Topological, smooth and holomorphic dynamics, ergodic theory, fractals
Simons Semester, IM PAN

Modern holomorphic dynamics and related fields
Mini-semester, MIM UW

Beyond Uniform Hyperbolicity

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Abstracts of minicourses

Collapsed Anosov flows
Thomas Barthelmé / Sérgio Fenley

Partially hyperbolic systems arise naturally when studying robust dynamical behavior such as robust transitivity and stable ergodicity. It is also a natural setting to understand statistical properties of dynamical systems beyond uniform hyperbolicity. In recent years, many fine dynamical and statistical properties have been obtained for certain classes of partially hyperbolic diffeomorphisms (such as derived from Anosov, skew-products, or perturbations of time one maps of Anosov flows). On the classification side, much effort was made to show that, at least under some topological restrictions, these settings encompassed all partially hyperbolic diffeomorphisms in dimension 3.

Recent examples have shown that there are many more types of partially hy- perbolic diffeomorphisms, with still mysterious dynamical properties. Motivated by these examples, as well as work on the classification problem, we introduced a new class of partially hyperbolic diffeomorphisms in 3-manifolds, that we called *collapsed Anosov flows*.

In this mini-course we will present this class and discuss some of their attributes. Our aim is to advertise the fact that this class can both represent a goal for the classification of partially hyperbolic diffeomorphisms in dimension 3, as well as a natural category for the study of some finer dynamical properties.

In particular, we aim to explain that: The class of collapsed Anosov flows is "large" in many sense; One can prove accessibility (and ergodicity) for partially hyperbolic diffeomorphisms in that class; And, finally, via the link with self orbit equivalences of Anosov flow, it gives us a map for where to look for "undiscovered" collapsed Anosov flows.

The mini-course will be split into four lectures, which will roughly cover the following topics:

- In lecture 1, we will cover some basic definitions, define collapsed Anosov flows, as well as stronger variants of this class, give examples and state the main results.
- In lecture 2 we will give ideas of the proofs behind the equivalent charac- terizations of (strong) collapsed Anosov flows.
- In lecture 3 we will explain some ideas behind the proofs that partially hyperbolic diffeomorphisms on hyperbolic 3-manifolds are collapsed Anosov flows, and explain how one can obtain accessibility in general.
- In lecture 4, we will talk about how the examples build via the Bonatti-Gogolev-Hammerlindl-Potrie method lead to collapsed Anosov flows, and explain the construction of a new example of a collapsed Anosov flow isotopic to the identity. If time permits, we will discuss more generally results about self-orbit equivalences of Anosov flows.

References

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Methods for studying abelian actions and centralizers

Danijela Damjanović / Disheng Xu

Given a dynamical system $f\colon M \to M$, where $f \in \mathrm{Diff}\,(M)$, generic situation should be that the only smooth coordinate changes of M under which f does not change at all, are powers of f. This is Smale's conjecture, it is proved in C^1 topology by Bonatti-Crovisier-Wilkinson. We will discuss here the non-generic situation, when the smooth centralizer $Z(f) = \{g \in \mathrm{Diff}\,(M)\colon g \circ f \circ g^{-1} = f\}$ of f is a larger group. The non-generic situation is in fact typical within the class of algebraic dynamical systems. Algebraic systems are defined via automorphisms and translations on homogeneous spaces G/Γ, where G is a Lie group and Γ a lattice. There has been a large body of work in the direction of showing that if $Z(f)$ is sufficiently large for an Anosov f, then f is essentially algebraic (i.e. f is smoothly conjugate to an algebraic system). This started with the local result of Katok-Spatzier and was then continued in two general directions: obtaining global results while assuming enough Anosov dynamics in $Z(f)$, and in the direction of weakening the Anosov condition to partially hyperbolic and obtaining local or, more recently, semi-local results.

The mini-course will consist of 6 lectures where we will cover the following topics:

Lectures 1 and 2. Introduction into local and global rigidity for abelian actions and centralizer rigidity. Examples and basic invariant structures for abelian actions.

