AMBIENT LIPSCHITZ GEOMETRY OF NORMALLY EMBEDDED SURFACES

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60th Anniversary of Lev Birbrair

Bedlewoo, 21th July 2023

Initial Definitions

Given $C \geq 1$, metric spaces (X, d_1) e (Y, d_2), a map $\varphi : X \to Y$ is **bi-Lipschitz** (or C-bi-Lipschitz) if φ is bijective and if, for all $p, q \in X$:

$$\frac{1}{C} \cdot d_1(p, q) \leq d_2(\varphi(p), \varphi(q)) \leq C \cdot d_1(p, q)$$

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2. **Inner Metric**: $d_X(p,q) = \inf\{l(p,q)\}, \forall\, p, q \in X$, where the infimum is considered over all rectifiable paths connecting p to q.

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Observation: Ambient Bi-Lipschitz Equivalent $\Rightarrow$ Outer Bi-Lipschitz Equivalent $\Rightarrow$ Inner Bi-Lipschitz Equivalent, but the converses in general are not true (BIRBRAIR, GABRIELOV, 2019).

Normally Embedded Sets: We say that a set $X \subseteq \mathbb{R}^n$ is **Lipschitz Normally Embedded (LNE)** if the outer metric and the inner metric are equivalent, i.e., there is a constant $C \geq 1$ such that:

$$\frac{1}{C} \cdot d_X(x,y) \leq d(x,y) \, \forall x, y \in X$$

In this case, we say that X is C-LNE. Analogous definitions can be done to germs of sets in a given point (generally the origin).

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Arcs: An arc in $\mathbb{R}^n$ is a germ at the origin of a semialgebraic map $\gamma : [0, t_0) \to \mathbb{R}^n$, for some $t_0 > 0$ sufficiently small, such that $\gamma(0) = 0$. Usually, the arc is identified with its image $\gamma(t)$ obtained by intersecting γ with a circle of small radius t (Local Conic Structure Theorem).

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2. **Inner**: $tord_X(\gamma_1, \gamma_2) = \beta$, where $d_X(\gamma_1(t), \gamma_2(t)) = ct^\beta + o(t^\beta)(c > 0)$

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Such orders are rational numbers satisfying $1 \leq tord(\gamma_1, \gamma_2) \leq tord_X(\gamma_1, \gamma_2)$. In addition, $(X, 0)$ is LNE if, and only if, $tord(\gamma_1, \gamma_2) = tord_X(\gamma_1, \gamma_2)$, for all $\gamma_1, \gamma_2 \in V(X)$ (BIRBRAIR; MENDES, 2018).

Research History Summary (Real Sets)

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(BIRBRAIR, SOBOLEVSKY, 1999): Every semialgebraic geometric Hölder complex can be realized in $\mathbb{R}^n$, for some n.

(SAMPAIO, 2016): If (X, x_0) and (Y, y_0) are ambient bi-Lipschitz equivalent, then the tangent cone germs $(C(X, x_0), x_0)$ and $(C(Y, y_0), y_0)$ are ambient bi-Lipschitz equivalent.

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(BIRBRAIR et al., 2020): Birbrair, Brandenbursky and Gabrielov proved that for any semialgebraic surface germ $(X, 0) \subset (\mathbb{R}^4, 0)$ there are infinite surface germs $(X_i, 0) \subset (\mathbb{R}^4, 0)$ such that $(X_i, 0)$ are topologically ambient equivalent to $(X, 0)$, outer bi-Lipschitz equivalent to $(X, 0)$, but are not bi-Lipschitz ambient equivalent to each other.

(BIRBRAIR et al., 2021): Birbrair, Fernandes and Jelonek proved that every semialgebraic and compact set X of dimension k is inner bi-Lipschitz equivalent to a normally embedded set $Y \subset \mathbb{R}^{2k+1}$.

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Conjecture: Let $(X, 0)$ and $(Y, 0)$ be two semialgebraic 2-dimensional surface germs on $(\mathbb{R}^n, 0)$ that are outer bi-Lipschitz equivalent and topologically ambient equivalent.

- If $n \geq 5$, then $(X, 0)$ and $(Y, 0)$ are ambient bi-Lipschitz equivalent.
- If $(X, 0)$ and $(Y, 0)$ are LNE, then $(X, 0)$ e $(Y, 0)$ are ambient bi-Lipschitz equivalent.

Preliminary Results in Lipschitz Geometry

Proposition

Let $U, V \subseteq \mathbb{R}^n$ be open and non-empty, and let $\psi : \overline{U} \to \overline{V}$, $\varphi : \overline{V} \to \overline{V}$ outer bi-Lipschitz maps. If $\varphi(p) = p$ for all $p \in \partial V$, then the map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, given by:

$$\Phi(p) = \begin{cases} p, & p \notin U \\ \psi^{-1} \circ \varphi \circ \psi(p), & p \in \overline{U} \end{cases}$$

is an outer bi-Lipschitz map.

Proposition

Let $X_1, X_2 \subseteq \mathbb{R}^n$ be path-connected sets such that $X_1 \cap X_2 \neq \emptyset$, and let $Y_1, Y_2 \subseteq \mathbb{R}^m$ such that $\varphi_1 : X_1 \to Y_1$ and $\varphi_2 : X_2 \to Y_2$ are inner bi-Lipschitz maps satisfying $\varphi_1(p) = \varphi_2(p)$ for all $p \in X_1 \cap X_2$. Then, if $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, the map $\varphi : X \to Y$, given by $\varphi(p) = \varphi_i(p)$, if $p \in X_i$ $(i = 1, 2)$, is an inner bi-Lipschitz map.

Proposition

Let $X \subseteq \mathbb{R}^n$ be a normally embedded semialgebraic set and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an outer bi-Lipschitz map. Then, $\Phi(X) \subseteq \mathbb{R}^n$ is a normally embedded semialgebraic set.

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Proposition

Let $x_1 < x_2$ be real numbers and let $f, g : [x_1, x_2] \to \mathbb{R}$ piecewise smooth functions satisfying $f(x) \geq g(x)$, for all $x \in [x_1, x_2]$. If $M > 0$ exists such that $|f'(x)|, |g'(x)| < M$, for all x where f, g are differentiable, then the set:

$$X = \{(x, y) \in \mathbb{R}^2 \mid x_1 \leq x \leq x_2 \; ; \; g(x) \leq y \leq f(x)\}$$

Is normally embedded.

Theorem (Pancake Decomposition)

Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set. Then there is a stratification $(X, 0) = \cup(X_k, 0)$ such that:

- *Each X_k is normally embedded in $\mathbb{R}^n$;*
- $\dim(X_i \cap X_j) < \min\{\dim X_i, \dim X_j\}$, *for each $i \neq j$.*

*Any decomposition satisfying these conditions is called a **Pancake Decomposition** of X.*

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Theorem (Mendes; Sampaio, 2021)

Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set, with $0 \in X$, such that $(X - 0, 0)$ is a connected germ. Then, X is LNE at 0 if and only if there exists a constant $C \geq 1$ such that $X \cap \{x \in \mathbb{R}^n : \|x\| = t\}$ is C-LNE for every small $t > 0$.

