Pairs of Lipschitz Normally Embedded Hölder Triangles:

Outer Lipschitz Classification

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All sets and maps are **definable** in a polynomially bounded o-minimal structure over $\mathbb{R}$ with the field of exponents $\mathbb{F}$, e.g., **real semialgebraic** or **subanalytic** with $\mathbb{F} = \mathbb{Q}$.

A set $X \subset \mathbb{R}^n$ inherits from $\mathbb{R}^n$ two metrics:
the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y.

The set X is **Lipschitz Normally Embedded** (**LNE**) if these two metrics on X are equivalent.

A **surface germ** X is a closed two-dimensional germ at the origin. The **link** of X is its intersection with a small sphere $\{|x| = \delta\}$.

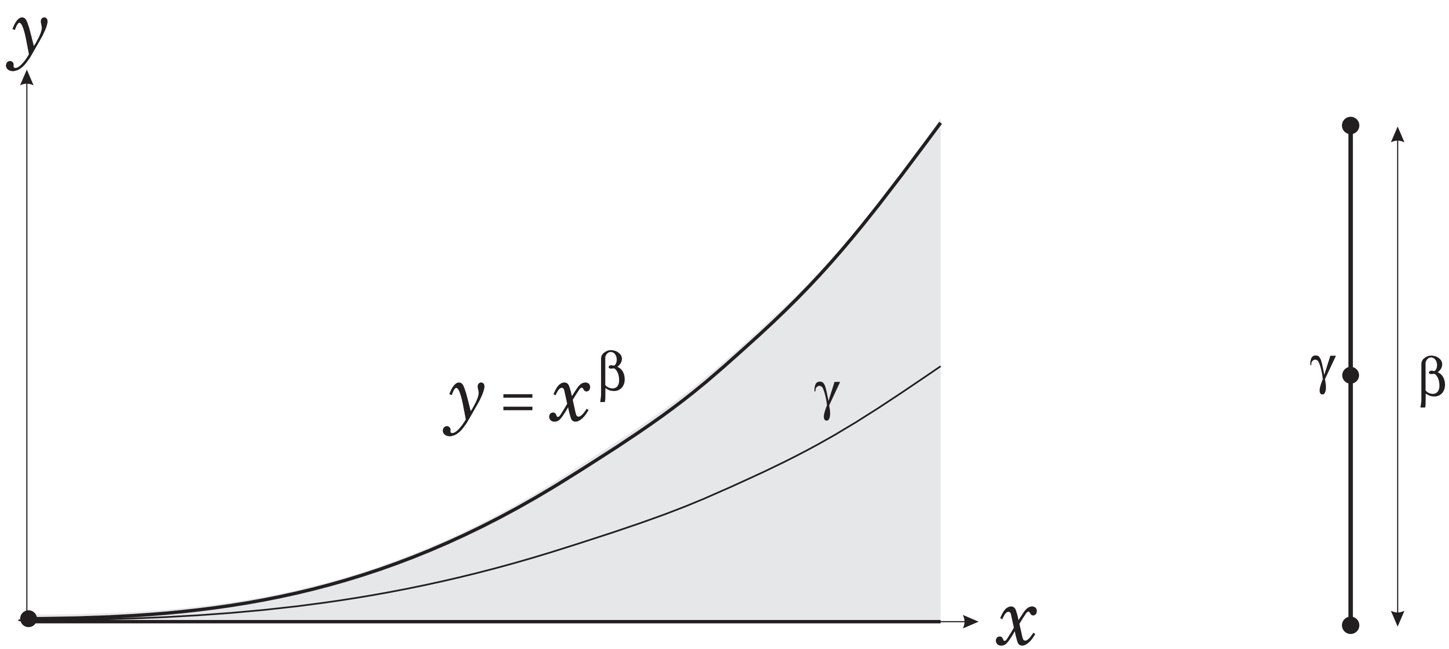
Surface germs X and Y are **outer (inner) Lipschitz equivalent** if there is an outer (inner) bi-Lipschitz homeomorphism $h : X \to Y$.

An **arc** $\gamma \subset X$ is the germ of a curve in X. The **tangency order** $tord(\gamma, \gamma') \in \mathbb{F} \cup \{\infty\}$ of two arcs is outer Lipschitz invariant.

A **standard β-Hölder triangle**, for $1 \le \beta \in \mathbb{F}$, is the surface germ

$$T_\beta = \{(x,y) \in \mathbb{R}^2 \mid x \ge 0,\ 0 \le y \le x^\beta\}.$$

The arcs $\{y = 0\}$ and $\{y = x^\beta\}$ are the **boundary arcs** of T_β.



A standard β-Hölder triangle T_β (left) and its link (right).

A β-**Hölder triangle** $T = T(\gamma_1, \gamma_2)$ with **boundary arcs** γ_1 and γ_2 is a surface germ inner Lipschitz equivalent to T_β.

Lipschitz Normally Embedded (LNE) Hölder triangles are **building blocks** of surface germs:

Given $\beta \in \mathbb{F}$, all LNE β-Hölder triangles are Lipschitz equivalent.

