Biologia matematyczna jako źródło trudnych problemów w równaniach różniczkowych cząstkowych

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Papers

1. B. Perthame, J. Skrzeczkowski,
   Fast reaction limit with nonmonotone reaction function,
   Communications on Pure and Applied Mathematics,
   doi: 10.1002/cpa.22042.

2. J. Skrzeczkowski,
   Fast reaction limit and forward-backward di usion:
   a Radon-Nikodym approach,
   Comptes Rendus Mathématique,
   tome 360, p. 189-203, 2022.

My target: explain in understandable way what is in these papers and why is it difficult?

Crash course on evolutionary PDEs

- We consider functions $u(t; x)$ of time t and space $x = (x_1; :::; x_d) \in \quad \subset \mathrm{R}^d$,
- We write $@_t u$ for derivative of u with respect to time,
- We write ∇u for gradient of u with respect to x

$$\nabla u = (u_{x_1}; :::; u_{x_d});$$

- For vector-valued function $u = (u^1; :::; u^d)$ we write div$u$ for

$$\mathrm{div} u = u^1_{x_1} + u^2_{x_2} + ::: + u^d_{x_d}:$$

- We write $\quad u$ for laplacian of u

$$u = u_{x_1;x_1} + u_{x_2;x_2} + ::: + u_{x_d;x_d}:$$

- usual setting: $u \in L^p((0,T) \times \Omega)$ if
$$\int_{(0,T)\times\Omega} |u(t,x)|^p \, dx \, dt < \infty.$$
- u is weakly differentiable if there exists function v such that
$$\int_{(0,T)\times\Omega} u \operatorname{div} \varphi \, dx \, dt = -\int_{(0,T)\times\Omega} v \cdot \varphi \, dx \, dt$$
for all smooth compactly supported test functions $\varphi = (\varphi_1, ..., \varphi_d)$,
- we write $v = \nabla u$; if u and v are in L^p we say that $u \in W^{1,p}(\Omega)$.

Crash course on neuroscience

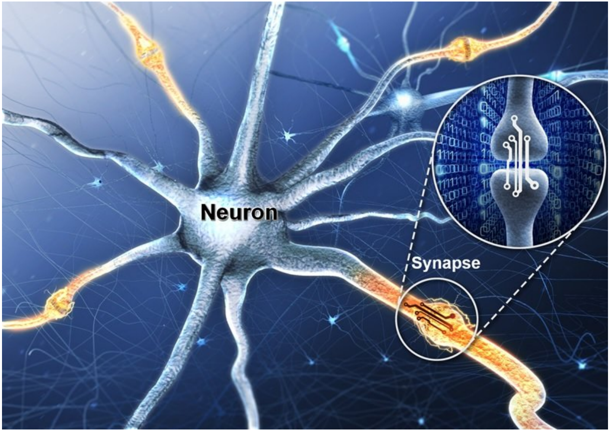
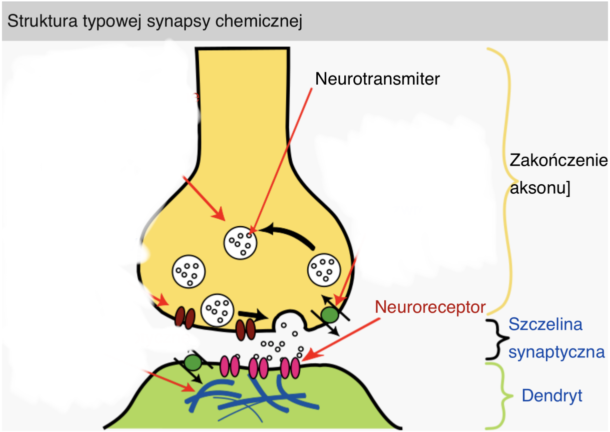


Figure: Di erent parts of the brain communicate by fast electric signals transmitted by neurons. Neurons communicate via synapses (electrical or chemical).

