Poincaré-Hopf for analytic singular varieties.

The M.-H.-Schwartz ideas in the Lipschitz framework.

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Będlewo, July 21, 2023

Celebrating Lev Birbrair on the occasion of his 60th birthday.

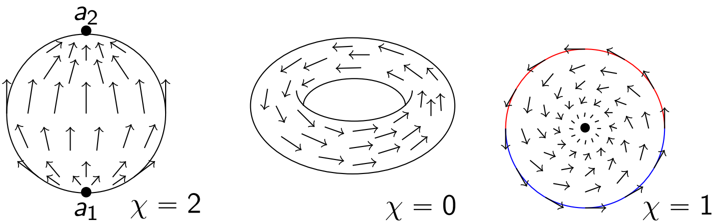
Theorem (Poincaré (1881), Hopf (1926))

Let M be a compact differentiable manifold. Let v a tangent vector field on M. with isolated zeroes a_i. If M has no boundary, the Euler-Poincaré characteristic of M is equal to :

$$\chi(M) = \sum_{a_i} I(v, a_i).$$

If M has boundary ∂M, one has the same formula if v is pointing in the outward normal direction along the boundary. If v is pointing in the inward normal direction along the boundary. then one has

$$\chi(M) - \chi(\partial M) = \sum_{a_i} I(v, a_i).$$



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We will consider real analytic varieties $X \subset \mathbb{R}^N$.
A stratification of a variety X, is a family of closed analytic subsets of X

$$\mathcal{S} \qquad \emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-2} \subset X^{n-1} \subset X = X^n$$

where each $\mathring{X}^\alpha = X^\alpha - X^{\alpha-1}$ is either empty or a smooth manifold of pure dimension α. The connected components S^α of $\mathring{X}^\alpha$ are the *strata*.

- Whitney-stratification, 1965 by Hassler Whitney,

Existence of stratifications satisfying conditions (a) and (b) for analytic varieties.

- Kuo-Verdier stratification, by Kuo 1971 and Verdier 1976.

Existence of stratifications satisfying condition (w) for analytic varieties.

- Lipschitz-stratification, 1985 by Tadeusz Mostowski,

Existence of stratifications satisfying Lipschitz L-conditions for complex analytic varieties – 1988 Existence for real analytic varieties ; Adam Parusiński.

Lipschitz $\Rightarrow$ (w) $\Rightarrow$ Whitney.

• X is a stratified variety with strata $\{\mathring{X}^\alpha\}$,
• v is a stratified vector field ($\mathcal{S}$-compatible in the terminology of Mostowski-Parusiński) i.e. for $x \in \mathring{X}^\alpha$, then $v(x) \in T_x(\mathring{X}^\alpha)$,
• v has isolated singular points a_i.

At each singular point a_i, the stratum $\mathring{X}^{\alpha(a_i)}$ containing the point a_i is smooth. The index $I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i)$ makes sense.

A first idea for a generalization of the Poincaré-Hopf Theorem, is to write

$$\chi(X) \overset{?}{=} \sum_{a_i} I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i),$$

where the points a_i are the isolated singular points of v.

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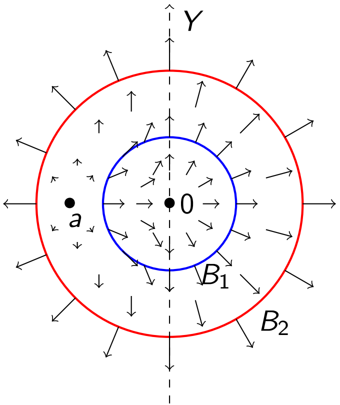
The formula is not true. We provide a Marie-Hélène counter-example.

M.-H. Schwartz's counterexample :



$B_1 = B(0;1)$

$X = B_2 = B(0;2)$

$Y = \{x = 0\}$ vertical axis

Vector fields :

- Inside B_1, $v_1(x,y) = (|x|, y)$.

One has $v_1(0) = 0$, index $I(v_1, 0) = 0$.