Classification of surface germs with respect to **inner** Lipschitz equivalence (Birbrair 1999) was based on the **Hölder complex**: canonical decomposition of a surface germ into Hölder triangles.

Our goal: To understand the **outer Lipschitz geometry** of a **pair** (T, T') of Lipschitz Normally Embedded Hölder triangles.

We want to decompose a surface germ X into **pizza slices**, Hölder subtriangles with simple metric properties.

The main difficulty: Hard to select boundary arcs of pizza slices **canonically**, uniquely up to outer Lipschitz equivalence.

Only **Lipschitz singular** arcs in X can be easily selected.

An arc γ in a surface germ X is **Lipschitz non-singular**
if it is an interior arc of a LNE Hölder triangle $T \subset X$,
otherwise γ is **Lipschitz singular**.

There are finitely many **Lipschitz singular** arcs in X.

A Hölder triangle is **non-singular** if all its interior arcs are Lipschitz non-singular.

Because of the difficulty of selecting arcs in X, we have to work in the **Valette link** $V(X)$, the set of all arcs in X, instead of X itself.

The tangency order of arcs defines a **non-archimedean metric** on $V(X)$: The distance between arcs γ and γ' is $1/tord(\gamma, \gamma')$.

There are plenty of outer Lipschitz invariant subsets in $V(X)$.

We are going to select outer Lipschitz invariant sets of arcs in $V(X)$, called **perfect zones**, so that any two choices of an arc in a perfect zone are outer Lipschitz equivalent.

Zonology (AG, Souza, 2022)

A set of arcs $Z \subset V(X)$ is a **zone** if, for any arcs $\gamma \neq \gamma'$ in Z, there is a non-singular Hölder triangle $T = T(\gamma, \gamma')$ such that $V(T) \subset Z$.

The **order** $ord(Z)$ of a zone Z is infimum of the tangency orders of arcs in Z. A single arc $\{\gamma\}$ is a **singular zone** of order ∞.

An arc γ in a LNE β-Hölder triangle $T = T(\gamma_1, \gamma_2)$ is **generic** if $tord(\gamma, \gamma_1) = tord(\gamma, \gamma_2) = \beta$.

A zone $Z \subset V(X)$ is **perfect** if, for any two arcs γ and γ' in Z, there is a Hölder triangle T, such that $V(T) \subset Z$ and both γ and γ' are generic arcs of T. By definition, a singular zone $Z = \{\gamma\}$ is perfect.

Pizza Hut (Birbrair et al, 2017)

Let T' be the graph of a non-negative Lipschitz function $f(x)$ on a LNE Hölder triangle T, such that $f(\mathbf{0}) = 0$.

For an arc $\gamma \in V(T)$, let $ord_\gamma f = tord(\gamma, \gamma')$ be the **order** of f on γ, where $\gamma' \in V(T')$ is the graph of $f|_\gamma$.

Let $Q(T) \subset \mathbb{F} \cup \{\infty\}$ be the set of exponents $ord_\gamma f$ for all $\gamma \subset T$. Then $Q(T)$ is a closed interval in $\mathbb{F} \cup \{\infty\}$.

A Hölder triangle T is **elementary** if $Z_q = \{\gamma \subset T,\ ord_\gamma f = q\}$ is a zone, for any $q \in Q(T)$. Let $\mu(q) \in \mathbb{F} \cup \{\infty\}$ be the order of Z_q. This defines a piecewise linear **width function** $\mu(q)$ on $Q(T)$.

All relations between exponents in a polynomially bounded o-minimal structure are piecewise linear. (van den Dries, 1997)

A Hölder triangle T is a **pizza slice** for f if $\mu(q) = aq + b$.

A **pizza** for a non-negative Lipschitz function f on T is a decomposition $\Lambda = \{T_j\}$ of T into **pizza slices** $T_j = T(\lambda_{j-1}, \lambda_j)$, such that $T_j \cap T_{j+1} = \{\lambda_j\}$, with the following **toppings:**

- exponents β_j of T_j,
- exponents $q_j = ord_{\lambda_j} f$,
- closed intervals $Q_j = Q(T_j) = [q_{j-1}, q_j]$ in $\mathbb{F} \cup \{\infty\}$,
- linear width functions $\mu_j(q)$ on Q_j (an exponent μ_j if Q_j is a point).

A pizza is **minimal** if the union of any two adjacent pizza slices is not a pizza slice.

Two pizzas $\Lambda = \{T_j\}$ and $\Lambda' = \{T'_j\}$ with the same toppings are **equivalent** if there is a bi-Lipschitz homeomorphism h of T such that $h(T_j) = T'_j$ for all j.

An **abstract pizza** is a combinatorial encoding of an equivalence class of minimal pizzas.