Reduction to Flat Links

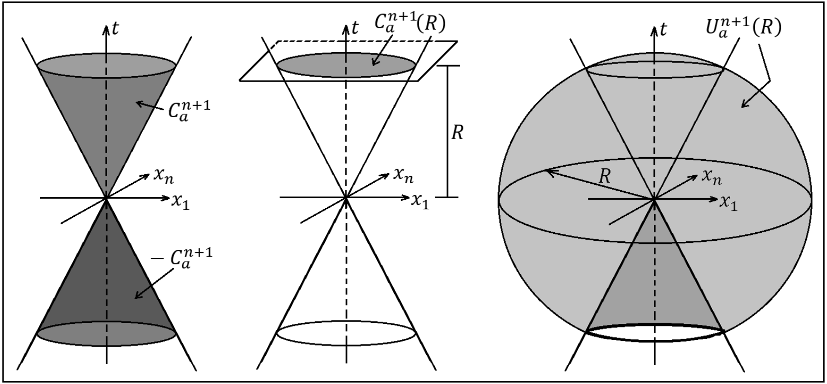
Given $n \in \mathbb{N}$ and $a, R > 0$, define the sets:

$$C_a^{n+1} = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} \mid t \geq 0;\ x_1^2 + \cdots + x_n^2 \leq (at)^2\}$$

$$-C_a^{n+1} = \{-p \mid p \in C_a^{n+1}\}\ ;\ C_a^{n+1}(R) = C_a^{n+1} \cap \{t = R\}$$

$$\mathbb{S}^n(0, R) = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 + t^2 = R^2\}$$

$$U_a^{n+1} = \mathbb{R}^{n+1} \setminus -C_a^{n+1}\ ;\ U_a^{n+1}(R) := U_a^{n+1} \cap \mathbb{S}^n(0, R)$$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t
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x_n
x_1

Stereographic Projection:

$$\psi_R : \mathbb{S}^n(0,R) - \{(0,\ldots,0,-R)\} \to \mathbb{R}^n \times \{R\}$$

$$(x_1,\ldots,x_n,x_{n+1}) \mapsto (\lambda x_1,\ldots,\lambda x_n, R)\,;\ \lambda = \frac{2R}{x_{n+1}+R}$$

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Inverse Difeomorfismo:

$$\psi_R^{-1} : \mathbb{R}^n \times \{R\} \to \mathbb{S}^n(0,R) - \{(0,\dots,0,-R)\}$$

$$(x_1,\dots,x_n,R) \mapsto (\lambda' x_1,\dots,\lambda' x_n,(2\lambda'-1)R)$$

$$\lambda' = \frac{4R^2}{x_1^2+\cdots+x_n^2+4R^2}$$

For every $a > 2$ large enough, there is $a' > 0$ small enough such that $\psi_R|_{\overline{U_{a'}^{n+1}(R)}}$ is a diffeomorphism over its image $C_a^{n+1}(R)$, and that a' depends only on a (more precisely, $a' = \frac{4a}{a^2+4}$).

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Proposition

The map $\psi : \overline{U_{a'}^{n+1}} \to C_a^{n+1}$ given by $\psi(x_1, \ldots, x_n, R) = \psi_R(x_1, \ldots, x_n, R)$, for all $(x_1, \ldots, x_n, R) \in \overline{U_{a'}^{n+1}}$, is an outer bi-Lipschitz map.

The previous Proposition allows reducing the proof of the Main Theorem to the case of surfaces in C_a^3. Indeed, given normally embedded $X \subset \mathbb{R}^3$, there is a non-zero vector:

$$u \notin S = \{\gamma'(0) \mid \gamma \in V(X)\}$$

We can assume that $u = (0, 0, -p)$, for some $p > 0$. Since S is closed, there is $a' > 0$ small enough such that $u \notin S, \forall u \in \overline{U_{a'}^3}$.

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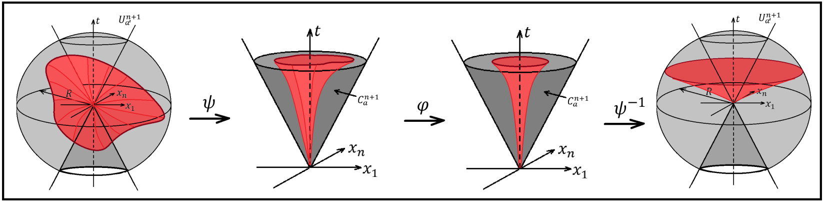


Definition

For each $X \subset C_a^{n+1}$ *and each* $\varepsilon > 0$*, define the sets:*

$$C_a^{n+1}[\varepsilon] = \{(x_1, \ldots, x_n, t) \in C_a^{n+1} \mid 0 < t < \varepsilon\} \cup \{0\}$$

$$C_a^{n+1}(\varepsilon) = \{(x_1, \ldots, x_n, \varepsilon) \in C_a^{n+1}\}$$

$$X[\varepsilon] = X \cap C_a^{n+1}[\varepsilon] \; ; \; X(\varepsilon) = X \cap C_a^{n+1}(\varepsilon)$$

The set $X(\varepsilon)$ *is the* ε *flat link of* X*.*

Definition

Given germs of sets $(X,0),(Y,0) \subset (C_a^{n+1},0)$, *we say that* $(X,0)$ *and* $(Y,0)$ *are ambient bi-Lipschitz equivalents in* $(C_a^{n+1},0)$ *if there is* $\varepsilon > 0$ *small enough and an outer bi-Lipschitz map* $\varphi : (C_a^{n+1},0) \to (C_a^{n+1},0)$ *such that:*

1. $\varphi(p) = p$, *for all* $p \in \partial C_a^{n+1} \cap C_a^{n+1}[\varepsilon]$.
2. $\varphi(X[\varepsilon]) = Y[\varepsilon]$

Some Technical Lemas

Lemma (Translation Lemma)

Let $a_0 > 0$, $X \subset C_{a_0}^{n+1}$ *be a surface and* $(\gamma, 0) \subset (C_{a_0}, 0)$, $\gamma(t) = (y_1(t), \ldots, y_n(t), t)$ *be an arc. Suppose that* $X' \subset C_{a_0}^{n+1}$ *is the translation of* X *by* γ, *that is:*

$$(x_1, \ldots, x_n, t) \in X' \Leftrightarrow (x_1 - y_1(t), \ldots, x_n - y_n(t), t) \in X$$

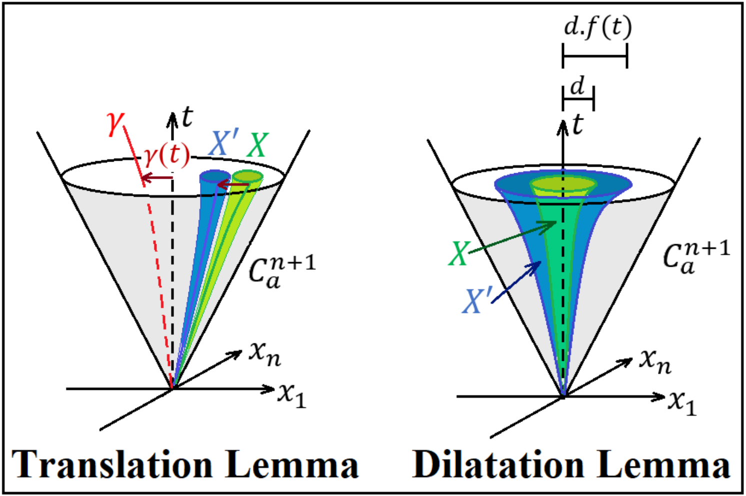
Then, there exists $a \geq a_0$ *such that* $(X, 0), (X', 0)$ *are bi-Lipschitz ambient equivalent in* $(C_a^{n+1}, 0)$.

