Tangent Cone of Medial Axis

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60 LB Geometry and Singularities - 60th anniversary of Lev Birbrair

21 July 2023

Main characters

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Conflict Set

Medial Axis

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Conflict Set

$\{X_i\}$ - finitely many, disjoint, closed, subsets of Ω

$$Conf\{X_i\} :=$$
$$= \{a \in \Omega \,|\exists i \neq j : \; d(a, X_i) = d(a, X_j) = d(a, \bigcup X_i)\}$$

"Points with at least two closest sets in $\{X_i\}$."

Medial Axis

X - closed, nonempty, subset of Ω

$$MA(X) :=$$
$$= \{a \in \Omega \mid \exists x \neq y \in X : \; d(a, x) = d(a, y) = d(a, X)\}$$

"Points with at least two closest points in X."

(Ω, d) - metric space; $d(a, X) := \min\{d(a, x) \mid x \in X\}$

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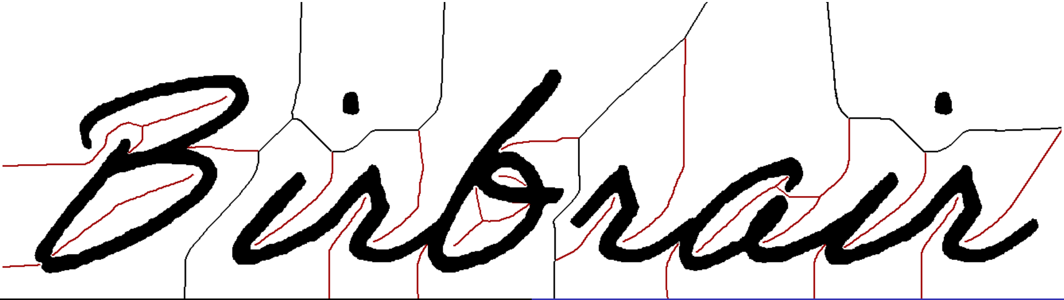

The tangent cone of a conflict set (extremely) local

Tangent Cone of a Conflict Set (Birbrair, Siersma)

For any $a \in Conf\{X_i\}$,

$$C_a Conf\{X_i\} = Conf\{m_{X_i}(a)\},$$

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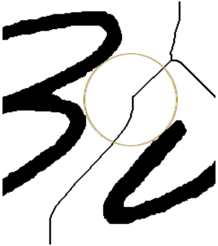

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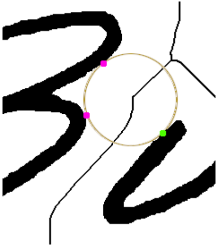

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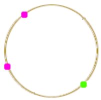

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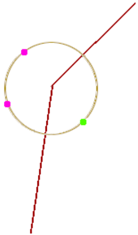

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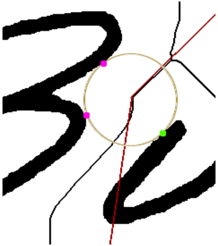

Problem:

The medial axis does not yield to the idea of Birbrair and Siersma's proof.

Yet they stabilise in a way

Definition (Kuratowski-Painlevé)

Let $\{X_t\}$ be a family of closed sets; we say that

- $v \in \limsup_{t\to 0} X_t$ if $\exists t_\nu \to 0, \exists x_\nu \in X_{t_\nu} : x_\nu \to v$,
- $v \in \liminf_{t\to 0} X_t$ if $\forall t_\nu \to 0, \exists x_\nu \in X_{t_\nu} : x_\nu \to v$.
- $X = \lim_{t\to 0} X_t$ if $\liminf X_t = X = \limsup X_t$.

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Theorem (B, Denkowska, Denkowski)

Let $\{X_t\}_{t\in\mathbb{R}}$ be a family of closed sets. If $X_0 = \lim_{t\to 0} X_t$ then

$$MA(X_0) \subset \liminf_{t\to 0} MA(X_t).$$

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The family doesn't have to be definable. The parameter space can be any topological space with a countable base at 0. The ambient space can be a Riemannian manifold.

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Corollary

In particular, setting $X_t = \frac{1}{t}X$*, we get, for a definable* X*,* $MA(C_0X) \subset \liminf MA(\frac{1}{t}X) = C_0MA(X)$.

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Corollary

Assuming that $C_0(X)$ *is nonconvex we get* $0 \in MA(C_0(X))$ *and, consequently,* $0 \in \overline{MA(X)} \cap X$*.*

Second problem

The tangent cone of X for $p \notin X$ is an empty set. So the previous theorem gives us a trivial inclusion

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This result is less than satisfactory.

A copy of the medial axis is hidden under the distance function graph

Lemma

Let Γ_d denote the graph of the distance function $d(\cdot, X)$.
For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ with $y < d(x, X)$ we have $(x, y) \in MA(\Gamma_d)$ if and only if $x \in MA(X)$.

The tangent cone of a medial axis is similar to the conflict set one

Theorem

Let X be a closed definable set, $p \in MA(X)$. Then

$$MA(m(p)) \subset C_p MA(X).$$

Due to the previous lemma, Lipschitzness of the distance function and the formula of its directional derivative.

Moreovers

1. The equality $C_p MA(X) = MA(m(p))$ for generic points of a definable MA.
2. The equality $C_p MA(X) = MA(m(p))$ for definable plane subsets
3. Definable planar medial axes have no cusps.
4. Inequality $\dim MA(m(p)) \leq \dim_p MA(X)$
5. More surprisingly, it follows
   $\dim_p MA(X) = \min\{\text{codim } m(a) \mid a \in MA(X) \cap U, U\text{-neighbr. of } p\} - 1$

Takeaways

Three similar results for three similar objects

Tangent cones of conflict sets:	$Conf\{m_{X_i}(p)\} = C_p Conf\{X_i\}$
Tangent cones of medial axes at $p \notin X$:	$MA(m(p)) \subset C_p MA(X)$
Tangent cones of medial axes at $p \in X$:	$MA(C_0 X) \subset C_p MA(X)$

with three different **noncompatible** proofs!

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with three different **noncompatible** proofs!
Is there one to rule them all? To bring them all and bind them?

Thank you for your attention.