Our problem



$$\partial_t u^\varepsilon = d_1 \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}$$

$$\partial_t v^\varepsilon = d_2 \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}$$

Fast reaction system

- We consider reaction-diffusion system on $[0;T] \times \Omega$

$$\partial_t u^\varepsilon = d_1 \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}$$

$$\partial_t v^\varepsilon = d_2 \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}$$

  equipped with initial conditions $u_0(x)$, $v_0(x)$ and Neumann no-flux boundary condition.

- This system may have nonconstant (oscillating) steady states only for nonmonotone F **motivating nonlinearity**.

- These equations can model some reactions in brain cell membrane where systems are rather unstable while reactions are fast

A. Moussa, B. Perthame, and D. Salort. *Backward parabolicity, cross-diffusion and Turing instability.* J. Nonlinear Sci., 29(1):139–162, 2019

Our problem: what happens when $\varepsilon \to 0$?

$$\partial_t u^\varepsilon = d_1 \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}$$

$$\partial_t v^\varepsilon = d_2 \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}$$

Cross-diffusion and fast-reaction limits...

1. D. Bothe and D. Hilhorst. *A reaction-di usion system with fast reversible reaction.* J. Math. Anal. Appl., 286(1):125–135, 2003.

2. D. Bothe, M. Pierre, G. Rolland. *Cross-di usion limit for a reaction-di usion system with fast reversible reaction.* Comm. Partial Differential Equations, 37(11):1940–1966, 2012.

3. E. S. Daus, L. Desvillettes, A. Jüngel. *Cross-di usion systems and fast-reaction limits.* Bull. Sci. Math.,159:102824, 29, 2020.

and many more...

Basics from the elementary school on compactness

Compactness = a fundamental thing in PDEs.

- In $\mathbb{R}^d$, any bounded sequence has a converging subsequence.
- In infinite dimensional spaces (like function spaces, L^p, $W^{1;p}$) this is no longer true (Riesz).
- To possible ways to overcome this problem:
  1. (Banach-Alaoglu) there exists a subsequence converging weakly, i.e. for all linear functionals , we have $(u_n) \to$ (u) - nice for linear PDEs
  2. impose stronger assumptions on the sequence rather than just boundedness - usual in nonlinear PDEs

Standard examples of the second way

1. (Arzela-Ascoli) $\{u_n\}$ is compact in $C(\overline{\ \ })$ if $\{u_n\}$ is bounded in $C(\overline{\ \ })$ and equicontinuous:

$$\forall_{">0} \quad \exists_{\ >0} \quad \forall_{x;y:jx \ \ yj} \qquad \forall_n \quad |u_n(x) - u_n(y)| \leq ":$$

(this is true if ∇u_n is uniformly bounded on   for instance

2. (Reillich-Kondrachov) $\{u_n\}$ is compact in $L^p(\ \ )$ if $\{\nabla u_n\}$ is bounded in $L^p(\ \ )$,

3. (Lions-Aubin) $\{u_n(t;x)\}$ is compact in $L^p((0;T) \times \ \ )$ if $\{@_x u_n(t;x)\}$ is bounded in L^p and $\{@_t u_n(t;x)\}$ is bounded as a functional on some Sobolev space.

To have compactness in L^p, we need to control derivatives in some sense

Kolmogorov theorem

Theorem (Fréchet–Kolmogorov theorem)

$\mathcal{F}$ is compact in $L^p(\mathbb{R}^d)$ if and only if it is a bounded set in $L^p(\mathbb{R}^d)$ and it is equicontinuous

$$\lim_{|h| \to 0} \int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \, dx = 0 \quad \textit{uniformly in } f \in \mathcal{F};$$

and equi-tight

$$\forall \varepsilon > 0 \qquad \exists K \subset \mathbb{R}^d \textit{ compact s.t.} \int_{\mathbb{R}^d \setminus K} |f|^p \leq \varepsilon.$$