Theorem (Birbrair *et al.*, 2017). **For any Lipschitz function f defined on a LNE Hölder triangle T, a minimal pizza exists, and is unique up to equivalence. Minimal pizzas for two Lipschitz functions f and g on T are equivalent if, and only if, f and g are Lipschitz contact equivalent.**

Remark: For a non-negative Lipschitz function f on a LNE Hölder triangle T, the **Lipschitz contact equivalence** class of f is the same as the **outer Lipschitz equivalence** class of a pair (T, T'), where T' is the graph of f.

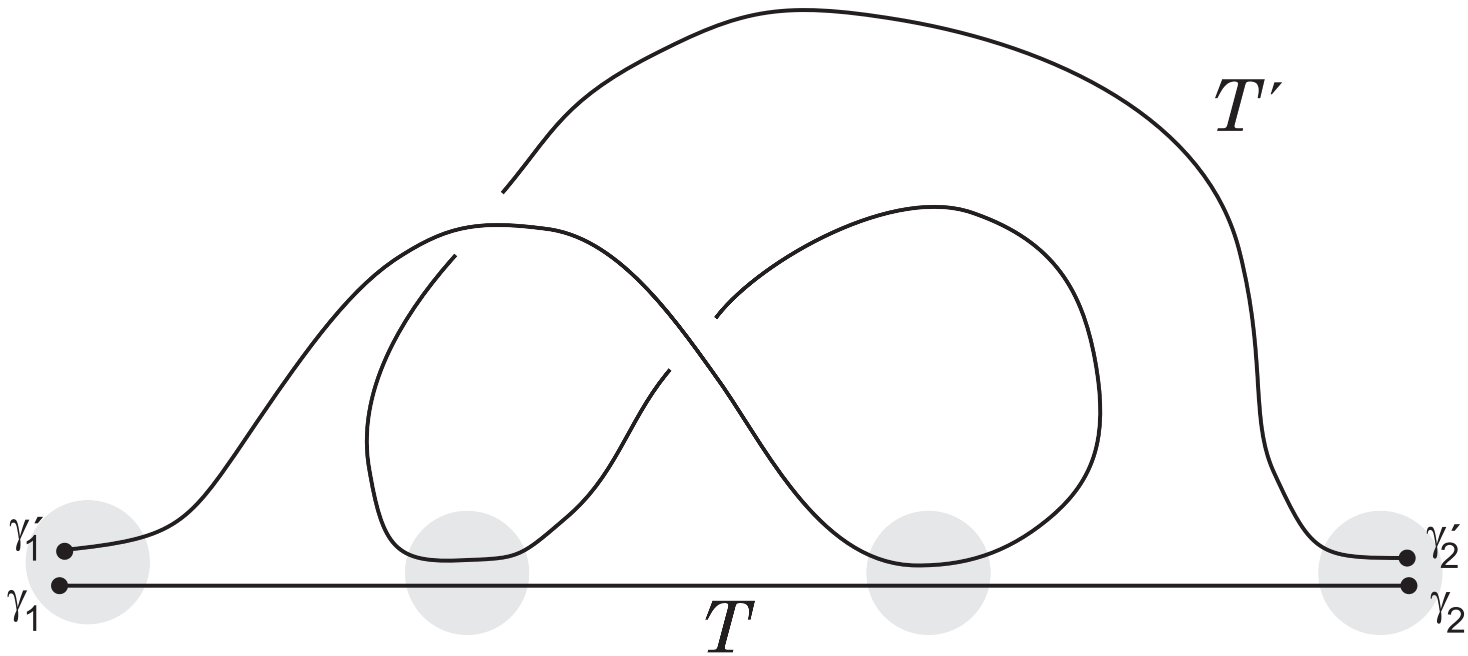
Normal pairs of Hölder Triangles

Given two Hölder triangles T and T', a pair of arcs $\gamma \subset T$ and $\gamma' \subset T'$, is **normal** if $tord(\gamma, T') = tord(\gamma, \gamma') = tord(\gamma', T)$. A pair (T, T') of LNE Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$ is **normal** if both pairs (γ_1, γ'_1) and (γ_2, γ'_2) of their boundary arcs are normal.

For example, if T' is a graph of a Lipschitz function f on T, then any pair of arcs (γ, γ'), where $\gamma \subset T$ and $\gamma' \subset T'$ is the graph of $f|_\gamma$, is normal, and the pair (T, T') is normal.

Theorem (Birbrair, AG). **Let (T, T') be a normal pair of Hölder triangles. If T is elementary with respect to $f(x) = dist(x, T')$, then the pair (T, T') is outer Lipschitz equivalent to the pair (T, Γ), where Γ is the graph of f. Moreover, T' is elementary with respect to $g(x') = dist(x', T)$, and a minimal pizza for g on T' is equivalent to a minimal pizza for f on T.**

If (T, T') is a normal pair of Hölder triangles such that T is not elementary with respect to $f(x) = dist(x, T')$, then $T \cup T'$ may be not equivalent to the union of T and a graph of a function on T.



The link of a normal pair (T, T') of Hölder triangles.