- On ∂B_2, $v_2(x,y) = (x,y)$.
- Inside B_2, continuous vector field v

$v = v_2$ along ∂B_2, $v = v_1$ inside B_1 and v tangent to the y-axis Y along Y.
For instance, on $B_2 \backslash B_1$, $v(x,y) = \left(2|x| - x + (x - |x|)\sqrt{x^2+y^2}, y\right)$.
The vector field v has another isolated singular point at $a = (-3/2, 0) \in B_2 \backslash B_1$.
By Poincaré-Hopf Theorem with boundary, we have

$$\chi(B_2) = +1 = I(v,0) + I(v,a),$$

that implies $I(v,a) = +1$.

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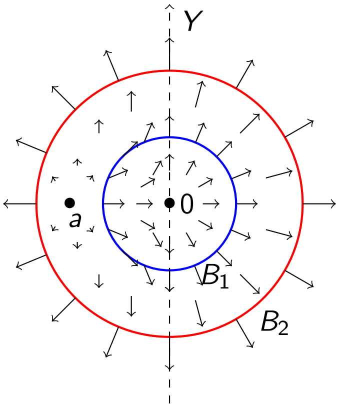


$$\chi(B_2) = +1$$

$$I(v,0) = 0,\ I(v,a) = +1$$

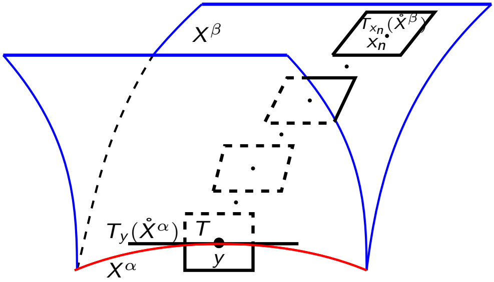
$$I(v|_Y, 0) = +1$$

Consider X stratified by Y and $X \backslash Y$.
v is a stratified vector field
$\chi(X) = +1 \neq I(v,a) + I(v|_Y, 0) = 2.$

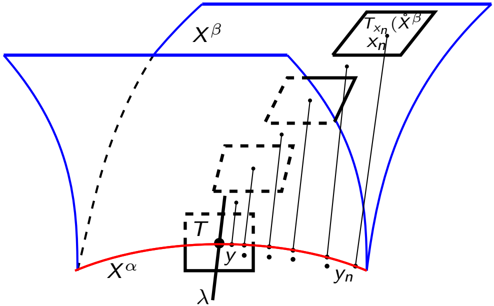
$$\chi(X) \neq \sum_{a_i} I(v|_{\mathring{X}^{\alpha(a_i)}}, a_i),$$

a_i isolated singular points.

Whitney stratifications.

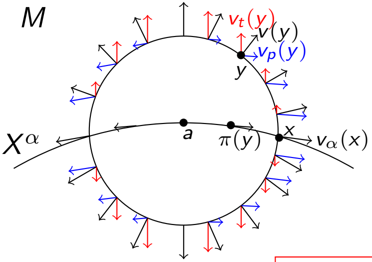


Whitney condition (a)



Whitney condition (b)

Local radial extension of a vector field.



v_α vector field along $\mathring{X}^\alpha$

v_p its local parallel extension

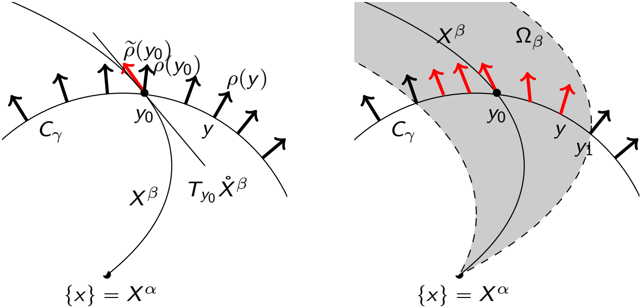
v_t the local transverse vector field

$$v = v_p + v_t$$

Main Property 1. $$I(v_\alpha, a; \mathring{X}^\alpha) = I(v, a; M)$$

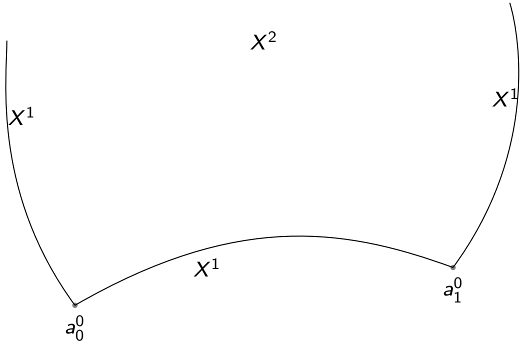
Difficulty : the radial extension is not necessarily a stratified vector field.
Solution (MHS) : use the Whitney condition (a) for the parallel vector field v_p, and Whitney condition (b) for the transverse vector field v_t.