Lemma (Dilatation lemma)

Let $n \in \mathbb{N}$, $a_0 > 0$ *and* $f : (\mathbb{R}, 0) \to \mathbb{R}$ *be a semialgebraic function germ whose Puiseux series at 0 is* $f(t) = c_0 + o(1)$; $c_0 > 0$. *If* $(X, 0) \subset (C_a^{n+1}, 0)$ *is the germ of a surface and if* $(X', 0) \subset (C_a^{n+1}, 0)$ *is the dilatation of* X *by* f, *that is:*

$$(x_1(t), \ldots, x_n(t), t) \in (X, 0) \Leftrightarrow (f(t)x_1(t), \ldots, f(t)x_n, t) \in (X', 0)$$

Then there is $a \geq a_0$ *such that* $(X, 0), (X', 0) \subset (C_a^{n+1}, 0)$ *are ambient bi-Lipschitz equivalent in* $(C_a^{n+1}, 0)$.


γ
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Translation Lemma
$d.f(t)$
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x_1
Dilatation Lemma

Basic Definitions

- $(\gamma_0, 0), (\gamma_1, 0) \subseteq (C_a^3, 0)$ distinct curve germs;
- $(T, 0) \subseteq (C_a^3, 0)$ Hölder triangle germ (boundary arcs $(\gamma_0, 0), (\gamma_1, 0)$).

We say that $(T, 0)$ is a **Synchronized Triangle Germ** if for every small t, $x_0(t) < x_1(t)$ and:

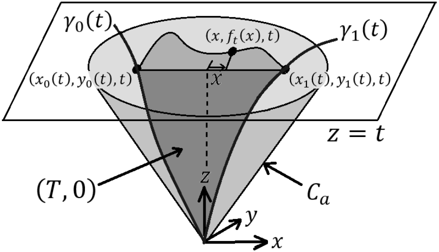
- $\gamma_0 = \gamma_0(t) = (x_0(t), y_0(t), t); \gamma_1 = \gamma_1(t) = (x_1(t), y_1(t), t)$
- $\pi_z(T \cap \{z = t\})$ is graph of a semialgebraic function $f_t : [x_0(t), x_1(t)] \to \mathbb{R}$ with $f_t(x_i(t)) = y_i(t)$ $(i = 1, 2)$.

The family of functions $\{f_t\}_{0<t<\varepsilon}$ is called **Generator of the Synchronized Germ** $(T, 0)$.

Arc Cover of Synchronized Triangle (T,0):
Arcs $\gamma_u \subset V(T)$ $(0 \leq u \leq 1)$, where:

$$\theta_u(t) = u \cdot x_1(t) + (1-u) \cdot x_0(t) \in [x_0(t), x_1(t)]$$

$$\gamma_u(t) = (\theta_u(t), f_t(\theta_u(t)), t) \; ; \; t > 0$$



- $(T_1,0),(T_2,0) \subset (C_a^3,0)$ synchronized triangle germs;
- $(T_1,0),(T_2,0)$ **are aligned on boundary arcs** if there are distinct curve germs

$$(\gamma_0,0),(\gamma_1,0),(\rho_0,0),(\rho_1,0) \subseteq (C_a^3,0)$$

  such that, for $i=0,1$, $(\gamma_i,0)$ and $(\rho_i,0)$ are the boundary arcs of $(T_1,0),(T_2,0)$, respectively, and:

$$\gamma_i = \gamma_i(t) = (x_i(t), y_i(t), t) \; ; \; \rho_i = \rho_i(t) = (x_i(t), w_i(t), t)$$

Curvilinear Rectangle Delimited by $(T_1,0),(T_2,0)$**:** If $\{f_t\},\{g_t\}$ are the families of generating functions of $(T_1,0),(T_2,0)$, respectively, then the curvilinear rectangle is the germ of:

$$R = \{(x,y,t) \in C_a^3 \mid x_0(t) \leq x \leq x_1(t) \; ; \; g_t(x) \leq y \leq f_t(x)\}$$

If $(\gamma_i,0)=(\rho_i,0)$ such a rectangle is called **Region Delimited by** $(T_1,0),(T_2,0)$.

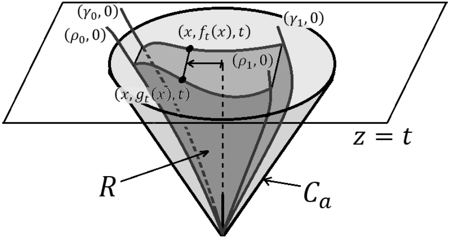
Arc Cover of the Curvilinear Rectangle (R,0):

Arcs $\gamma_{u,v} \subset V(R)$ $(0 \le u, v \le 1)$, where:

$$\theta_u(t) = u \cdot x_1(t) + (1-u) \cdot x_0(t) \in [x_0(t), x_1(t)]$$

$$\sigma_{u,v}(t) = v \cdot f_t(\theta_u(t)) + (1-v) \cdot g_t(\theta_u(t)) \in [g_t(\theta_u(t)), f_t(\theta_u(t))]$$

$$\gamma_{u,v}(t) = (\theta_u(t), \sigma_{u,v}(t), t)\ ;\ t > 0$$



Properties

Proposition

Let $a, M > 0$ *and let* $(T, 0) \subset (C_a^3, 0)$ *be a synchronized triangle germ* C^1*, with derivative M-bounded. If* $\{f_t\}$ *is the family of generating functions associated with* $(T, 0)$*, then there are* $\varepsilon > 0$ *small and* $N > 1$ *large such that, for all* $0 < t < \varepsilon$*,* $x \in (x_0(t), x_1(t))$ *and* $u \in [0, 1]$*:*

$$\left|\frac{\partial}{\partial t}(f_t)(x)\right| \ , \ |\gamma_u'(t)| < N$$

Corollary

Let $a, M > 0$ be real numbers and let $(T_1, 0), (T_2, 0) \subset (C_a^3, 0)$ be germs of C^1 synchronized triangles, with derivative M-bounded and aligned on boundary arcs. Let $\{\gamma_{u,v}\}_{(u,v)\in[0,1]^2}$ be the arc cover of the curvilinear rectangle delimited by $(T_1, 0)$ and $(T_2, 0)$. Then, there are $\varepsilon > 0, N > 1$ such that, for all $(u, v, t) \in [0,1]^2 \times (0, \varepsilon)$:

$$\left|\gamma'_{u,v}(t)\right| < N$$

Proposition

Let $(T_1, 0), (T_2, 0) \subset (C_a^3, 0)$ *be germs of synchronized triangles of class* C^1*, with derivative M-bounded and aligned on the boundary arcs. Then the curvilinear rectangle bounded by* $(T_1, 0)$ *and* $(T_2, 0)$ *is a germ of a normally embedded semialgebraic set.*