Maximal exponent zones (maximum zones)

Let (T, T') be a normal pair of Hölder triangles $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma'_1, \gamma'_2)$. Let $\{D_\ell\}_{\ell=0}^p$ be pizza zones in $V(T)$ of a minimal pizza for $f(x) = dist(x, T')$, ordered from $D_0 = \{\gamma_1\}$ to $D_p = \{\gamma_2\}$. The **exponent** $q_\ell = tord(D_\ell, T')$ of the zone D_ℓ is defined as $ord_\gamma f$ for $\gamma \in D_\ell$ (it is the same for all $\gamma \in D_\ell$).

A zone D_ℓ is a **maximal exponent zone**, or a **maximum zone**, if $0 < \ell < p$ and $q_\ell \geq \max(q_{\ell-1}, q_{\ell+1})$, or $\ell = 0$ and $\beta < q_0 \geq q_1$, or $\ell = p$ and $\beta < q_p \geq q_{p-1}$.

Maximum zones in $V(T')$ are some of the pizza zones D'_ℓ of a minimal pizza for $g(x') = dist(x', T)$, defined exchanging T and T'.

Theorem (Birbrair, AG). **Let (T,T') be a normal pair of Hölder triangles.**

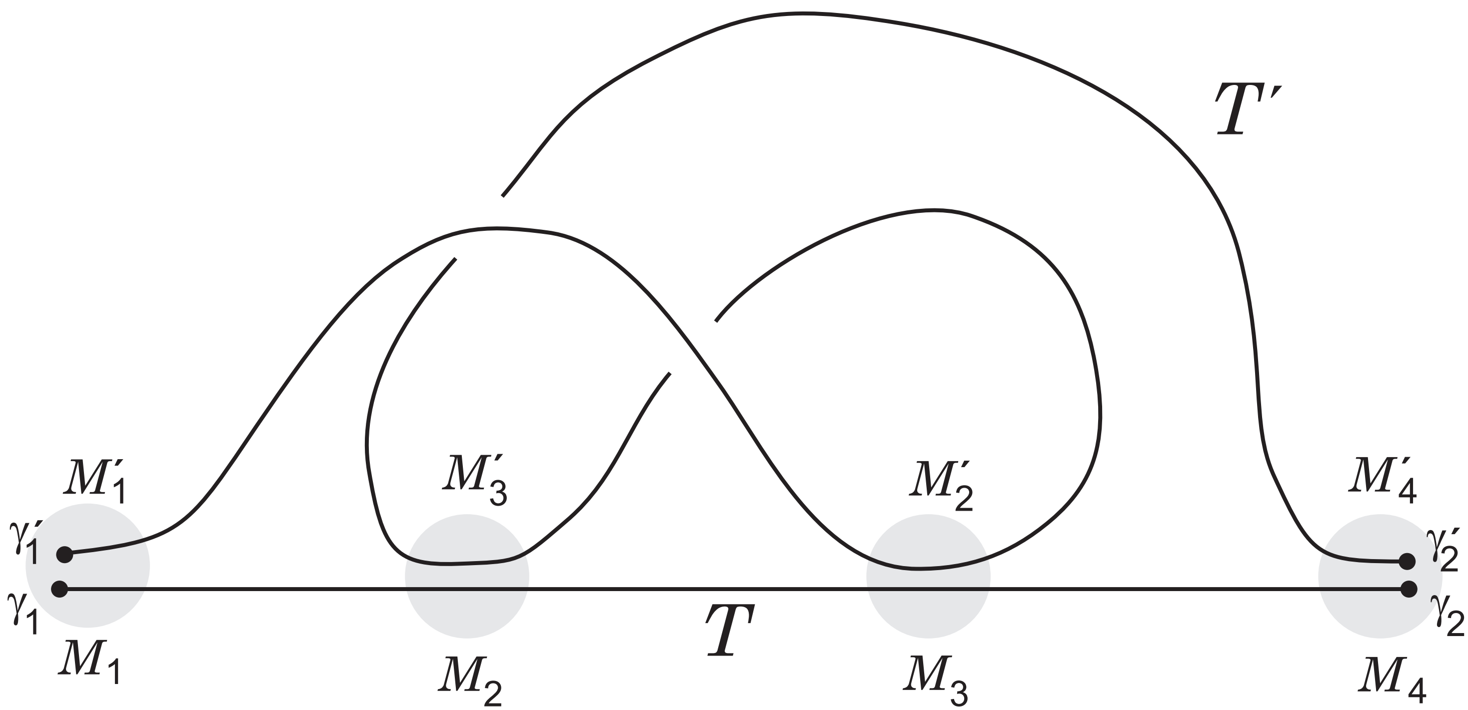
Let $\{M_i\}_{i=1}^m$ and $\{M'_j\}_{j=1}^n$ be maximum zones in $V(T)$ and $V(T')$. Let $\bar{q}_i = tord(M_i,T')$ and $\bar{q}'_j = tord(M'_j,T)$.