Lecture 3. Two methods: to linearize or not to linearize. Cohomology.

Lecture 4. Focusing on partially hyperbolic conservative dynamics, disintegration of volume along foliations, large centralizer vs pathological center foliation.

Lecture 5. Transitive centralizer and fibered partially hyperbolic systems.

Lecture 6. Focusing on accessible examples: centralizer rigidity and centralizer classification.

Dimension of stationary measures

François Ledrappier / Pablo Lessa

The goal of these lectures is to present results and methods for studying the dimension of stationary measures of random walks. The setting will be the action of matrix groups over flag spaces. Exact dimension is proven for dynamical (Oseledets) measures. Other problems have only partial answers. A good part of the study is understanding the fine structure of the flag space.

I. Generalities about dimension of measures, random walks on matrices. Statement of results (F).
II. Strategy of the proof in dimension 3. Example of Hitchin representations (F).
III. Admissible topologies on $1, \dots, N$ and configuration spaces (P).
IV. Change of coordinates and dynamics of configuration spaces (P).
V. One-step formula (P).
VI. Addition of entropy and dimension (F).

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Classifying pseudo-Anosov flows on 3-manifolds
Kathryn Mann

I will describe work with Barthelmé, Fenley, and Frankel towards classifying (pseudo)-Anosov flows on a manifold of dimension 3. The approach is to study the dynamics of the action of the fundamental group on the *orbit space* of the flow, a bifoliated plane, and its compactification by a circle at infinity. This translates the flow to a discrete dynamical system, where individual elements exhibit some hyperbolic-like behavior. We introduce a toolkit to study these, in the more general framework of "Anosov-like actions on bifoliated planes". The minicourse will present many of these tools, their applications to the study of flows, and some next directions for study on Anosov-like actions on the plane.

On the algebraic structure of groups area-preserving homeomorphisms
Sobhan Seyfaddini

In an influential article from the 1970s, Albert Fathi, having proven that the group of compactly supported volume-preserving homeomorphisms of the n-ball is simple for $n \geq 3$, asked if the same statement holds in dimension 2. In a joint work with Cristofaro-Gardiner and Humiliére, we proved that the group of compactly supported area-preserving homeomorphisms of the 2-disc is not simple. This answers Fathi's question and settles what is known as "the simplicity conjecture" in the affirmative.

In fact, Fathi posed a more general question about all compact surfaces: is the group of "Hamiltonian homeomorphisms" (which I will define) simple? The main goal of these lectures will be to review recent joint work with Cristofaro-Gardiner, Humiliére, Mak and Smith answering this more general question of Fathi. The solution relies on a collection of numerical invariants of area-preserving maps which we call **link spectral invariants**. In the first talk, I will show how these invariants lead to the solution of Fathi's question. The following talks will be dedicated to the construction of these invariants which relies on a version of Lagrangian Floer homology. We will be reviewing the construction of **Lagrangian Floer homology** and the associated spectral invariants.

Spectrum rigidity and joint integrability for Anosov systems on tori
Yi Shi

In this minicourse, we address on the strong rigidity properties from joint integrability in the setting of Anosov diffeomorphisms on tori. More specifically, for an irreducible Anosov diffeomorphism with splitted stable bundle, the joint integrability of the strong stable and full unstable subbundles implies existence of fine dominated splitting along the weak stable subbundle as well as Lyapunov exponents rigidity. This builds an equivalence bridge between the geometric rigidity (joint integrability) and dynamical spectral rigidity (Lyapunov exponents rigidity) for Anosov diffeomorphisms on tori. Moreover, we will show that the topological conjugacy of non-invertible Anosov maps on 2-torus implies the smooth conjuacy along stable foliations. The course will be split into four lectures:

1. We study local perturbation of generic irreducible linear automorphisms on $\mathbb{T}^d$ and prove the joint integrability of the strong stable and full unstable subbundles of these Anosov diffeomorphisms implies smooth conjugacy along weak stable bundles.