Convex Decomposition

Proposition (Synchronized Decomposition)

Let $a > 0$ and let $X \subset C_a^3$ be a pure, semialgebraic, closed 2-dimensional surface. Then, there exists $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}_{\geq 1}$ and $M > 0$ such that:

- *$(X, 0)$ is the union of Hölder triangles $(X_1, 0), \ldots (X_n, 0)$, where, for $i, j \in \{1, \ldots, n\}; i \neq j$, we have that either $(X_i, 0) \cap (X_j, 0) = \{0\}$ or $(X_i, 0) \cap (X_j, 0)$ is an arc. Furthermore, the elements of $\Gamma - \{0\}$ are boundary arcs of $(X_1, 0), \ldots (X_n, 0)$, where:*

$$\Gamma := \{(X_i, 0) \cap (X_j, 0) : i, j \in \{1, \ldots, n\}; i \neq j\}$$

Proposition (Synchronized Decomposition)

- *$(sing(X), 0) \subset \Gamma$;*
- *There are angles $\theta_1, \ldots, \theta_n$ such that $(T_i, 0)$ is a germ of a C^1 synchronized triangle and derivative M-limited, where $T_i = r_{\theta_i}(X_i)$, for each $i = 1, \ldots, n$.*

Any decomposition $(X, 0) = \bigcup_{i=1}^{n}(X_i, 0)$ of the germ $(X, 0)$ satisfying these conditions is called a ***Synchronized Decomposition of*** *$(X, 0)$.*

Proposition (δ-Convex decomposition)

Let $a > 0$ and let $X \subset C_a^3$ be a pure, semialgebraic, closed 2-dimensional surface. Given a synchronized decomposition of $(X, 0)$, for every $\delta > 0$, there is a synchronized decomposition $(X, 0) = \bigcup_{i=1}^{n} (X_i, 0)$ which is a refinement of the initial decomposition, such that:

1. *If $\{f_t\}$ is the family of generating functions of $(T_i, 0) = (r_{\theta_i}(X_i), 0)$, then each $f_t : [x_0(t), x_1(t)] \to \mathbb{R}$ is a convex function;*
2. *For every $t > 0$ small enough, we have:*

$$\left| \frac{\partial f_t}{\partial x_+}(x_0(t)) - \frac{\partial f_t}{\partial x_-}(x_1(t)) \right| < \delta$$

Ambient Bi-Lipschitz Isotopy

Definition

*Let $X, X_0, X_1 \subseteq \mathbb{R}^n$ sets such that $X_1, X_2 \subseteq X$. We say that X_1, X_2 are **Ambient Bi-Lipschitz Isotopic in** X if there is a continuous map $\varphi : X \times [0,1] \to X$ such that, if we denote $\varphi_\tau(p) = \varphi(p, \tau)$, then:*

1. $\varphi_\tau : X \to X$ *is a bi-Lipschitz map (with respect to the induced metric of* $\mathbb{R}^n$*), for all* $0 \leq \tau \leq 1$.
2. $\varphi_0 = id_X$.
3. $\varphi_1(X_1) = X_2$.

Definition

*The map φ is called **Ambient Bi-Lipschitz Isotopy in** X**, taking** X_1 **in** X_2. We also say that the isotopy φ is **Invariant on the Boundary of** X if $\varphi_\tau|_{\partial X} = id_{\partial X}$, for all $0 \leq \tau \leq 1$.*

Definition

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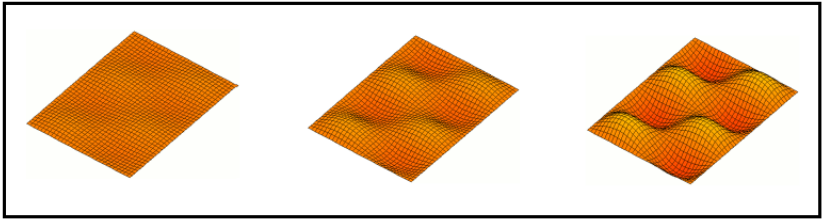


Theorem (Ambient bi-Lipschitz Isotopy in Curvilinear Rectangles)

Let:

$$(T_1,0),(T_2,0),(W_1,0),(W_2,0)\subset(C_a^3,0)$$

be germs of synchronized triangles, two by two alligned on the boundary arcs. If for all $t>0$ *small, there is* $M>1$ *such that:*

- $(T_1,0),(T_2,0),(W_1,0),(W_2,0)$ *have* M*-bounded derivative and that* $\{f_t\},\{g_t\},\{a_t\},\{b_t\}$ *are their respective families of generating functions;*
- $(T_1,0)$ *has* $(\gamma_0,0),(\gamma_1,0)$ *as boundary arcs, where:*

  $$\gamma_0=\gamma_0(t)=(x_0(t),y_0(t),t)\ ;\ \gamma_1=\gamma_1(t)=(x_1(t),y_1(t),t)$$

  and $x_0(t)<x_1(t)$*;*

Theorem

- *for all* $x \in (x_0(t), x_1(t))$, *the inequalities are satisfied:*

$$g_t(x) < a_t(x), b_t(x) < f_t(x)$$

$$\frac{1}{M} \leq \frac{a_t(x) - g_t(x)}{f_t(x) - g_t(x)}, \frac{b_t(x) - g_t(x)}{f_t(x) - g_t(x)} \leq 1 - \frac{1}{M}$$

If $(R, 0)$ *is the curvilinear rectangle bounded by* $(T_1, 0), (T_2, 0)$, *then there is a continuous map* $\varphi : (R, 0) \times [0, 1] \to (R, 0)$ *such that, if we denote* $\varphi_\tau(p) = \varphi(p, \tau)$, *then for all* $0 \leq \tau \leq 1$*:*

Theorem

1. $\varphi_\tau : (R,0) \to (R,0)$ *is an outer bi-Lipschitz map;*
2. $\varphi_0 = id((R,0))$ *and* $\varphi_\tau|_{(T_1,0)\cup(T_2,0)} = id((T_1,0)\cup(T_2,0))$*;*
3. *For all small* $t > 0$ *and* $x \in [x_0(t), x_1(t)]$*:*

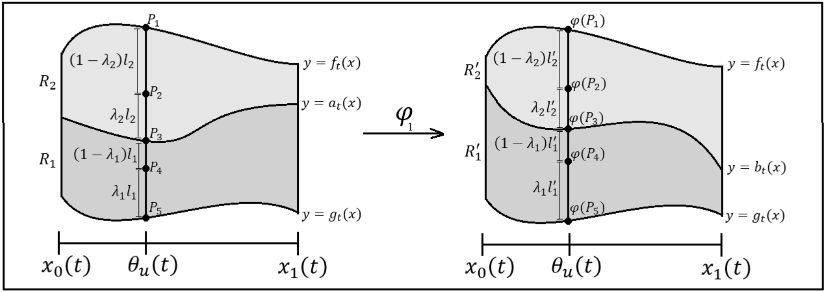
$$\varphi_1(x, f_t(x), t) = (x, f_t(x), t)$$

$$\varphi_1(x, g_t(x), t) = (x, g_t(x), t)$$

$$\varphi_1(x, a_t(x), t) = (x, b_t(x), t)$$

In particular, φ *is an isotopy on* $(R,0)$ *taking* $(W_1,0)$ *into* $(W_2,0)$*. Furthermore, if* $(T_1,0)$ *and* $(T_2,0)$ *have the same boundary arcs,* φ *is invariant on the boundary of the region* $(R,0)$ *delimited by* $(T_1,0)$ *and* $(T_2,0)$*.*

