Contributions of Lev Birbrair to the Lipschitz Geometry of Singularities

Alexandre Fernandes UFC

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Definition:

(X, p) and (Y, q) are called **bi-Lipschitz homeomorphic** *if there exist neighborhoods U of p in X and V of q in Y such that U and V are bi-Lipschitz homeomorphic.*

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Lipschitz Geometry of Singularities: Study of germs (X, p) up to bi-Lipschitz homeomorphisms ($X \subset \mathbb{R}^n$ semialgebraic/subanalytic or $X \subset \mathbb{C}^n$ algebraic or analytic).

Real surface singularities

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Theorem: (Birbrair 1999)

If (X, p) is a real 2D isolated singularity with connected link, then there exists a unique rational number $\beta \geq 1$ such that (X, p) is inner bi-Lipschitz homeomorphic to the germ of the β-horn

$$\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^{2\beta}; \ z \geq 0\}$$

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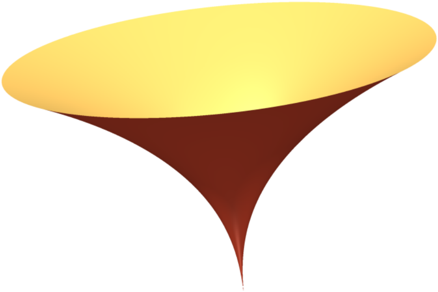

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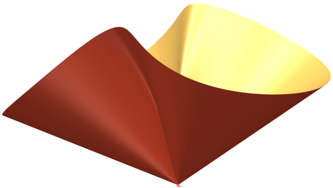

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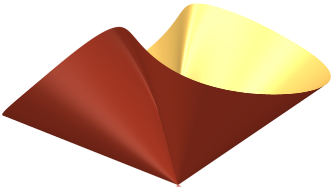

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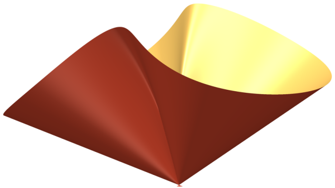

• (Birbrair, Gabrielov, Grandjean, F. 2017) Lipschitz contact equivalence of function germs in $\mathbb{R}^2$.

• (Birbrair, Gabrielov 2023) Lipschitz geometry of pairs of normally embedded Hölder triangles.

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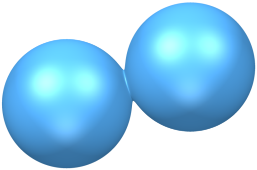

Figure: Link of $(X, \mathbf{0})$.

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Example

Let $\beta > 2$. *Let* $X \subset \mathbb{R}^4$ *be defined by*

$$[(z-t)^2 + x^2 + y^2 - t^2] \cdot [(z+t)^2 + x^2 + y^2 - t^2] = t^{2\beta} \ ; \ t \geq 0.$$

The point $\mathbf{0} = (0,0,0,0)$ *is an isolated singular point of* X *which the link is homeomorphic to the sphere* $\mathbb{S}^3$.

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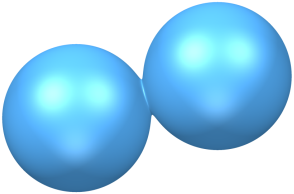

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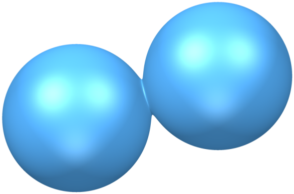

If $3 < \nu < \beta + 1$, then $MH_2^{\nu}(X, \mathbf{0})$ (with coefficient in $\mathbb{R}$) contains a vector space isomorphic to $\mathbb{R}^2$.

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Theorem: (Birbrair, Brasselet 2002)

Let $X \subset \mathbb{R}^n$ *be a semialgebraic closed subset with isolated singularity at* $p \in X$*.*

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The singularity in the previous example is not semialgebraically inner bi-Lipschitz conical.

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If (X, p) is semialgebraically inner bi-Lipschitz conical, then $\lambda_k = k+1$.

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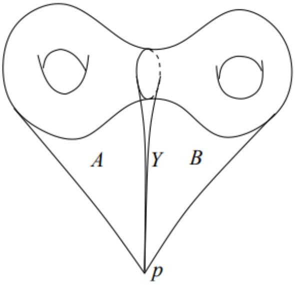

Figure: $A : |x| \geq |y|$ and $B : |x| \leq |y|$ and $Y : |x| = |y|$

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- $length(\gamma_t) \ll t^a$ as $t \to 0^+$, for any $0 \leq a < \alpha$.

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α-Fast loops. Let (X, p) be a singularity. A **loop** of (X, p) is a (Lipschitz) map $\gamma\colon \mathbb{S}^1 \to X \setminus p$; X is a small representative of (X, p). It is said a **trivial loop** of (X, p) if it is null-homotopic in $X \setminus p$.

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A loop γ of (X, p) is said **α-fast** if there is a homotopy $\gamma_t\colon \mathbb{S}^1 \to X$ $(0 \leq t \leq 1)$, that deforms $\gamma \to p$, such that:

- $\{\gamma_t\}_{t>0}$ is a family of loops of (X, p).
- $dist(\gamma_t, p) \approx t$ as $t \to 0^+$.
- $length(\gamma_t) \ll t^a$ as $t \to 0^+$, for any $0 \leq a < \alpha$.

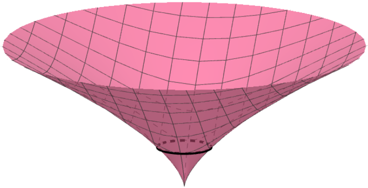

Figure: An α-fast loop; $\alpha > 1$.

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Let X *and* Y *be the weighted homogeneous algebraic surfaces in* $\mathbb{C}^3$ *given below:*

$$X\colon x^2 + y^{51} + z^{102} = 0 \quad \textit{and} \quad Y\colon x^{12} + y^{15} + z^{20} = 0.$$

Then, $(X, \mathbf{0})$ *and* $(Y, \mathbf{0})$ *are homeomorphic but they are not inner bi-Lipschitz homeomorphic.*

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The links of $(X, \mathbf{0})$ and $(Y, \mathbf{0})$ are orientable 3-dimensional manifolds which are circle bundle over a orientable surface of genus $g = 25$ and both have Euler Class equal to 1. On the other hand, $\frac{4}{3} \leq \lambda(Y, \mathbf{0}) \leq \frac{5}{3} < 2 \leq \lambda(X, \mathbf{0}) \leq 51$.

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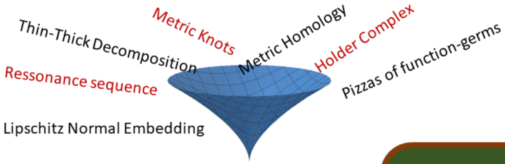
Theorem: (Birbrair, Neumann, Pichon 2014)

Two normal complex algebraic surface singularities are inner bi-Lipschitz homeomorphic if, and only if, they have combinatorial isomorphic thin-thick decomposition.

Contributions of Birbrair



Prof. Lev Birbrair

Many thanks for your attention