2. We show some global rigidity of irreducible Anosov diffeomorphisms on $\mathbb{T}^d$ satisfying center bunching condition. The joint integrability of the strong stable and full unstable subbundles of these Anosov diffeomorphisms implies dynamical coherence, full leaf conjugacy, and matching finest dominated splittings and spectral rigidity in weak stable bundles.

3. Let $A \in \mathrm{Sp}(4, \mathbb{Z})$ be an irreducible and non-conformal Anosov automorphism. For any symplectic diffeomorphism f which is C^1-close to A, we show that the extremal symplectic bundle of f is integrable if and only f is smoothly conjugate to A.

4. We show that if two non-invertible Anosov maps on 2-torus are topological conjugate, then they also admit spectral rigidity along stable bundles, i.e. they have the same Lyapunov exponents on corresponding periodic points. In particular, the conjugacy is smooth along stable foliations.

References

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Abstracts of talks

Robustly transitive endomorphism and strong volume expansion for toral endomorphisms
Martin Andersson

In this talk I will present a new criterion for robust transitivity for partially hyperbolic endomorphisms of the two torus whose action in the first homology has two integer eigenvalues greater than one. The criterion states that if the Jacobian is everywhere greater than the largest eigenvalue, then the map is robustly transitive. For this we introduce Blichfedt's theorem as a tool for extracting dynamical information from the action of a map in homology.

Minimality of the action on the circle at infinity of non-R-covered Anosov flows
Christian Bonatti

An Anosov flow on a 3-manifold defines an action of the fundamental group on the plane preserving a pair of transverse foliations. This action extends on the circle at infinity associated to this pair of foliations. I recently proved that the minimality of this action is equivalent to the exisitence of non-separated leaves for the foliation (which means that the flow is non-R-covered). This holds in particular for non-transitive Anosov flows, which is somewhat surprising.

Strong positive recurrence for diffeomorphisms
Jérôme Buzzi

We introduce a new form of hyperbolicity for diffeomorphisms which we call strong positive recurrence. This new property implies exponential decay of correlations, CLT, etc. for measures maximizing the entropy. Strong positive recurrence turns out to be equivalent to a continuity property of Lyapunov exponents. In particular, it is satisfied by all smooth diffeomorphisms with positive topological entropy on compact surfaces. This is a joint work with Sylvain CROVISIER and Omri SARIG.

Smooth Models for Certain Partially Hyperbolic Systems
Margaret Doucette

We prove that under restrictions on the fiber, any fibered partially hyperbolic system over a nilmanifold is leaf conjugate to a smooth model that is isometric on the fibers and descends to a hyperbolic nilmanifold automorphism on the base.

Uniformly Expanding Random Walks on Manifolds
Rose Elliott Smith

In this talk we will study a dynamical property called uniform expansion— this property tells us that a system is, on average, expanding everywhere. An in-prep work of Brown, Eskin, Filip, and Rodriguez Hertz proves strong Ratner-like orbit and measure classification theorems specific to this class of random walks. With motivation from this work, we ask and answer a few natural questions: how common are uniformly expanding random walks, and in what settings can they exist?

Non-hyperbolic measures of (almost) maximally possible entropy
Katrin Gelfert

We quantify the "lack of hyperbolicity" in transitive nonhyperbolic partially hyperbolic diffeomorphisms. We first study skew products of $\mathcal{C}^1$ circle diffeomorphism. They capture the key mechanism of nonhyperbolic behavior of robustly transitive dynamical systems. They naturally arise, for example, from the projective action of certain 2x2 elliptic matrix cocycles. They are also paradigmatic models for a rich class of $\mathcal{C}^1$ partially hyperbolic diffeomorphisms with a one-dimensional center bundle such as perturbations of the time-one map of transitive Anosov flows and some DA-Anosov diffeomorphisms. The coexistence of saddles of different types of hyperbolicity is described in terms of fiber-expanding and -contracting regions which are mingled by the dynamics. It gives also rise to nonhyperbolic ergodic measures which are characterized in terms of a zero Lyapunov exponent in the circle fiber direction. We perform a multifractal analysis for fiber-Lyapunov exponents and establish restricted variational principles of the topological entropy of the level set of each exponent in terms of the metric entropy of ergodic measures. In particular, we describe the maximal entropy of ergodic nonhyperbolic measures. Moreover, we construct ergodic nonhyperbolic measures with "as large as possible" entropy.