Then $m=n$, and there is a canonical permutation

$$\sigma : [1,\ldots,m] \to [1,\ldots,m]$$

such that $ord(M_i) = ord(M'_{\sigma(i)})$ and $tord(M_i, M'_{\sigma(i)}) = \bar{q}_i = \bar{q}'_{\sigma(i)}$.

If $\{\gamma_1\} = M_1$ is a maximum zone, then $\{\gamma'_1\} = M'_1$ and $\sigma(1)=1$.
If $\{\gamma_2\} = M_m$ is a maximum zone, then $\{\gamma'_2\} = M'_m$ and $\sigma(m)=m$.



A normal pair (T, T') of Hölder triangles with four pairs of maximum zones and $\sigma = (1, 3, 2, 4)$.

Transversal and coherent Hölder triangles and pizza slices

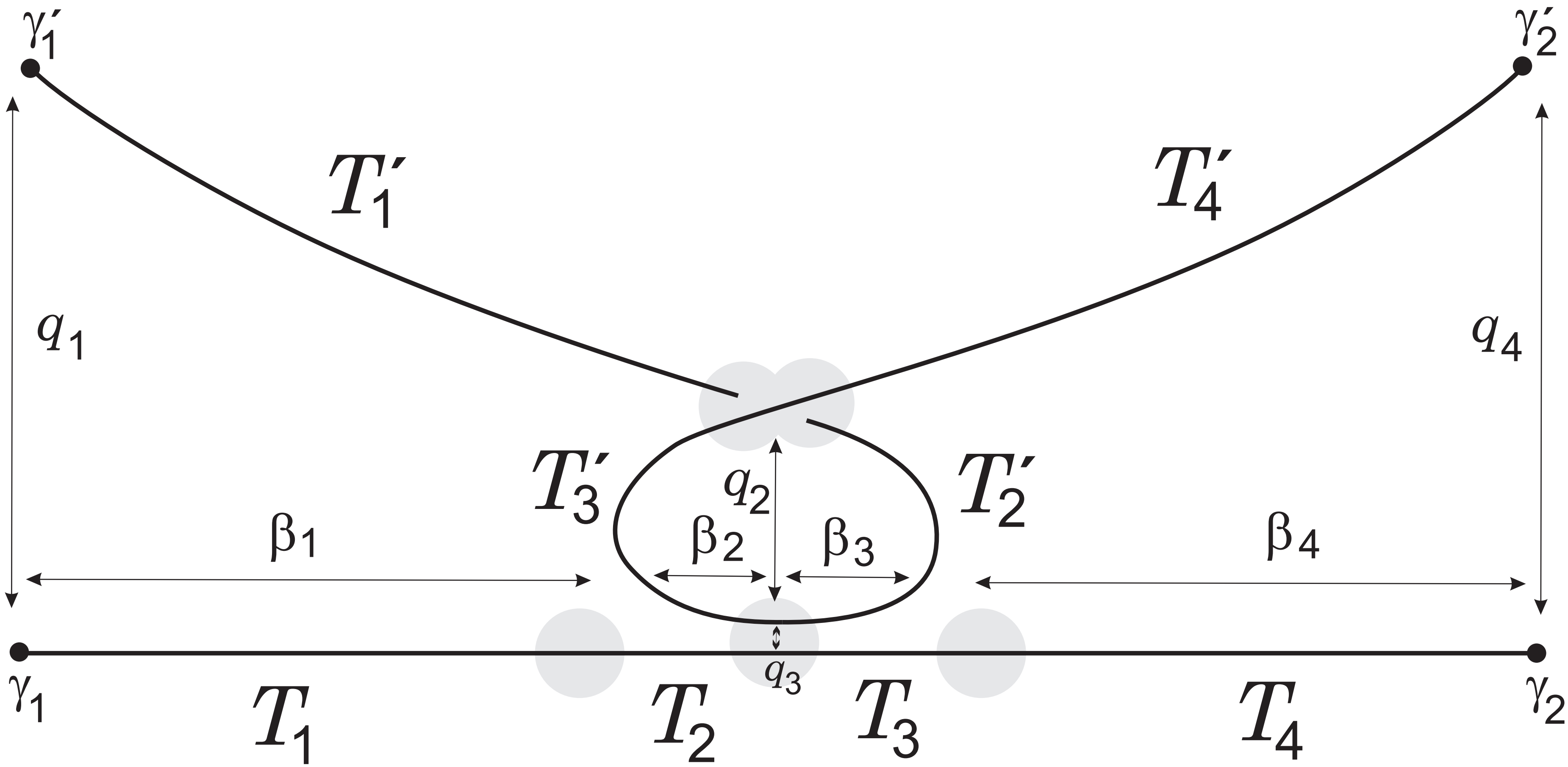
Two LNE Hölder triangles T and T' are **transversal** if there is a boundary arc $\tilde{\gamma}$ of T and a boundary arc $\tilde{\gamma}'$ of T', such that $tord(\gamma, T') = tord(\gamma, \tilde{\gamma}')$ for any arc γ of T and $tord(\gamma', T) = tord(\gamma', \tilde{\gamma})$ for any arc γ' of T'.

