Singularity Theory on Spider Spaces

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Geometry and Singularities
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Outline

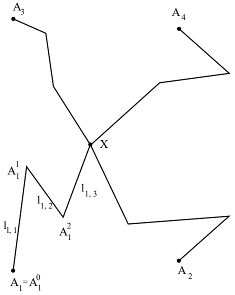
Joint work with M. Denkovski, G. Khimshiashvili, G. Panina

Spiders

An n-legged *spider* in $\mathbb{R}^2$ consists of:

- pairwise distinct points $A_1, \ldots, A_n \in \mathbb{R}^2$
- a center point $X \in \mathbb{R}^2$ (the *body* of the spider)
- legs $\mathcal{L}_i$, that are connected chains of finitely many *edges* (rigid bars) from foot to body with given edge-length $\ell_{i,j+1} > 0$.

The legs are allowed to rotate freely around their joints.



The collection of all spiders form the *spider space* $\mathcal{S}$.
The set of all points in $\mathbb{R}^2$ attainable by X is the *workspace* $\mathcal{W}$.

Hooke Energy

The Hooke Energy $H : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$H(x) = \sum_{i=1}^{n} ||x - a_i||^2$$

Unique stationary points of H: $\nabla H = nx - 2\sum_{i=1}^{n} ||x - a_i| = 0$, so $x = \frac{1}{n}\sum_{i=1}^{n} a_i$, the centre of gravity

Proposition

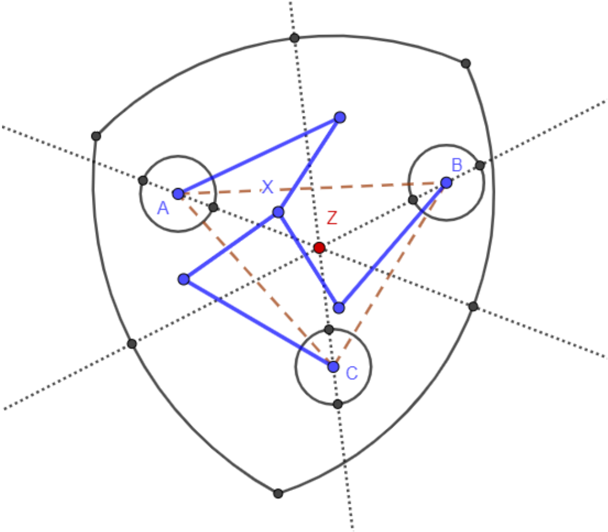
The Hooke energy H is an affine linear transform of the squared distance function to the centre of gravity Z.

$$H(x) = n||x - z||^2 + \kappa.$$

The Hooke function is minimal in the centre of gravity. Both functions have the same critical point theory.

Note $H(X) = ||x - z||^2 + \kappa$, where z is the centre of gravity Z and $\kappa = \sum_{i=1}^{n} ||a_i||^2 - ||z||^2$.

All level curves are circles.



X
Z
A
B
C

The Goal

To study the critical point theory of Hooke Energy and more general the Squared Distance Function on the configuration space of spiders.

This is joint work in progress with
Marcej Denkovski, George Khimshishvil and Gaiane Panina

The geometry and topology of spider spaces was considered before by: N. Shvalb-M.Shoham-B. Blanc, P. Mounoud, J.-L. O'Hara.

Smoothness of Spider Space

Proposition ([SSB])

Generically $\mathcal{S}$ is an orientable, compact, algebraic submanifold of $\mathbb{R}^{2N+2}$ of dimension

$$2(1+\sum_{i=1}^{n} p_i - n) - \sum_{i=1}^{n} p_i.$$

FACT: For planar spiders the singular points of spider space $\mathcal{S}$ correspond to:

- Two aligned legs with the same direction,
- Three aligned legs

Robot Arms

Robot arm $=$ Spider with only one leg

With start point A and end point X, each edge is oriented from A to X. The configuration the torus $(S^1)^p$, where p is the number of edges. Modulo the oriented rotation group around A we have the reduced configuration space, the torus $(S^1)^{p-1}$, where p is the number of edges. For an aligned configuration ξ, denote by $Pos(\xi)$ the number of positively directed edges.

Proposition

([KM]) The function $||a - x||^2$ (defined on the reduced configuration space) is a Morse function on the set $||a - x|| > 0$, whose critical points are aligned configurations. The Morse index of an aligned configuration ξ equals $Pos(\xi) - 1$.

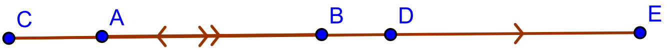
NB. Notice that the set $\{x \in \mathbf{R}^2 | \; ||a - b|| = 0\}$ is also critical.