This is joint work with L.J. Díaz, M. Rams, B. Santiago, and J. Zhang.

Non-classifiability of K-automorphisms
Marlies Gerber

Within the collection of measure-preserving transformations, Bernoulli shifts have the ultimate mixing property, and K-automorphisms have the next-strongest mixing properties of any widely considered family of transformations. In particular, K-automorphisms have positive entropy and are mixing of all orders. It is known that, unlike Bernoulli shifts, the family of K-automorphisms cannot be classified up to isomorphism by a complete numerical Borel invariant. This left open the possibility of classifying K-automorphisms with a more complex type of Borel invariant. We show that this is impossible, by proving that the isomorphism equivalence relation restricted to K-automorphisms is complete analytic, and hence not Borel. This work is joint with Philipp Kunde.

Total renormalizations of perturbations of identity
Nicolas Gourmelon

A *total renormalization* of $f\colon X \to X$ is a transformation $F\colon X \to X$ conjugate to the first return of f to a domain Δ whose orbit covers X, e.g. the Rauzy-Veech induction on IETs. We show any $F \in \mathrm{Diff}_0^\infty(\mathbb{S}^1 \times M)$ is a total renormalization C^∞-close to identity maps. This roughly means that all global dynamics exist arbitrarily C^∞-close to identity, It generalizes previous local results by Turaev.

Markovian families in the bifoliated plane of an Anosov flow in dimension 3
Ioannis Iakovoglou

In this talk, we will introduce a new approach to the problem of classification of transitive Anosov flows on 3-manifolds up to orbital equivalence. Generalizing the notion of Markov partition, we will first define the notion of Markovian family in the bifoliated plane of an Anosov flow. We will then see that any such family can be canonically associated to two combinatorial invariants: one that describes the flow up to Dehn-Goodman-Fried surgeries on specific periodic orbits and another that describes the flow up to orbital equivalence. Finally, we will discuss some open questions concerning the two previous combinatorial invariants and how those can lead to a classification of Anosov flows in dimension 3.

Symplectic blenders and persistence of homoclinics to saddle-center periodic orbits
Dongchen Li

A blender is a hyperbolic basic set such that non-transverse intersections with its invariant manifolds can be unremovable by small perturbations. We consider blenders for C^r $(r > 4)$ symplectic diffeomorphisms and show that, up to an arbitrarily small symplectic perturbation, symplectic blenders exist arbitrarily close to any one-dimensional whiskered torus (KAM curve) which has a homoclinic tangency. Using this result, we prove that non-transverse homoclinic intersections between invariant manifolds of a generic saddle-center periodic orbit are persistent (for both symplectic diffeomorphisms and Hamiltonian systems), in the sense that the original system is on the C^r boundary of a C^1 open region where systems having such homoclinic intersections are dense.

Hölder continuity of the Lyapunov exponent for Markov cocycles
Aline Melo

An important problem in ergodic theory concerns the regularity of the Lyapunov exponents as a function of the data. In this talk, we establish the joint Hölder continuity of the maximal Lyapunov exponent as a function of the Markov cocycle and the transition kernel. Our approach provides a more computable Hölder exponent. This is a joint work with Ao Cai, Marcelo Durães and Silvius Klein.

Old and new on Lorenz attractors
Maria José Pacifico

Ever since its discovery in 1963 by Lorenz [1], the Lorenz attractor has been playing a central role in the research of singular flows, i.e., flows generated by smooth vector fields with singularities. In this talk I shall survey about old and new results describing the dynamics of this kind of attractors from the topological as well as the ergodic point of view. I will end sketching the proof of my newest result establishing that that in a C^1-open and densely family of vector fields (including the classical Lorenz attractor), if the point masses at singularities are not equilibrium states, then there exists a unique equilibrium state supported on Λ. In particular, there exists a unique measure of maximal entropy for the flow $X|_\Lambda$. This corresponds to a joint work with Fan Yang and Jiagang Yang.