Let $\{T_j\}$ be a pizza decomposition of a LNE Hölder triangle T for a Lipschitz function f on T. Then a pizza slice T_j is called **transversal** if T_j and the graph of $f|_{T_j}$ are transversal and **coherent** otherwise.

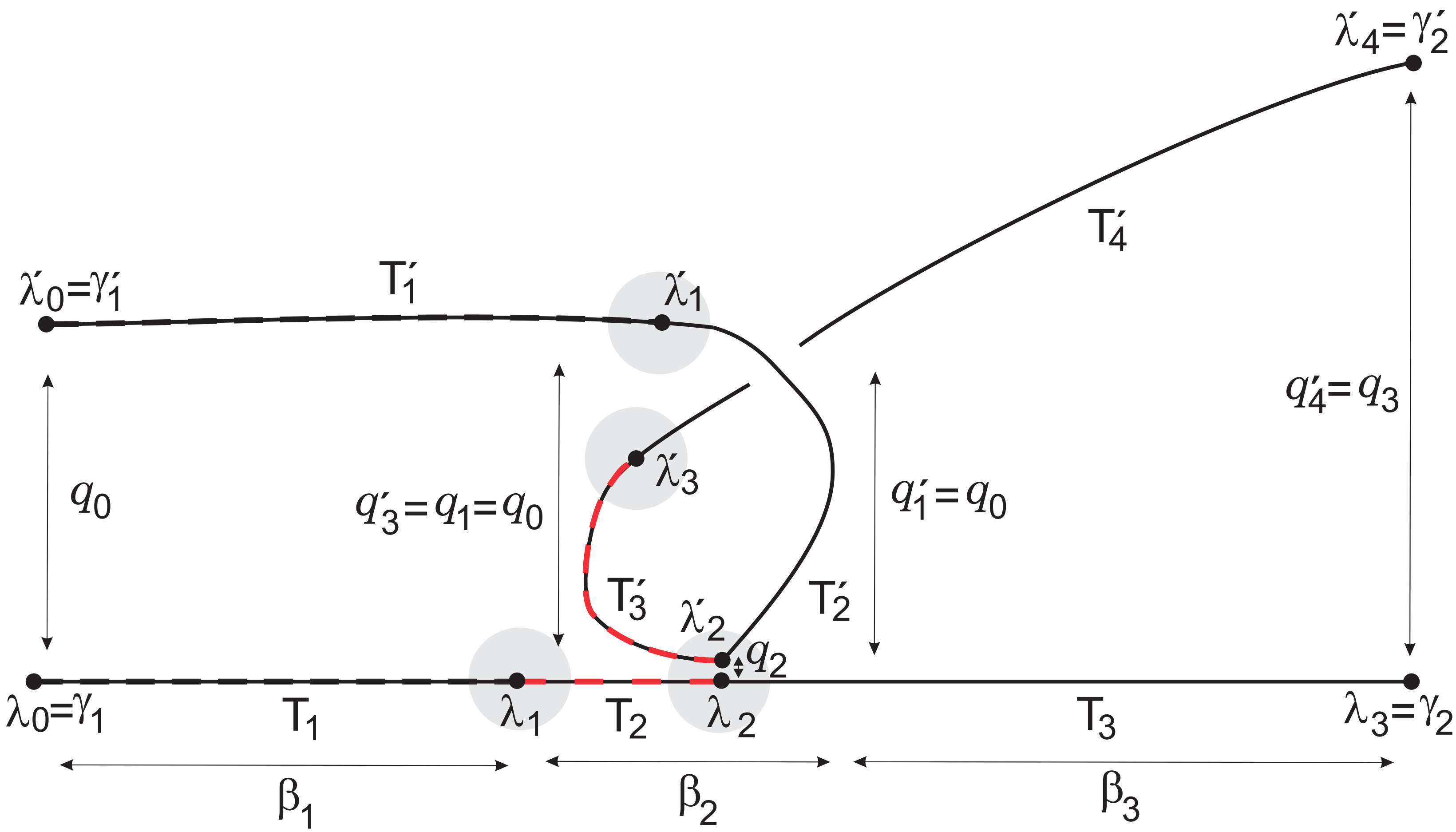
Alternatively, a pizza slice T_j with exponent β_j is **transversal** if either $Q_j = \{q_j\}$ where $q_j \leq \beta_j$, or $\mu_j(q) \equiv q$, where $\mu_j : Q_j \to \mathbb{F} \cup \{\infty\}$ is the width function on Q_j.