For the proof ; Let $h : A(L) \to \mathbb{R}$ defined by the squared length of the connecting vector:

$$h = (p_1 + \cdots + p_n)^2.$$

Statement: The critical points of h are

- the zero set of h
- all direction vectors p_i are dependent .



Conclusion: For any non-zero value of h: Fix the connecting edge with the corresponding length and we get the moduli space of a closed linkage. This space is smooth exactly when the edges are not collinear.

Linkages

A *planar linkage* L is a closed polygon whose consecutive edge lengths are $l_1, ..., l_p$. It may have self-intersections and/or self-overlaps.

The *configuration space*, or the *polygon space* for short, is the set of all configurations of L modulo orientation preserving isometries of $\mathbb{R}^2$. It does not depend on the ordering of l_i; however, it does depend on the values of the l_i.

Proposition

The polygon space is a closed $(p-3)$*-dimensional manifold iff no configuration of L fits in a straight line [KM].*

The work map

Next we look for fibers of the work map $\pi : \mathcal{S} \to \mathcal{W}$. We fix the point X in workspace, different from the feet points A_i. We are left with the freedom of n legs with fixed starting point A_i and end point X. Each of them constitute a linkage if we add the edge XA_i to $\mathcal{L}_i$ with length $l_{i,0} = ||x - a_i||$: *the closure of the leg*. Each of these linkages has a polygon space. All these polygon spaces are independent. It follows:

Proposition

The fiber of the work map above $X \neq A_i$ is the product of the polygon spaces of the linkages, which are the closures of the legs with endpoint X.

REMARK: If $X = A_i$ the leg itself is closed en determines a linkage with one vertex less. The fibre of the work map above A_i becomes the product of the the moduli space of the corresponding linkage with a circle S^1 and the moduli spaces of the other leg closures. See also the remark below.

Disc - Annulus decomposition

Fix a foot A_i and assume for the moment the leg $\mathcal{L}_i$ it is moving freely in $\mathbb{R}^2$. The critical points of $||a_i - x||^2$. Its critical points corresponds to an *aligned leg*. The critical points project via π to circles, with centre A_i and *critical* radius $r_i^{\epsilon_i}$:

$$C(A_i, r_i^{\epsilon_i}) := \{z \in \mathbb{R}^2 \mid ||x - a_i|| = r_i^{\epsilon_i}\}.$$

We denote by r_i, resp R_i the minimal and the maximal values of $||x - a_i||$ .They define the zone:

$$\mathcal{Z}(A_i) := \{x \in \mathbb{R}^d \mid 0 \leq r_i \leq ||x - a_i|| \leq R_i\},$$

which can be a disc or an annulus with centre A_i.

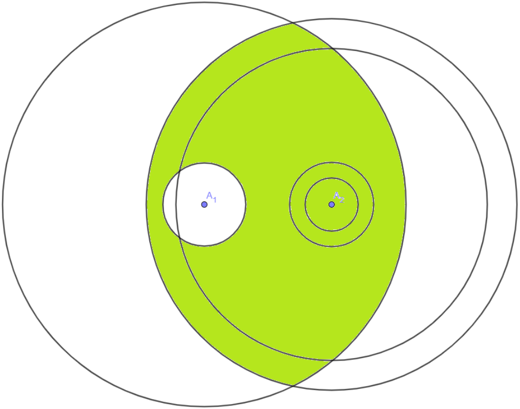


Figure: The critical circles and the workspace (green)

The following two facts have obvious proofs.

Proposition

In the situation considered, $\mathcal{W} = \bigcap_{i=1}^{n} \mathcal{Z}(A_i)$.

Proposition

Assume $\mathcal{S}$ *is smooth. The critical locus of the projection* $\pi : \mathcal{S} \to \mathcal{W}$ *consists of the fibers with at least 1 alligned leg. The discriminant set coincides with those points of the workspace* $\mathcal{W}$*, that lie on one of the circles* $C(A_i, r_i^{\epsilon_i})$.

Remark:
Generically 0 is not a critical radius for any of the legs. In that case the fibration above A_i is locally trivial and the fibre above A_i is diffeomorphic to the nearby fibres.

Critical points of the quadratic distance $||x - z||^2$

For any spider $\mathcal{S}$ and the point Z for any leg $\mathcal{L}_i$ its extension $z\mathcal{L}_i$ as the robot arm by adding the edge ZA_i before the leg $\mathcal{L}_i$. This new robot arm ZX has $p_i + 1$ edges.