Ambient Bi-Lipschitz Isotopy



Kneadable Triangles

Let $a > 0$, $\gamma_1, \gamma_2 \subset C_a^3$ be two arcs satisfying $tord(\gamma_1, \gamma_2) \neq \infty$, with $\gamma_i(t) = (x_i(t), y_i(t), t)$ $(i = 1.2)$, for every $t > 0$ small enough. We define the **Linear Triangle Delimited by** γ_1, γ_2 as the germ at the origin of the set:

$$\overline{\gamma_1\gamma_2} = \{\lambda\gamma_1(t) + (1-\lambda)\gamma_2(t) \mid t > 0 \; ; \; 0 \leq \lambda \leq 1\}$$

For each $t > 0$, also define the unit vectors:

$$\overrightarrow{\gamma_1\gamma_2}(t) = \frac{\gamma_2(t) - \gamma_1(t)}{\|\gamma_2(t) - \gamma_1(t)\|} \; ; \; \overrightarrow{\gamma_1\gamma_2} = \lim_{t \to 0^+} \frac{\gamma_2(t) - \gamma_1(t)}{\|\gamma_2(t) - \gamma_1(t)\|}$$

Given three arcs $\gamma_1, \gamma_2, \gamma_3 \subset C_a^3$ satisfying $tord(\gamma_i, \gamma_j) \neq \infty$, for $i \neq j$, we define, for each $t > 0$, the angle $\angle\gamma_1\gamma_2\gamma_3(t)$ as the angle formed by $\overrightarrow{\gamma_2\gamma_1}(t)$ and $\overrightarrow{\gamma_2\gamma_3}(t)$. Similarly, we define the angle $\angle\gamma_1\gamma_2\gamma_3$ as the angle formed by $\overrightarrow{\gamma_2\gamma_1}$ and $\overrightarrow{\gamma_2\gamma_3}$.

Observation

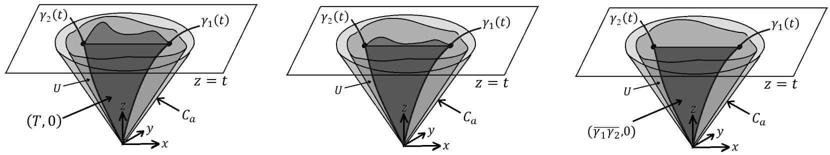
If $(\overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3}, 0)$ *is normally embedded surface germ, then* $\angle\gamma_1\gamma_2\gamma_3 > 0$.

Definition

Let $(T,0) \subset (C_a^3,0)$ *be a Hölder triangle with main vertex at the origin,* γ_1,γ_2 *its boundary arcs and* $(U,0)$ *be a germ of a closed set containing* $(T,0)$*. We say that* $(T,0)$ *is* ***kneadable in*** $(U,0)$ *if there is an ambient bi-Lipschitz isotopy in* U *that takes* $(T,0)$ *in* $(\overline{\gamma_1\gamma_2},0)$*, invariant on the boundary of* $(U,0)$*.*

Definition

Let $(T,0) \subset (C_a^3,0)$ *be a Hölder triangle with main vertex at the origin,* γ_1,γ_2 *its boundary arcs and* $(U,0)$ *be a germ of a closed set containing* $(T,0)$*. We say that* $(T,0)$ *is* ***kneadable in*** $(U,0)$ *if there is an ambient bi-Lipschitz isotopy in* U *that takes* $(T,0)$ *in* $(\overline{\gamma_1\gamma_2},0)$*, invariant on the boundary of* $(U,0)$*.*



Definition

Let $a, M, \delta > 0$ *and* $(T, 0) \subset (C_a^3, 0)$ *be the germ of a synchronized triangle with* M*-bounded derivative and* $\gamma_i(t) = (x_i(t), y_i(t), t)$, $i = 0, 1$ *be the boundary arcs of* $(T, 0)$, *with* $x_0(t) < x_1(t)$, *and* $\{a_t\}$ *the family of generating functions of* $(T, 0)$. *Define, for each small* $t > 0$ *and for* $i = 0, 1$:

$$m_{t,i} = \inf\left\{\frac{a_t(x) - y_i(t)}{x - x_i(t)}\ ;\ x_0(t) < x < x_1(t)\right\}$$

$$M_{t,i} = \sup\left\{\frac{a_t(x) - y_i(t)}{x - x_i(t)}\ ;\ x_0(t) < x < x_1(t)\right\}$$

Note that, since $(T, 0)$ *has a* M*-bounded derivative, then* $|m_{i,t}|, |M_{i,t}| \leq M$.

Definition

*We define the δ-**supporting envelope of** $(T, 0)$ as the germ, at the origin, of the following semialgebraic set:*

$$U_\delta(T) = \{(x, y, t) \mid t > 0\ ;\ x_0(t) \leq x \leq x_1(t); g_t(x) \leq y \leq f_t(x)\}$$

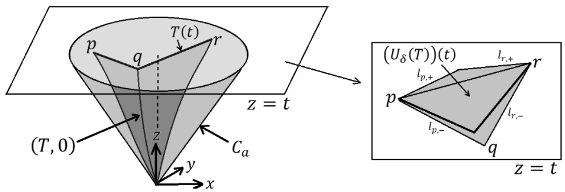
Where:

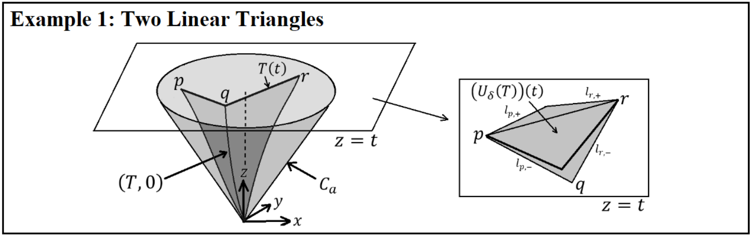
$$g_t(x) = max\{y_0(t) + (m_{t,0} - \delta)(x - x_0(t)); y_1(t) + (M_{t,1} + \delta)(x - x_1(t))\}$$

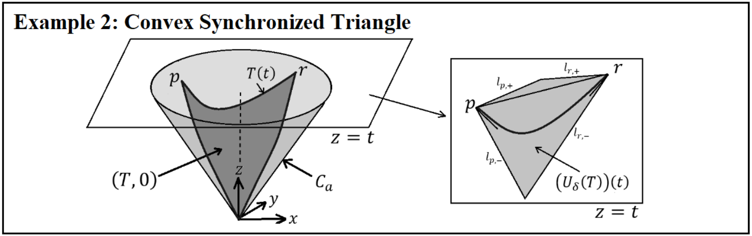
$$f_t(x) = min\{y_0(t) + (M_{t,0} + \delta)(x - x_0(t)); y_1(t) + (m_{t,1} - \delta)(x - x_1(t))\}$$

Kneadable Triangles

Example 1: Two Linear Triangles




Example 1: Two Linear Triangles
p
q
T(t)
r
z = t
(T, 0)
z
y
x
C_a
(U_δ(T))(t)
l_{r,+}
l_{p,+}
r
p
l_{r,-}
l_{p,-}
q
z = t


Example 2: Convex Synchronized Triangle
p
T(t)
r
z = t
(T, 0)
z
y
x
C_a
l_{r,+}
r
l_{p,+}
p
l_{r,-}
l_{p,-}
(U_δ(T))(t)
z = t

Applications

Proposition

For all $M, \delta > 0$ *and for all synchronized triangle germ* $(T, 0)$ *with derivative M-bounded, we have that* $(T, 0)$ *is kneadeable in its* δ*-supporting envelope* $U_\delta(T)$.