Theorem (Birbrair, AG). **Let (T,T') be a normal pair of Hölder triangles. Let $\{T_i\}_{i=1}^p$ and $\{T'_j\}_{j=1}^s$ be minimal pizza decompositions of T and T' for the distance functions $f(x)=dist(x,T')$ and $g(x')=dist(x',T)$.**

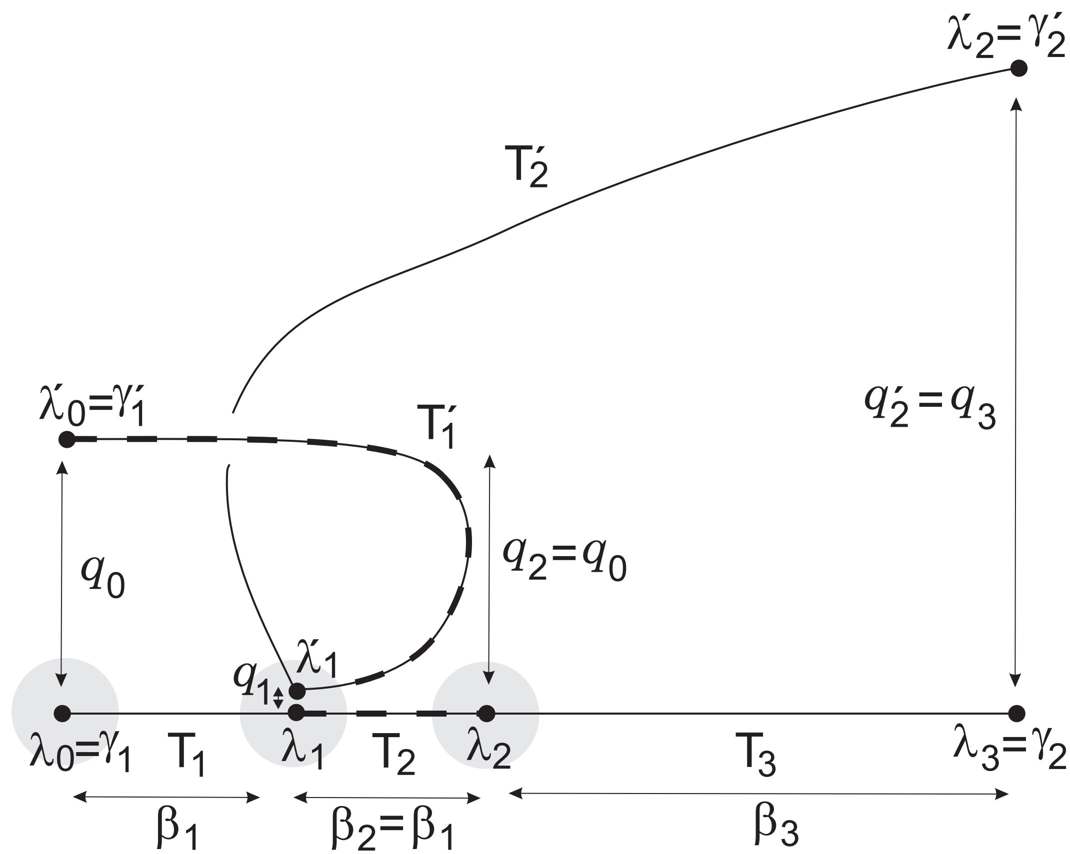
There is a canonical one-to-one correspondence $j=\tau(i)$ between coherent pizza slices T_i of T and coherent pizza slices T'_j of T', such that each pair (T_i,T'_j), where $j=\tau(i)$, is outer Lipschitz equivalent to (T_i,Γ_i), where Γ_i is the graph of $f|_{T_i}$, and to (T'_j,Γ'_j), where Γ'_j is the graph of $g|_{T'_j}$.



A normal pair of Hölder triangles with four pairs of coherent pizza slices, $\tau(1) = 1$, $\tau(2) = 3$, $\tau(3) = 2$, $\tau(4) = 4$.



A pair (T, T') of Hölder triangles with 3 pizza slices of T, 4 pizza slices of T', two pairs of coherent pizza slices, $\tau(1) = 1$, $\tau(2) = 3$.



A normal pair (T, T') of Hölder triangles with 3 pizza slices of T, 2 pizza slices of T', one pair of coherent pizza slices, $\tau(2) = 1$.

Theorem (Birbrair, AG). **If two normal pairs** (T, T') **and** (S, S') **of Hölder triangles are outer Lipschitz equivalent, then**

1. Minimal pizzas on T **and** T' **for the distance functions** $f(x) = dist(x, T')$ **and** $g(x') = dist(x', T)$ **are equivalent to minimal pizzas on** S **and** S' **for the distance functions** $\phi(y) = dist(y, S')$ **and** $\psi(y') = dist(y', S)$**, respectively.**

2. The numbers of maximum zones for the pairs (T, T') **and** (S, S') **are equal, and the permutations** σ **of these zones, are the same.**

3. The numbers of coherent pizza slices for the pairs (T, T') **and** (S, S') **are equal, and the correspondences** τ **between these pizza slices, are the same.**

Conversely, if the items 1, 2, 3 are satisfied, then the pairs (T, T') and (S, S') are outer Lipschitz equivalent.

Thus the two pizzas, together with the permutation σ and the correspondence τ, constitute a complete invariant of the outer Lipschitz equivalence class of normal pairs of Hölder triangles.