M.-H. Schwartz's radial extension - local construction

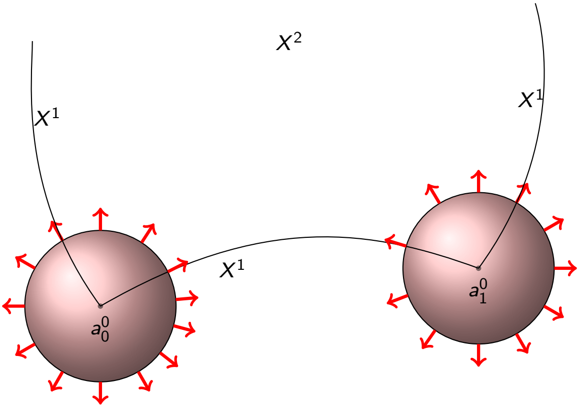


Difficulty : Even being stratified, the radial extension is not continuous.
Solution (MHS) : Use "tapered" neighbourhoods in which homotopy is performed. In the picture, homotopy is performed for the radial vector field v_p denoted ρ.

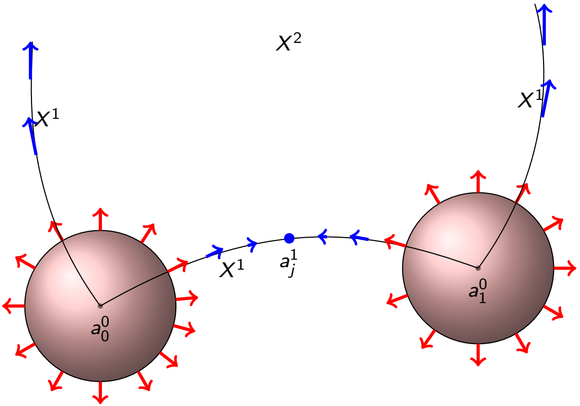
M.-H. Schwartz's radial extension - global construction - 1



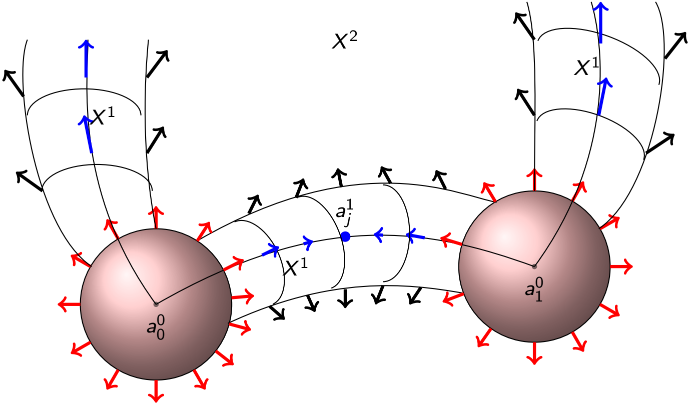
M.-H. Schwartz's radial extension - global construction - 2



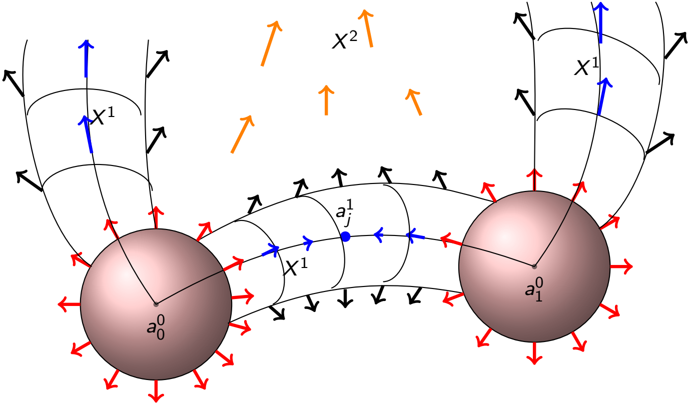
M.-H. Schwartz's radial extension - global construction - 3