Applications

Proposition

For all $M, \delta > 0$ and for all synchronized triangle germ $(T, 0)$ with derivative M-bounded, we have that $(T, 0)$ is kneadeable in its δ-supporting envelope $U_\delta(T)$.

Proposition

Let $a > 0$, $\gamma_1, \gamma_2 \subset C_a^3$ be two arcs such that $tord(\gamma_1, \gamma_2) = \alpha \neq \infty$. Then $(\overline{\gamma_1\gamma_2}, 0)$ is ambient bi-Lipschitz equivalent to the standard α-Hölder's triangle embedded in $\mathbb{R}^3$:

$$T_\alpha = \{(x, 0, t) \mid t \geq 0 \; ; \; 0 \leq x \leq t^\alpha\}$$

Proposition

Let $(\gamma_i, 0) \in (C_a^3, 0)$ *(*$i = 1, 2, 3$*) be distinct arcs such that:*

$$(X, 0) = (\overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3}, 0)$$

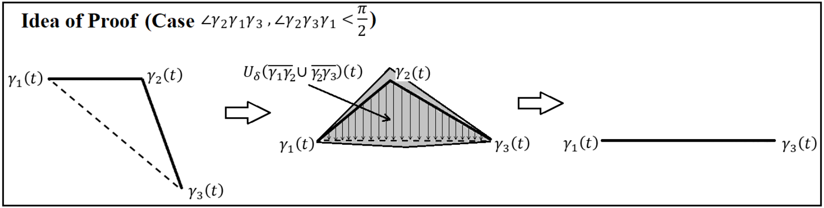
is LNE. Then, $(X, 0)$ *is ambient bi-Lipschitz equivalent to* $(\overline{\gamma_1\gamma_3}, 0)$*.*

Proposition

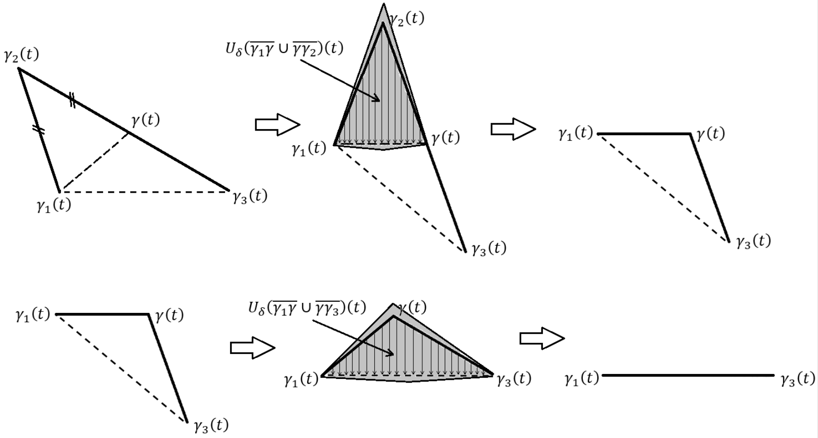
Let $(\gamma_i, 0) \in (C^3_a, 0)$ $(i = 1, 2, 3)$ *be distinct arcs such that:*

$$(X, 0) = (\overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3}, 0)$$

is LNE. Then, $(X, 0)$ *is ambient bi-Lipschitz equivalent to* $(\overline{\gamma_1\gamma_3}, 0)$.



Idea of Proof (Case $\angle\gamma_2\gamma_1\gamma_3, \angle\gamma_2\gamma_3\gamma_1 \geq \frac{\pi}{2}$)



Definition

Let $\gamma_1, \gamma_2, \gamma_3$ *be arcs and let* $Y = \overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3}$ *be a surface whose germ is LNE. Given* $\theta > 0$*, for each* $t > 0$ *and for* $i = 1.3$*, let* $r_{i,+}(t), r_{i,-}(t)$ *be lines passing through* $\gamma_i(t)$ *and external to the triangle* $\gamma_1\gamma_2\gamma_3(t)$ *such that:*

$$\angle(r_{i,-}(t), \overline{\gamma_1\gamma_3}(t)) = \angle(r_{i,+}(t), \overline{\gamma_i\gamma_2}(t)) = \theta$$

For θ *small enough, the lines* $r_{1,+}(t), r_{3,+}(t)$ *intersect at a point* $\gamma_+(t)$ *and the lines* $r_{1,-}(t), r_{3,-}(t)$ *intersect at a point* $\gamma_-(t)$*. Let the arcs* $\gamma_+ = \gamma_+(t)$*,* $\gamma_- = \gamma_-(t)$*,* $V_\theta(t)$ *be the quadrilateral delimited by* $\gamma_1(t)$*,* $\gamma_+(t)$*,* $\gamma_3(t)$ *and* $\gamma_-(t)$ *and* $V_\theta(Y) = \cup_{t>0} U_\theta(t) \cup \{0\}$*.*

Definition

*We define the θ-**kneading envelope of** Y as the germ of the set $V_\theta(Y)$ and denote γ_+ as the boundary arc of this θ-envelop closest to γ_2.*

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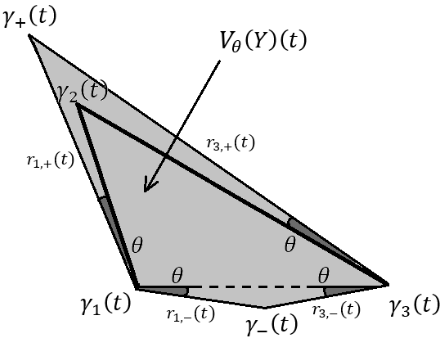

Proposition

Let $(\gamma_i, 0) \in (C_a^3, 0)$ *(*$i = 1, 2, 3$*) be distinct arcs such that:*

$$(X, 0) = (\overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3} \cup \overline{\gamma_3\gamma_1}, 0)$$

is LNE. Then, there exists $\beta \in \mathbb{Q}_{\geq 1}$ *such that* $(X, 0)$ *is ambient bi-Lipschitz equivalent to the germ of the standard* β*-horn* $(H_\beta, 0)$.

