Higher order boundary Harnack principle via degenerate equations

We prove higher order Schauder estimates for solutions to singular/degenerate elliptic equations of type:

$$-\mathrm{div}\,(\rho^a A\nabla w) = \rho^a f + \mathrm{div}\,(\rho^a F) \quad \text{in } \Omega$$

for exponents $a > -1$, where the weight ρ vanishes in a non degenerate manner on a regular hyper-surface Γ which can be either a part of the boundary of Ω or mostly contained in its interior. As an application, we extend such estimates to the ratio v/u of two solutions to a second order elliptic equation in divergence form when the zero set of v includes the zero set of u which is not singular in the domain (in this case $\rho = u$, $a = 2$ and $w = v/u$). We prove first $C^{k,\alpha}$-regularity of the ratio from one side of the regular part of the nodal set of u in the spirit of the higher order boundary Harnack principle. Then, by a gluing lemma, the estimates extend across the regular part of the nodal set. Eventually, using conformal mapping in dimension $n = 2$, we provide local gradient estimates for the ratio which hold also across the singular set.