Proposition

Given one-leg spider $\xi = \mathcal{L}$ and a point $Z \in \mathbb{R}^2$.
For smooth $\mathcal{S}$ the critical points of $||x - z||^2$ are:

1. *$X \neq Z$; Isolated critical points:*
   A one-leg spider ξ is critical for $||x - z||^2$ iff it is aligned, and the point Z lies on this line. In this case the critical point is Morse and the Morse index of a critical spider is
   $$M(\xi, Z) = Pos(z\mathcal{L}) - 1$$
2. *$X = Z$. The (possibly empty) submanifold of $\mathcal{S}$ defined by the condition $X = Z$. Here we have the the global minimum. It is Morse-Bott with transversal index 0.*
   The critical submanifold is the configuration space of the polygonal linkage obtained by adding the bar ZA to $\mathcal{L}$.

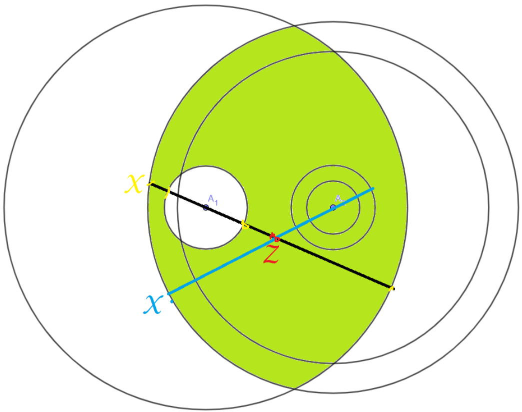


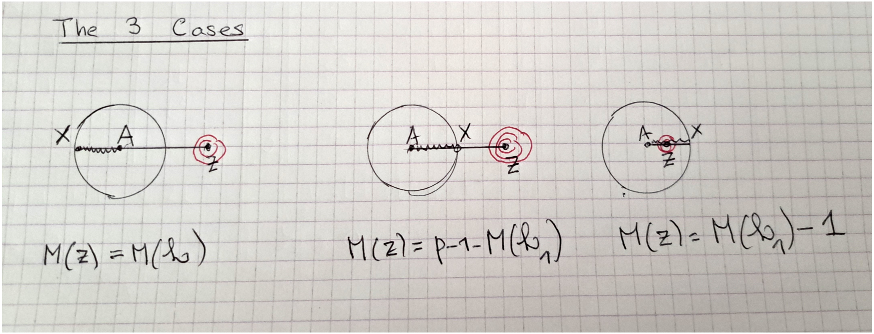
Figure: Some critical positions of type 1.

In terms of the leg $\mathcal{L}$

$$M(\xi, Z) = Pos(z\mathcal{L}) - 1$$

can be written as

$$M(\xi, Z) = \begin{cases} p - Pos(\mathcal{L}), & \text{if } X \text{ lies between } A \text{ and } Z; \\ Pos(\mathcal{L}) - 1, & \text{if } Z \text{ lies between } A \text{ and } X; \\ Pos(\mathcal{L}), & \text{if } A \text{ lies between } Z \text{ and } X. \end{cases}$$



Critical points

For a spider (with any number n of legs) and a point $Z \in \mathbb{R}^2$ consider the function $||x - z||^2$ defined on $\mathcal{S}$.

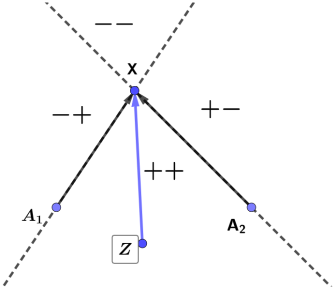
Theorem

For a smooth spider space, a configuration $\xi \in \mathcal{S}$ is critical for $||x - z||^2$ iff it satisfies at least one of the following conditions:

1. $X = Z$;
2. *One leg is aligned, and Z lies on this line;*
3. *Two different legs are aligned.*

Let us have a deeper look at case 3. We will assume that the two aligned legs are transversal at X. Assume they are numbered 1 and 2. The three points A_1, A_2, X define a partition in the plane into four quadrants seperated by the lines $\mathbb{R}[A_1 X]$ and $\mathbb{R}[A_2, X]$. Let Z be in one of the quadrants and define the following signs, depending on Z for the 2 aligned legs at X:

$$\sigma(Z) = (\sigma_1(Z), \sigma_2(Z)) = (sign(A_1X \cdot ZX), sign(A_2X \cdot ZX))$$



Morse indices

Theorem

For a strongly generic spider and a point Z, then $||x - z||^2$ is a Morse-Bott function. The Morse indices are:

1. *The point X coincides with Z:* The Morse index $M = 0$.
2. *One leg is algined and Z lies on this line:* The Morse index $M = Morse(\xi_1, Z)$, where ξ_1 is a one-leg spider obtained by eliminating of all the legs except for the aligned one.
3. *Two legs are aligned;* Then the Morse index $M = M_1 + M_2$, where M_i is the Morse index of the robot arm $\mathcal{L}_i$ or its dual:

$$M_i = \begin{cases} M(\mathcal{L}_i), & \text{if } \sigma_i(Z) > 0 \\ p_i - M(\mathcal{L}_i), & \text{if } \sigma_i(Z) < 0. \end{cases}$$

The critical manifolds are in these cases diffeomorphic to:

1. The point X coincides with Z: the Cartesian product of n polygon spaces (by adding the bar $A_i X$ to the leg $\mathcal{L}_i$).
2. One leg is algined and Z lies on this line: the Cartesian product of $n-1$ polygon spaces (arising from non-aligned legs by closing them with $A_i X$).
3. Two legs are aligned: the product of $n-2$ polygon spaces, also arising from closing non-aligned legs with $A_i X$.

Spiders with only 2 bars for each leg

The workmap has finite fibers and the workspace as intersection of annuli with different types of boundary points:

- *arcs* : circle arcs (possibly of length 0),
- *vertices* : intersection of at least 2 arcs.

Considered by Mounoud. Let K be a connected component of $\mathcal{S}$ and $\pi(K)$ its image in workspace. The leg starting from A_i is called *active* for K if $\mathcal{L}_i$ meets the boundary of $\pi(K)$.

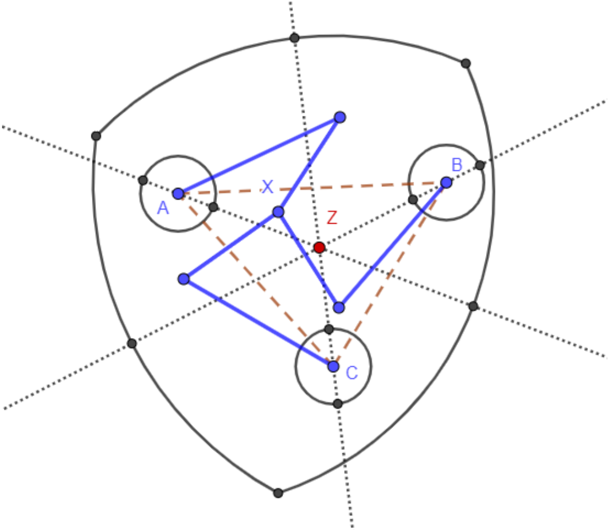
Proposition (Mounoud)

Generically $\mathcal{S}$ is smooth. Let K be a smooth connected component of $\mathcal{S}$. Let p be the number of vertices of $\delta\pi(K)$, k its number of connected components and n the number of active legs for K. Then K is diffeomorphic to a compact connected orientable surface with genus $g = 1 + 2^{n-3}(p + 4k - 8)$.

Corrolary : The Euler characteristic of K is :

$$\chi(K) = -2^{n-2}(p + 4k - 8)$$

.



Mounoud formula: $\chi = -22$

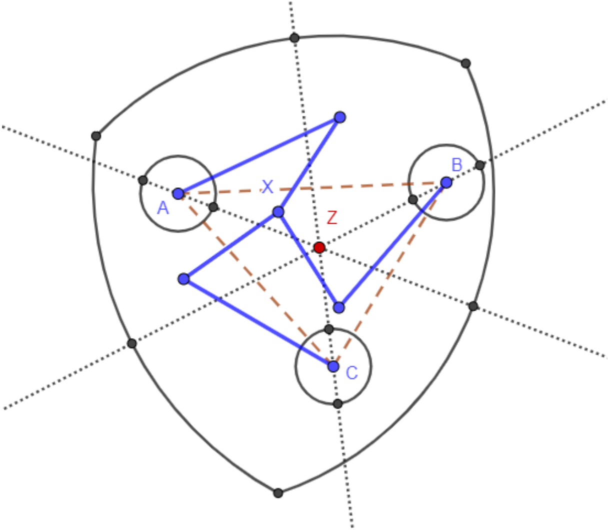
Corrolary : The Euler characteristic of K is :

$$\chi(K) = -2^{n-2}(p + 4k - 8)$$

. At the other hand

$$\chi(\mathcal{S}) = \sum (-1)^k \mu_k(||x - z||^2)$$

where μ_k denotes the number of critical points of index k.
Let us compute this in an example.