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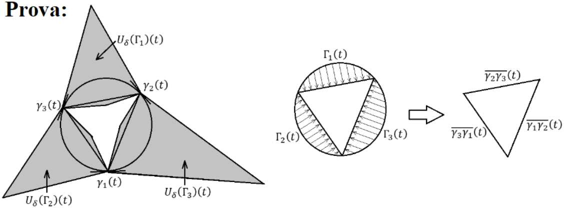
Let $(\gamma_i, 0) \in (C_a^3, 0)$ $(i = 1, 2, 3)$ *be distinct arcs such that:*

$$(X, 0) = (\overline{\gamma_1\gamma_2} \cup \overline{\gamma_2\gamma_3} \cup \overline{\gamma_3\gamma_1}, 0)$$

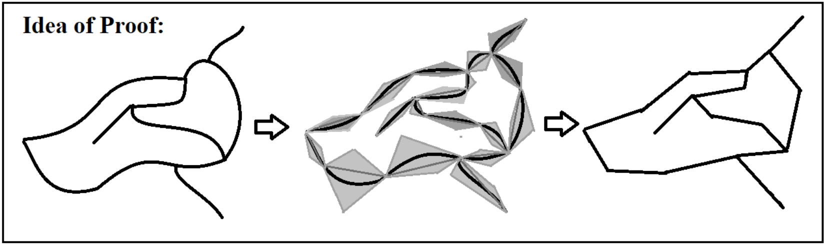
is LNE. Then, there exists $\beta \in \mathbb{Q}_{\geq 1}$ *such that* $(X, 0)$ *is ambient bi-Lipschitz equivalent to the germ of the standard* β*-horn* $(H_\beta, 0)$*.*

Ideia da Prova:



Proposition

Let $a > 0$ and let $(X, 0) \subset (C_a^3, 0)$ be a pure, closed, semi-algebraic, 2-dimensional LNE surface germ with connected link. Then, $(X, 0)$ is ambient bi-Lipschitz equivalent to a germ of a surface formed by a finite union of linear triangles delimited by arcs.



Normally Embedded Polygonal Surfaces

Let $a > 0$ and let $(X, 0) \subset (C^3_a, 0)$ be a closed semialgebraic surface germ. We say that X is a **Polygonal Surface** if there are $n \in \mathbb{N}_{\geq 2}$, distinct arcs $\gamma_1, \ldots, \gamma_n \subset X$ and one of the two following situations occurs:

1. The link of X is homeomorphic to $[0, 1]$ and $(X, 0) = (\cup_{i=1}^{n-1} \overline{\gamma_i \gamma_{i+1}}, 0)$. In this case, we say that X is a **Open Polygon (or** $(n-1)$**-Open Gonal)**, and that n is the number of vertices of X.
2. The link of X is homeomorphic to $\mathbb{S}^1$, $n \geq 3$ and $(X, 0) = (\cup_{i=1}^{n} \overline{\gamma_i \gamma_{i+1}}, 0)$, where $\gamma_{n+1} = \gamma_1$. In this case, we say that X is a **Closed Polygonal (or** n**-Gronal Closed)**, and that n is the number of vertices of X.

In any case, we denote X as a polygonal surface (open or closed) delimited by the arcs $\gamma_1, \ldots, \gamma_n$. The surfaces $\overline{\gamma_1\gamma_2}, \ldots, \overline{\gamma_{n-1}\gamma_n}$ are defined as **Edge Surfaces of** X (if X is closed, $\overline{\gamma_1\gamma_n}$ is also an edge surface of X).

We also say that X is a **Non-Degenerate** n**-Gonal Surface** if $n \geq 3$ and the following conditions are satisfied:

- X is open n-gonal and $\angle\gamma_{i-1}\gamma_i\gamma_{i+1} < \pi$, for $i = 2, \ldots, n-1$, or X is closed n-gonal and $\angle\gamma_{i-1}\gamma_i\gamma_{i+1} < \pi$, for $i = 1, \ldots, n$ ($\gamma_0 = \gamma_n$, $\gamma_1 = \gamma_{n+1}$);
- for all $t > 0$ small enough and all $1 \leq i < j < k \leq n$, $\gamma_i(t), \gamma_j(t), \gamma_k(t)$ are not collinear.

Lemma

Let $(X,0)$ *be a polygonal surface germ (open or closed) LNE delimited by* $\gamma_1,\dots,\gamma_n$ *and let* $i\in\{1,\dots,n\}$ *be such that:*

- $\overline{\gamma_{i-1}\gamma_i}$, $\overline{\gamma_i\gamma_{i+1}}$ *are surfaces of edges of* X;
- $\alpha = tord(\gamma_{i-1},\gamma_i) = tord(\gamma_i,\gamma_{i+1})$;
- $0 < \angle\gamma_{i-1}\gamma_i\gamma_{i+1} < \pi$.

Then there is $\varepsilon_0>0$ *such that, for all* $\varepsilon\in(0,\varepsilon_0)$, *if* $(\gamma_\varepsilon,0)$ *is the arc defined by:*

$$\gamma_\varepsilon(t)\in\overline{\gamma_{i-1}(t)\gamma_i(t)}\ ;\ \|\gamma_\varepsilon(t)-\gamma_i(t)\|=\varepsilon.t^\alpha\ (t>0)$$

Then $(X,0)$ *is ambient bi-Lipschitz equivalent to the germ of the polygonal surface (open or closed) delimited by* $\gamma_1,\dots,\gamma_{i-1},\gamma_\varepsilon,\gamma_{i+1},\dots,\gamma_n$.

Lemma

Let $(X,0)$ be a polygonal surface germ (open or closed) LNE delimited by $\gamma_1,\ldots,\gamma_n$ and let $i\in\{1,\ldots,n\}$ be such that:

- $\overline{\gamma_{i-1}\gamma_i}$, $\overline{\gamma_i\gamma_{i+1}}$ *are edge surfaces of X;*
- $tord(\gamma_{i-1},\gamma_i) > tord(\gamma_i,\gamma_{i+1})$;
- $0<\angle\gamma_{k-1}\gamma_k\gamma_{k+1}<\pi$, *for all k.*

Then the following statements are true:

1. *If $X(t)\simeq[0,1]$ and $i=2$, then $(X,0)$ is ambient bi-Lipschitz equivalent to the germ of the open polygonal surface delimited by $\gamma_1, \gamma_3, \ldots, \gamma_n$.*
2. *If $\overline{\gamma_{i-2}\gamma_{i-1}}$ is an edge surface of X and $tord(\gamma_{i-2},\gamma_{i-1})\le tord(\gamma_i,\gamma_{i+1})$, then $(X,0)$ is ambient bi-Lipschitz equivalent to the germ of the polygonal surface (open or closed) delimited by $\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n$.*

Edge Reduction on Polygonal Surfaces

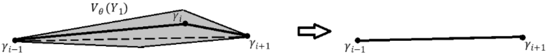
Proposition

Let $a > 0$, $n \geq 3$ be an integer and let $(X, 0) \subset (C_a^3, 0)$ be a polygonal LNE surface germ delimited by the arcs $\gamma_1, \ldots, \gamma_n$.