M.-H. Schwartz's radial extension - global construction -4

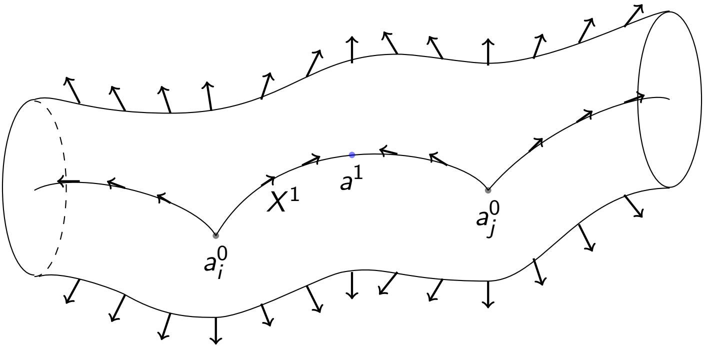


M.-H. Schwartz's radial extension - global construction -5



Observation : In order to obtain a continuous and stratified vector field, the process requires, at each step, the delicate and technical construction of systems of nested tubular neighbourhoods and tapered neighbourhoods in which the field is built by suitable homotopies.

M.-H. Schwartz's radial extension - global construction



Important property 2 : **The radial vector field is pointing outwards the tubular neighbourhoods of strata.**

Proof of the Poincaré-Hopf Theorem

The proof is made by induction on the dimension of the strata.
+ For the 0-dimensional strata : $\chi(X^0) = \sum_{i=1}^{k} I(v, a_i^0; M)$ where $I(v, a_i^0; M) = 1$.
+ The obtained vector field v is pointing inwards the 1-dimensional strata along a neighbourhood of their boundary ∂X^1. By the classical Poincaré-Hopf Theorem for manifold with boundary and for a vector field pointing inwards the strata along the boundary :

$$\chi(X^1) = \sum_j I(v, a_j^1; M) + \chi(\partial X^1) = \sum_j I(v, a_j^1; M) + \sum_{i=1}^{k} I(v, a_i^0; M).$$

One continues till reaching the dimension of X and obtain :

$$\chi(X) = \sum_i I(v, a_i; M)$$

where a_i denotes all singularities of v in X and $I(v, a_i; M)$ denotes also the index of v at its singular point a_i computed in the stratum of the point a_i.

The M.-H. Schwartz method

Advantages

+ Clear use of Whitney conditions.
+ Clear use of radiality properties 1 and 2.

Note : Some authors tried to avoid radiality, but they use a different notion of index, which is to compensate the lack of radiality.

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+ The method is quite technical, it requires a lot of delicate constructions (of tubular neighbourhoods) which imply a lot of necessary attention.
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Idea

+ Keep the M.-H. Schwartz method (Main properties) **BUT**
+ Use the Lipschitz framework : Lipschitz stratifications and Lipschitz vector fields.

Lipschitz stratifications.

T. Mostowski (complex analytic varieties)
A. Parusiński (real analytic varieties)

- c is a constant $c > 1$,
- $q = q_{j_1}$ is a point in a j_1-dimensional stratum $\mathring{X}^{j_1}$

A chain is a sequence of points $q_{j_s} \in \mathring{X}^{j_s}$, such that

$$X^{j_1} \supset X^{j_2} \supset \cdots \supset X^{j_s} \supset \cdots X^{j_r} = X^{\ell}$$

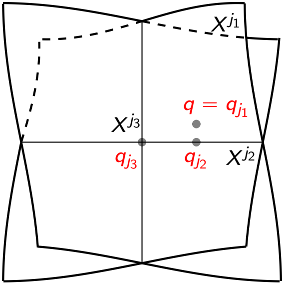
(with dimensions $j_1 > j_2 > \cdots > j_s > \cdots > j_r = \ell$)
and j_s is the bigger integer such that

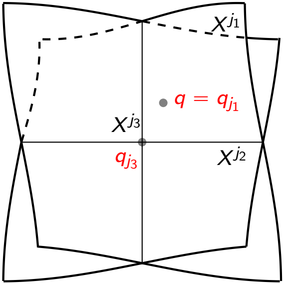
$d(q, X^{j_k}) \geqslant 2c^2\, d(q, X^{j_s})$ for all k for which $j_s > j_k \geqslant \ell$

and

$|q - q_{j_s}| \leqslant c\, d(q, X^{j_s})$.