Derivatives for our problem

We have to multiply equations by something **good**

$$\partial_t u^{\varepsilon} = d_1 \Delta u^{\varepsilon} + \frac{v^{\varepsilon} - F(u^{\varepsilon})}{\varepsilon} \qquad \cdot F(u^{\varepsilon})$$

$$\partial_t v^{\varepsilon} = d_2 \Delta v^{\varepsilon} + \frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\varepsilon} \qquad \cdot v^{\varepsilon}$$

With $G(t) = \int_0^t F(s)\, ds$ we have

$$\partial_t G(u^{\varepsilon}) = d_1 \Delta u^{\varepsilon} F(u^{\varepsilon}) + \frac{v^{\varepsilon} - F(u^{\varepsilon})}{\varepsilon} F(u^{\varepsilon})$$

$$\frac{1}{2}\partial_t (v^{\varepsilon})^2 = d_2 \Delta v^{\varepsilon} v^{\varepsilon} + \frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\varepsilon} v^{\varepsilon}$$

Integrating in space,

$$\int_{\Omega} \Delta u^{\varepsilon} F(u^{\varepsilon}) = - \int_{\Omega} \nabla u^{\varepsilon} F'(u^{\varepsilon}) \nabla u^{\varepsilon} + \int_{\partial\Omega} F(u^{\varepsilon}) \frac{\partial u^{\varepsilon}}{\partial \mathbf{n}} =$$

$$= - \int_{\Omega} F'(u^{\varepsilon}) \, |\nabla u^{\varepsilon}|^2$$

With $G(t) = \int_0^t F(s)\, ds$ we have

$$\partial_t G(u^\varepsilon) = d_1 \quad u^\varepsilon F(u^\varepsilon) + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon} F(u^\varepsilon)$$

$$\frac{1}{2}\partial_t (v^\varepsilon)^2 = d_2 \quad v^\varepsilon v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon} v^\varepsilon$$

Therefore,

$$\partial_t \int_\Omega G(u^\varepsilon) + (v^\varepsilon)^2/2 =$$

$$= -d_1 \int_\Omega F'(u^\varepsilon)\, |\nabla u^\varepsilon|^2 - d_2 \int_\Omega |\nabla v^\varepsilon|^2 - \int_\Omega \frac{(v^\varepsilon - F(u^\varepsilon))^2}{\varepsilon}$$

If F' can be negative, this identity is useless...

But there is a trick to get something...

Simplified problem

New system:

$$\partial_t u^\varepsilon = \cancel{d_1 \Delta u^\varepsilon} + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon},$$

$$\partial_t v^\varepsilon = d_2 \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}.$$

Now, energy equality reads:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G(u^\varepsilon) + \frac{1}{2}(v^\varepsilon)^2 \,\mathrm{d}x =$$

$$= \cancel{-d_1 \int_\Omega |\nabla u^\varepsilon|^2 F'(u^\varepsilon)\,\mathrm{d}x} - d_2 \int_\Omega |\nabla v^\varepsilon|^2 \,\mathrm{d}x - \frac{1}{\varepsilon} \int_\Omega (F(u^\varepsilon) - v^\varepsilon)^2 \,\mathrm{d}x.$$

