 190-204.

Theorem

There exists a Valette variety $V_{\mathcal{P}}$ associated to the Pinchuk map $\mathcal{P}$ such that

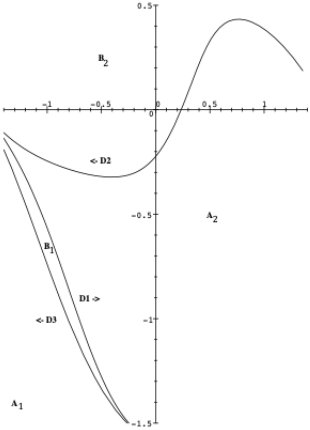
$$IH_1^{\bar{0},c}(V_{\mathcal{P}}) = IH_1^{\bar{0},cl}(V_{\mathcal{P}}) = 0.$$

Remark

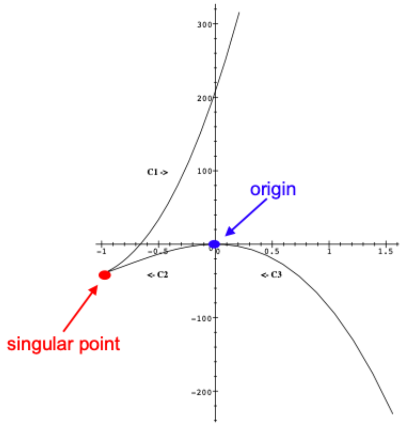
The "*real version*" of Valettes' result *is no longer true!*

A Valette variety associated to a Pinchuk's map

The asymptotic variety and the asymptotic flower of the Pinchuk's map (L. A. Campbell, 1996-2001).

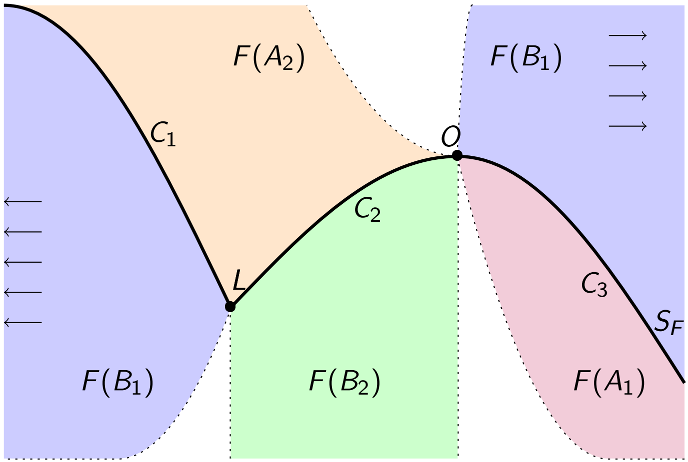


The asymptotic flower



The asymptotic variety

A Valette variety associated to a Pinchuk's map



Theorem (- , Maria Aparecida Soares Ruas, 2018)

Let $G = (G_1, \dots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geq 3$, be a generic polynomial mapping such that $K_0(G) = \emptyset$ and the parts at infinity of the fibres of G are complete intersections. Then there exist singular varieties $\mathcal{V}_G$ associated to G such that if $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R})$ is trivial then the bifurcation set $B(G)$ is empty (here $\bar{t}$ is the total perversity).

Nguyen, T.B.T. and Ruas, M.A.S., *On singular varieties associated to a polynomial mapping from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*, Asian Journal of Mathematics, v.22, p.1157-1172, 2018.

Theorem (Hà Huy Vui and Nguyễn Tất Thắng, 2011).

Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geq 2$, be a *local submersion* polynomial mapping. If the parts at infinity of the fibres of G are complete intersections, then the bifurcation set $B(G)$ of G is empty if and only if the Euler characteristic of $G^{-1}(t)$ is a constant for any $t \in \mathbb{C}^{n-1}$.

V. H. Ha, T. T. Nguyen *On the topology of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*, International Journal of Mathematics, Vol. 22, No. 3 (2011) 435-448.

H. Ha, *Nombres de Łojasiewicz et singularités à l'infini des polynômes de deux variables complexes*, C. R. Acad. Sci. Paris Ser. I **311** (1990) 429-432.

V. H. Ha and L.D. Tráng, *Sur la topologie des polynomes complexes*, Acta Math. Vietnamica **9** (1984) 21-32.