The notion of chain. (Mostowski).





Here, $j_1 = 2, j_2 = j_s = 1, j_3 = 0$,

$$d(q, X^{j_3}) \geqslant 2c^2\, d(q, X^{j_2})$$

$$|q - q_{j_2}| \leqslant c\; d(q, X^{j_2}).$$

Chain : $q_{j_1},\ q_{j_2},\ q_{j_3}$

Here, $j_1 = 2, j_3 = j_s = 0$,

$$|q - q_{j_3}| \leqslant c\; d(q, X^{j_3}).$$

Chain : $q_{j_1},\ q_{j_3}$

Lipschitz stratifications.

For $q \in \mathring{X}^j$,
$P_q : \mathbb{R}^n \to T_q\mathring{X}^j$ be the orthogonal projection onto the tangent space
$P_q^\perp = Id - P_q$ the orthogonal projection onto the normal space $T_q^\perp \mathring{X}^j$.

The stratification is *L*-stratification if, for some constant $C > 0$ and every chain $q = q_{j_1}, q_{j_2}, \dots, q_{j_r}$ and every k, $1 \leqslant k \leqslant r$,

$$|P^\perp_{q_{j_1}} P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q_{j_1}, -q_{j_2}|/d(q_{j_1}, X^{j_k-1})$$

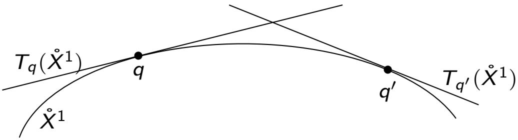
If, further, $q' \in \mathring{X}^{j_1}$ and $|q' - q| \leqslant (1/2c)\ d(q, X^{j_1-1})$, then

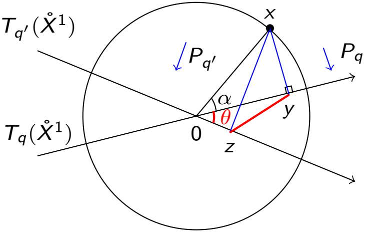
$$|(P_{q'} - P_q)P_{q_{j_2}} \cdots P_{q_{j_k}}| \leqslant C|q' - q|\ /d(q, X^{j_k-1}),$$

in particular (for q and q' in $\mathring{X}^{j_1}$),

$$|P_{q'} - P_q| \leqslant C|q' - q|\ /d(q, X^{j_1-1}).$$

$\|P_{q'} - P_q\|$ geometrically is roughly the angle between $T_{q'}\mathring{X}^j$ and $T_q\mathring{X}^j$.



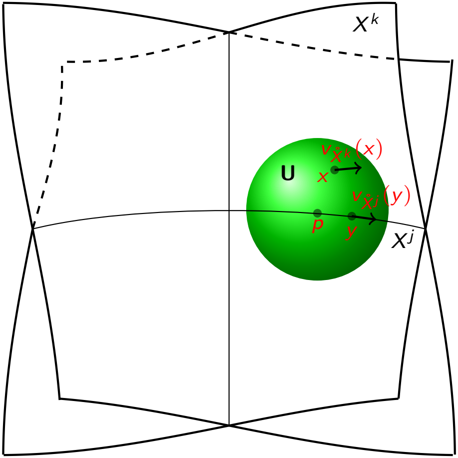


$$\|P_q - P_{q'}\| = \sup_{\|x\|=1} |(P_q - P_{q'})(x)| = \sin(\theta).$$

Let X stratified subset of the smooth manifold M.