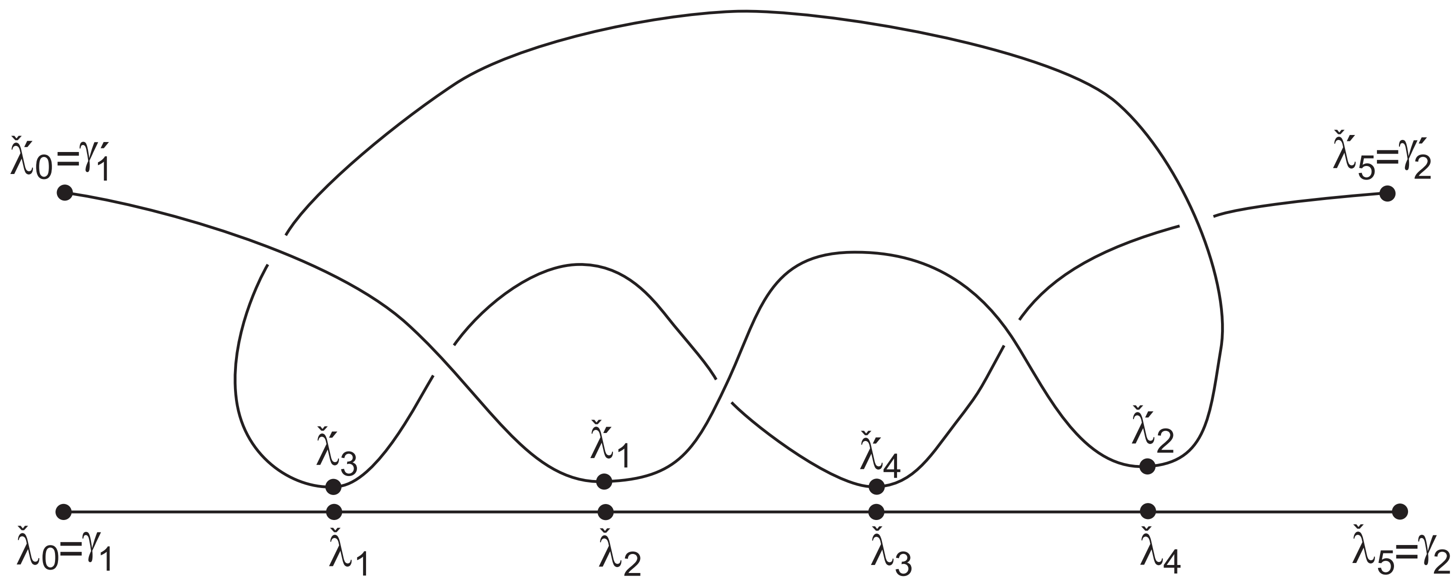
Moreover, given any one of the two pizzas, and given the permutation σ and correspondence τ satisfying some explicit admissibility conditions, the normal pair (T, T') exists and is unique up to outer Lipschitz equivalence.

Blocks

Let π be a permutation of the set $[n] = \{0, \ldots, n-1\}$ of n elements. A **segment** of $[n]$ is a non-empty set of consecutive indices $\{i, \ldots, k\}$. A segment B of $[n]$ is a **block** of π if $\pi(B)$ is also a segment of $[n]$ (not necessarily in increased order). Each non-empty subset J of $[n]$ is contained in a unique minimal block $B_\pi(J)$ of π.

Given a set $\lambda_0, \ldots, \lambda_{n-1}$ of n arcs in an oriented Hölder triangle T, ordered according to orientation of T, a permutation π of $[n]$ is **admissible** if $tord(\lambda_i, \lambda_j) \leq tord(\lambda_i, \lambda_k)$ for any indices $i \neq j$ in $[n]$ and all $k \in B_\pi(\{i, j\})$.

This relation between combinatorial and metric properties of a normal pair of Hölder triangles, applied to permutations related to the $\sigma\tau$-invariant, is an important part of the existence and uniqueness conditions for the normal pairs of Hölder triangles.



Example. A normal pair of Hölder triangles with the permutation $\pi = (0, 3, 1, 4, 2, 5)$.

$B_\pi(\{0,1\}) = \{0,\dots,4\}$, $B_\pi(\{4,5\}) = \{1,\dots,5\}$, $B_\pi(\{1,2\}) = B_\pi(\{2,3\}) = B_\pi(\{3,4\}) = \{1,\dots,4\}$. Accordingly,

$$tord(\check\lambda_0,\check\lambda_1)=tord(\check\lambda'_0,\check\lambda'_1)\le tord(\check\lambda_1,\check\lambda_2)=tord(\check\lambda_2,\check\lambda_3)=tord(\check\lambda_3,\check\lambda_4)=$$
$$=tord(\check\lambda'_1,\check\lambda'_2)=tord(\check\lambda'_2,\check\lambda'_3)=tord(\check\lambda'_3,\check\lambda'_4)\ge tord(\check\lambda_4,\check\lambda_5)=tord(\check\lambda'_4,\check\lambda'_5).$$

HAPPY BIRTHDAY, LEV!