$\mu_0 = 8 \times 1 = 8$; $\mu_1 = 4 \times 3 \times 3 = 36$; $\mu_2 = 2 \times 3 = 6$; $\chi = -22$

$$\text{Morse polynomial} \quad MP(\mathcal{S})(t) = \sum \mu_k t^k$$

where μ_k denotes the number of critical points of index k. and recall $MP(\mathcal{S})(t) = P(\mathcal{S})(t) + (1+t)Q(t)$ (Morse inequalities).

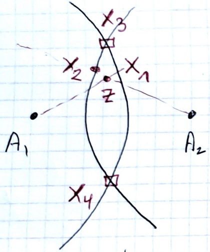
- μ_k depends on the position of z, but is locally constant on the complement of a bifuraction set.
- For $t = -1$ we get $\chi(\mathcal{S})$: independent of z.

Next consider $t = 1$ and $\mu_{tot} = \sum \mu_k$ and ask

- what is the minimum of μ_{tot} over z ?
- what is the average value of μ_{tot} ?

This has been a successful scenario in other cases and relates it to the concept of total absolute curvature and (real)Euclidean Distance degree. An other question is to complexify and use the same formula's with complex numbers. The total number of critical points can serve as upper bound for the real case.

Longer legs



$\boxed{\text{Morse-Bott}}$

$$MP[t] = \sum_k P(\Sigma_k) \cdot t^{i(\Sigma_k)}$$

2 x 2 bones

	i	Σ_k	$MP[t]$
$\cancel{Z}$	0	$S^0 \times S^0$	4
X_1	1	S^0	$2t$
X_2	1	S^0	$2t$
X_3	2	$\bullet$	t^2
X_4	2	$\bullet$	t^2
MP		$4 + 4t + 2t^2$	

$\chi = 2$

2 dim. mfd

2 bones ⊕ 3 bones

i	Σ_k	$MP[t]$
0	$S^0 \times S^1$	$2+2t$
1	S^1	$(1+t)t$
2	S^0	$2t^2$
3	$\bullet$	t^3
3	$\bullet$	t^3

$2 + 3t + 3t^2 + 2t^3$

$\chi = 0$

3-dim-mfd

3 bones ⊕ 3 bones

i	Σ_k	$MP[t]$
0	$S^1 \times S^1$	$(1+t)^2$
2	S^1	$(1+t)t^2$
2	S^1	$(1+t)t^2$
4	$\bullet$	t^4
4	$\bullet$	t^4

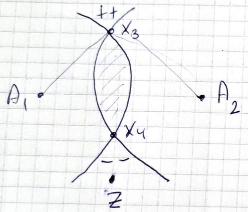
$1 + 2t + 3t^2 + 2t^3 + t^4$

$\chi = 2$

4 dim. mfd

Longer legs

Another Choice of Z :



Only two critical Points !

on a d-dimensional manifold $d=2,3,4$

Maximum and Minimum

Conclusion $S \cong S^d$ d-sphere!

$d=2,3,4,5 \cdots \to \infty$

Less critical points $\Rightarrow$ More about the topology

Happy Birthday Lev !

References

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Proof for diagonal

proof

$\vec{p}_i \in r_i S^1$

$$\vec{p} = \vec{p}_1 + \cdots + \vec{p}_n$$

$$\frac{\partial}{\partial\theta_i}(\vec{p}\cdot\vec{p}) = 2\vec{p}\cdot\frac{\partial p_i}{\partial\theta_i} = 0 \qquad \frac{\partial p_i}{\partial\theta_i} \perp p_i$$

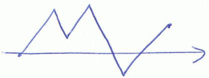
$$\Rightarrow \vec{p} \,/\!/\, p_i \ \ \forall i \quad \text{or} \quad \vec{p} = \vec{0}$$

NB. Proof generalizes to arms in $\mathbb{R}^d$

Remark: similar statement and proof for

$\vec{a}\cdot\vec{p}$

(projection on a line)

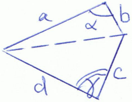


$\mathbb{R}a$

$$\vec{a}\cdot\frac{\partial p_i}{\partial\theta_i} = 0$$

Proof for linkages

critical points of A on linkages



$\boxed{\text{4 gon}}$

$$A = ab \sin\alpha + cd \sin\gamma$$

$$x^2 = a^2 + b^2 - 2ab\cos\alpha$$
$$= c^2 + d^2 - 2cd\cos\gamma$$

Lagrange multipliers:

$$\begin{vmatrix} ab\cos\alpha & cd\cos\gamma \\ ab\sin\alpha & -cd\sin\gamma \end{vmatrix} = 0$$

$\sin(\alpha+\gamma) = 0 \quad \Rightarrow$ vertices on circle.

$\boxed{\text{4 gon} \Rightarrow \text{n-gon}}$