1. *If $(X, 0)$ is open $(n-1)$-gonal, then $(X, 0)$ is ambient bi-Lipschitz equivalent to the germ surface 1open-gonal delimited by γ_1, γ_n.*
2. *If $(X, 0)$ is closed n-gonal, then $(X, 0)$ is ambient bi-Lipschitz equivalent to a germ of a closed 3-gonal LNE surface.*

Idea of Proof:

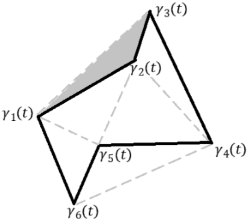
Caso $\angle\gamma_{i-1}\gamma_i\gamma_{i+1} = \pi$



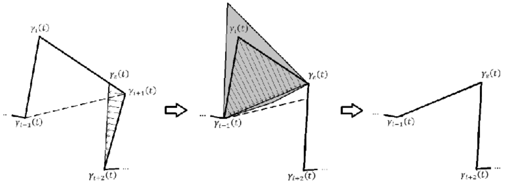
If $\angle\gamma_{i-1}\gamma_i\gamma_{i+1} < \pi$, let $\alpha_i = tord\left(\gamma_i, \gamma_{i+1}\right)$.

Case 1: $\alpha_i = \alpha$, for all i

Case 1.1: The surface is polygonal non degenerated. Take a triangulation of the link



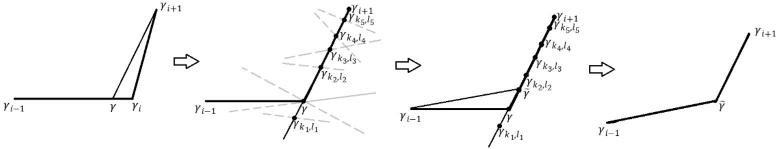
Reducing Number of Edges:



Idea of Proof:

Case 1.2: The surface is polygonal degenerated.

We can prove that such surface is ambient bi-Lipschitz equivalent to a polygonal non degenerated surface, by small deformations along the edges surfaces, by an algorithm that always avoids each vertex from being collinear with the lines determined by the other pair of segments.



Idea of Proof:

Case 2: There are i,j such that $\alpha_i \neq \alpha_j$.

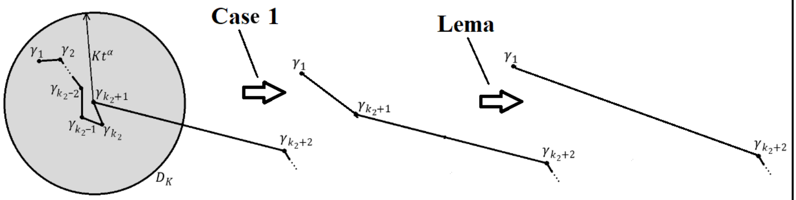
Case 2.1: $X(t) \simeq [0,1]$; $\alpha_1 > \alpha_2$ or $\alpha_{n-1} > \alpha_{n-2}$

Is equivalent to the lemma.

Case 2.2: $X(t) \simeq [0,1]$; $\alpha_1 \leq \alpha_2$ e $\alpha_{n-1} \leq \alpha_{n-2}$

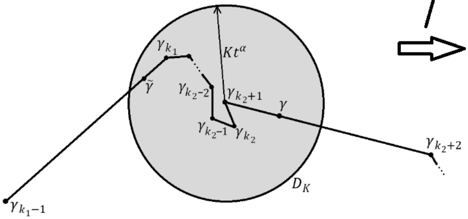
Let $\alpha = \max\{\alpha_i\}$ e take a maximal sequence of indices such that $\alpha_{k_1} = \ldots = \alpha_{k_2} = \alpha$.

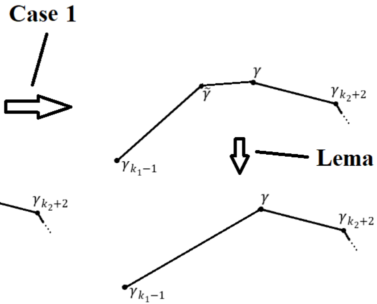
Case 2.2.1: $k_1 = 1$ or $k_2 = n-1$.



Idea of Proof:

Case 2.2.2: $1 < k_1$ e $k_2 < n - 1$





Case 2.3: $X(t) \simeq \mathbb{S}^1$

It's possible to find at least 3 edges surfaces with minimum contact order. Therefore, between two of then there is at least a maximal sequence of indices and the case follows as the case 2.2.2.

Main Theorem

Theorem

Let $(X,0) \subset (\mathbb{R}^3,0)$ be a normally embedded semi-algebraic surface germ.

1. *If the link of X is homeomorphic to $[0,1]$, then $(X,0)$ is ambient bi-Lipschitz equivalent to the germ of a standard α-Hölder triangle embedded in $\mathbb{R}^3$, with principal vertex at the origin, for some rational $\alpha \geq 1$.*
2. *If the link of X is homeomorphic to $\mathbb{S}^1$, then $(X,0)$ is ambient bi-Lipschitz equivalent to the germ of the standard β-horn $(H_\beta,0)$, for some rational $\beta \geq 1$.*

Observation

Although all the concepts of synchronized triangles, regions and polygonal surfaces can be defined naturally for higher dimensions, reducing edge surfaces via triangulations does not work for higher dimensions. For example, the following set:

Observation

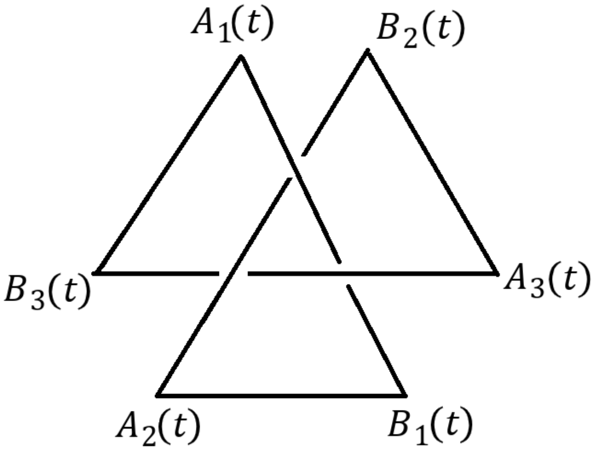
Although all the concepts of synchronized triangles, regions and polygonal surfaces can be defined naturally for higher dimensions, reducing edge surfaces via triangulations does not work for higher dimensions. For example, the following set:

$$X(t) = \overline{A_1(t)B_1(t)} \cup \overline{B_1(t)A_2(t)} \cup \overline{A_2(t)B_2(t)} \cup \overline{B_2(t)A_3(t)}$$
$$\cup \overline{A_3(t)B_3(t)} \cup \overline{B_3(t)A_1(t)}; X = (\cup_{t>0} X(t)) \cup \{0\}$$

where:

$$A_1(t) = (5t\sqrt{3}, 3t, 3t, t)\ ;\ A_2(t) = (-4t\sqrt{3}, 6t, 3t, t)$$
$$A_3(t) = (-t\sqrt{3}, -9t, 3t, t)\ ;\ B_1(t) = (-4t\sqrt{3}, -6t, -3t, t)$$
$$B_2(t) = (5t\sqrt{3}, -3t, -3t, t)\ ;\ B_3(t) = (-t\sqrt{3}, 9t, -3t, t)$$

X is LNE, and is outer bi-Lipschitz equivalent to the 1-horn embedded in $\mathbb{R}^4$. However, X is not topologically equivalent to the 1-horn, so X is not ambient bi-Lipschitz equivalent to the 1-horn.



Happy Birthday, Lev!!!