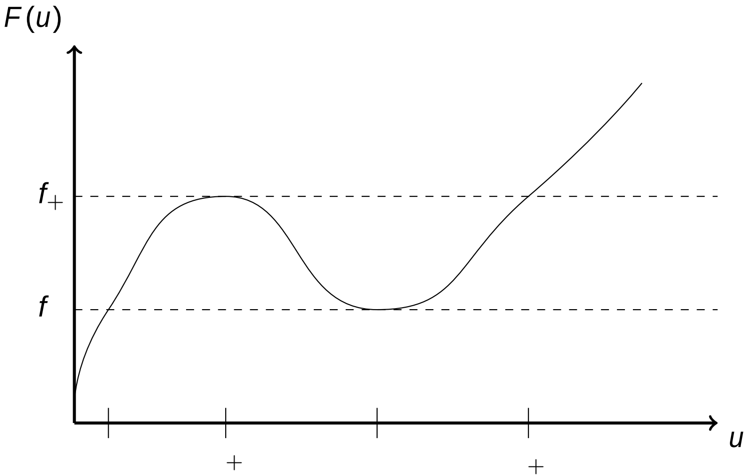


Figure: Plot of a typical function F. It is strictly increasing in intervals $(-\infty; \ _+) \cup (\ _-; \infty)$ and strictly decreasing in $(\ _+; \ _-)$.

Evolution of u″ and v″

Figure: Evolution of u″ (continuous line) and v″ (dash-dotted line) solving the system above with fixed and small value of ″ > 0.

Results

For the system

$$\partial_t u^{\varepsilon} = \frac{v^{\varepsilon} - F(u^{\varepsilon})}{\varepsilon},$$

$$\partial_t v^{\varepsilon} = \quad v^{\varepsilon} + \frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\varepsilon}$$

we have a priori estimates in L^2 for

$$\{\nabla v^{\varepsilon}\}_{\varepsilon \in (0,1)} \qquad \qquad \frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\sqrt{\varepsilon}} \quad {}_{\varepsilon \in (0,1)}$$

and we prove

$$v^{\varepsilon} \to v \text{ in } L^2 \qquad \text{and} \qquad F(u^{\varepsilon}) \to v \text{ in } L^2.$$

This means

$$\int_0^T \int_{\Omega} |v^{\varepsilon}(t,x) - v(t,x)|^2 \, dx \, dt \to 0 \text{ as } \varepsilon \to 0.$$

.

What about $u^{"}$ as " $\to 0$?

Oscillations and why we don't like it?

Sequences that oscillates do not converge a.e. (even after passing to the subsequence...)

In particular, we cannot say a priori anything about $F(u_n)$ if u_n is oscillating...

Strong vs. weak convergence in $L^2(0,1)$

$f_n \to f$ strongly in $L^2(0,1)$ if

$$\int_0^1 |f_n(x) - f(x)|^2 \, dx \to 0.$$

$f_n \rightharpoonup f$ weakly in $L^2(0,1)$ if for all $\varphi \in L^2(0,1)$

$$\int_0^1 (f_n(x) - f(x))\, \varphi(x) \, dx \to 0$$

Exercise: let $f : (0,1) \to \mathbb{R}$ be continuous with $f(0) = f(1)$, $f_n = f(nx)$. Then, $f_n \rightharpoonup \int_0^1 f(x)\, dx$ in $L^2(0,1)$.

$\sin(2\pi nx) \rightharpoonup 0$ but $\sin^2(2\pi nx) \rightharpoonup \frac{1}{2} \neq 0^2$

Young measures to describe oscillations

1. For smooth $G : \mathbb{R} \to \mathbb{R}$ we have

$$G(u^{\varepsilon}) = \int_{\mathbb{R}} G(\ )\, d_{\ u^{\varepsilon}(t;x)}(\ ) \to \int_{\mathbb{R}} G(\ )\, d_{\ t;x}(\ )$$

where $_{u^{\varepsilon}(t;x)} \overset{*}{\ } \ _{t;x}$.

If it is a single Dirac mass, the convergence is strong.