A stratified vector field $v = \{v_{\mathring{X}^j} : \mathring{X}^j \text{ are strata}\}$ is said rugose if for each point $p \in \mathring{X}^j$, there is a constant $C > 0$ and a neighbourhood U of p in M such that for each point $y \in \mathring{X}^j \cap U$, and each point $x \in X \cap U$, if $\mathring{X}^k$ denotes the stratum of X containing x, then

$$\|v_{\mathring{X}^k}(x) - v_{\mathring{X}^j}(y)\| < C|x - y|.$$



For v to be Lipschitz
one need to allow y not only
to belong to $\mathring{X}^j$
but also to any stratum
incident to $\mathring{X}^j$.

The parallel Lipschitz vector field

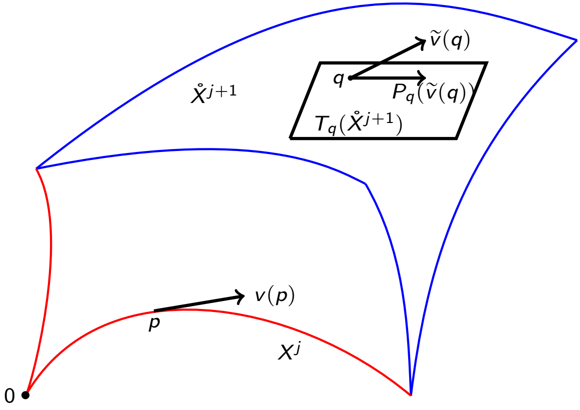
Proposition (Mostowski (1985) Prop. 2.1, Parusiński (1988) Sec 1.)

Let $\{\mathring{X}^j\}_{j=1}^m$ be a Lipschitz-stratification of X and let v be a Lipschitz stratified vector field on $\mathring{X}^j$ bounded on $\mathring{X}^\ell (\ell \leqslant j \leqslant m)$. Then v can be extended to a Lipschitz stratified vector field on $\mathring{X}^{j+1}$.

In a first step, one extends the Lipschitz vector field v, defined on $\mathring{X}^j$ in a Lipschitz vector field $\tilde{v}$ defined on M.
One defines a vector field w on X^{j+1} by

$$w(q) = \begin{cases} v(q) & \text{if } q \in X^j \\ P_q \tilde{v}(q) & \text{if } q \in \mathring{X}^{j+1} \end{cases}$$

where $P_q : \mathbb{R}^n \to T_q(\mathring{X}^{j+1})$. The obtained vector field is Lipschitz.



The parallel extension, à la Lipschitz : Let v a Lipschitz vector field tangent to a stratum $\mathring{X}^j$, then there is a neighbourhood of $\mathring{X}^j$ such that the Lipschitz extension of v provides a stratified vector field parallel to v.

The transverse Lipschitz vector field

Lemma

For each stratum $\mathring{X}^{j_0}$, there is a neighbourhood of $\mathring{X}^{j_0}$ and a transverse vector field which is a stratified Lipschitz vector field.

Idea of proof : We work in a neighbourhood of some point in $\mathring{X}^{j_0}$ and assume that $X^i = \emptyset$ for $i < j_0$ and that $\mathring{X}^{j_0}$ is a linear subspace : $x_\mu = 0, (\mu > j_0)$. Put $r_0 = \sum_{\mu > j_0} x_\mu \frac{\partial}{\partial x_\mu}$.

For every $\varepsilon > 0$, there is a tubular neighbourhood $\mathcal{N}$ of $\mathring{X}^{j_0}$ in $\mathbb{R}^n$ and a vector field r in it, s.t. :

1. r is Lipschitz, tangent to the strata,
2. $\measuredangle(r, r_0) < \varepsilon$.

By increasing dimension on j, we construct a Lipschitz vector field r_j on $\mathcal{N}_j \cap X^j$ (for some tubular neighbourhood $\mathcal{N}_j$ of X^j) with the same properties.

Conclusion

The sum of the parallel and transverse Lipschitz vector fields provides a Lipschitz vector field v such that the main properties are fulfilled :
If a is a singularity of v in a stratum $\mathring{X}^\alpha$, then

$$\boxed{I(v_\alpha, a; \mathring{X}^\alpha) = I(v, a; M)}$$

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One concludes by the same argument than M.-H. Schwartz.

Thanks a lot for your attention

Dziękuję za uwagę

Merci pour votre attention

Happy birthday Lev !

Joyeux anniversaire, Lev

Wszystkiego najlepszego, Lev

Feliz aniversário, Lev