2. Example: Consider sequence of functions $\{u_n\}_{n \in \mathbb{N}}$ oscillating between 1, -1.

The Young measure reads: $\frac{1}{2}\ _{1} + \frac{1}{2}\ _{\ 1}$

Results (once again)

For the system

$$\partial_t u^{\varepsilon} = \frac{v^{\varepsilon} - F(u^{\varepsilon})}{\varepsilon},$$

$$\partial_t v^{\varepsilon} = \quad v^{\varepsilon} + \frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\varepsilon}$$

we have a priori estimates in L^2 for

$$\{\nabla v^{\varepsilon}\}_{\varepsilon 2(0;1)} \qquad \frac{F(u^{\varepsilon}) - v^{\varepsilon}}{\sqrt{\varepsilon}} \quad {}_{\varepsilon 2(0;1)}$$

and we prove

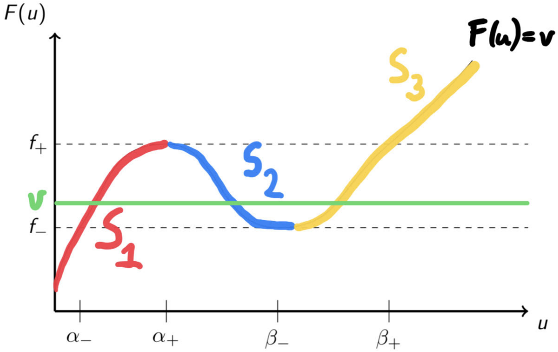
$$v^{\varepsilon} \to v \text{ in } L^2 \qquad \text{and} \qquad F(u^{\varepsilon}) \to v \text{ in } L^2.$$

(without any bound on $\partial_t v^{\varepsilon} \approx \frac{1}{\sqrt{\varepsilon}}$)

Results: representation of u

From $v^{"}; F(u^{"}) \to v$, Young measure generated by $\{u^{"}\}_{">0}$ reads

$$_{t;x} = \sum_{i=1}^{3} {}_i(t;x) \quad S_i(v(t;x)) \qquad \sum_{i=1}^{3} {}_i(t;x) = 1$$



$u(t;x) = \sum_{i=1}^{3} {}_i(t;x)\, S_i(v(t;x))$ so $F(u) \neq v$ so u is not a stationary solution but rather its combination!

Proof of strong convergence $v^\varepsilon \to v$ - STRATEGY

- **STEP 1:** Obtain a lot of energy inequalities of the form

$$\partial_t(\quad(u^\varepsilon) + \quad(v^\varepsilon)) - \qquad (v^\varepsilon) \leq 0$$

for many functions and .

- **STEP 2:** Use *compensated compactness* and energy identities to deduce that for many functions , and ′ weak limits behave nicely:

$$\text{w-}\lim_{\varepsilon \to 0}(\quad(u^\varepsilon) + \quad(v^\varepsilon))\ '(v^\varepsilon) = \text{w-}\lim_{\varepsilon \to 0}(\quad(u^\varepsilon) + \quad(v^\varepsilon))\ \text{w-}\lim_{\varepsilon \to 0}\ '(v^\varepsilon).$$

- **STEP 3:** As above holds for sufficiently many functions, use this to characterize Young measure and conclude that the Young measure of v^ε is Dirac mass.

OPEN PROBLEM: The case of two diffusions

For the general problem

$$\partial_t u^\varepsilon = d_1 \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon},$$

$$\partial_t v^\varepsilon = d_2 \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}$$

we have bounds in L^2 for:

- $d_1 \nabla u^\varepsilon + d_2 \nabla v^\varepsilon$
- $d_1 \sqrt{\varepsilon} \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\sqrt{\varepsilon}}$
- convergence $d_1 \varepsilon \Delta u^\varepsilon + v^\varepsilon - F(u^\varepsilon) \to 0$ in L^2

Hypothesis: u^ε, v^ε both oscillate but $d_1 u^\varepsilon + d_2 v^\varepsilon$ converges strongly.

Y. Morita and N. Shinjo. *Reaction-diffusion models with a conservation law and pattern formations.* Josai Mathematical Monographs, 9:177–190, 2016.

Conclusion

- There are simple PDEs for which we cannot use standard compactness arguments.
- For some of them, we can develop rather difficult ways to deduce compactness; for some of them **not**.
- New compactness criteria???

DZIĘKUJĘ ZA UWAGĘ!