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Volume entropy for flat surfaces
Mark Pollicott

Volume entropy is a useful quantity for studying surfaces of negative curvature, where it coincides with the topological entropy of the geodesic flow. We will recall a corresponding quantity in the context of flat surfaces (i.e., translation surfaces). We will discuss the minimum value of this entropy on a $\mathrm{SL}(2,\mathbb{R})$-orbit for some particular cases. This is joint work with Paul Colognese.

Metric and ergodic properties of partially hyperbolic diffeomorphisms with topological neutral center
Gabriel Ponce

The theory of partially hyperbolic maps has been extensively studied in several directions including, for example, the study of ergodic properties under suitable center conditions. Recent new examples of partially hyperbolic dynamics, brought to attention a class of partially hyperbolic maps $f : M \to M$ where the central direction, say E^c, has a very "controlled" behavior in the sense that given any $\varepsilon > 0$, there exists $\delta > 0$ for which given any C^1 arc γ tangent to the center direction E^c, we have:

$$\text{length}(\gamma) < \delta \Rightarrow \text{length}(f^n(\gamma)) < \varepsilon,$$

for any $n \in \mathbb{Z}$. This class is called the class of partially hyperbolic maps with topological neutral center. In this talk we will present several metric and ergodic properties recently obtained for this class assuming the center direction is one-dimensional. These properties include a dichotomy between atomicity of the center direction and the Bernoulli property, and a type of invariance principle for ergodic invariant measures.

Birkhoff sums as distributions
Daniel Smania

We study Birkhoff sums as distributions. We obtain regularity results on such distributions for various dynamical systems with hyperbolicity, such as hyperbolic linear maps on the torus and piecewise expanding maps on the interval. We also give applications to deformation theory. Often topological classes of one-dimensional dynamical

systems are finite codimension smooth manifolds. We describe a method to prove this sort of statement that we believe can be applied in many settings. The most important step will be the identification of infinitesimal deformations with primitives of Birkhoff sums, that allows us to use the ergodic properties of piecewise expanding maps to study the regularity of infinitesimal deformations. Joint work with Clodoaldo Ragazzo.

Prediction of dynamical systems from time-delayed measurements with self-intersections
Adam Śpiewak

We study prediction of dynamical systems from measurements performed via a one-dimensional observable along an orbit of the system. We give new versions of the Takens time-delay embedding theorem, both in the deterministic and probabilstic setting. In the latter case, we obtain upper bounds on the decay rate of predicition errors, as conjectured by Schroer, Sauer, Ott and Yorke.

Infinite Renormalizability and Non-Uniform Partial Hyperbolicity
Jonguk Yang

Loosely speaking, a dynamical system is renormalizable if it exhibits self-similarity at a smaller scale. Understanding this phenomenon often yields deep results about the combinatorial, topological and geometric nature of the dynamics. However, the existing techniques are largely limited to one-dimensional cases (e.g. maps on an interval, a circle, a domain in $\mathbb{C}^1$, etc).

In this talk, we generalize the renormalization theory of unimodal interval maps to a two-dimensional setting (which includes real Hénon maps). A key step will be to identify the appropriate notion of a "critical point" using a quantitative reformulation of Pesin theory. This allows us to give an explicit description of the structure of the non-uniform partial hyperbolicity of infinitely renormalizable "unimodal" diffeomorphisms. Our main result will be *a priori* bounds for these systems: a certain uniform control on the geometry of the dynamics at arbitrarily small scales.

This talk is based on joint work with S. Crovisier, M. Lyubich and E. Pujals.

Homotopically unbounded disks for generic surface diffeomorphisms
Salvador Zanata

In this talk, we consider closed orientable surfaces S of positive genus and C^r-diffeomorphisms $f : S \to S$ isotopic to the identity ($r \geq 1$). The main objective is to study periodic open topological disks which are homotopically unbounded (i.e. which lift to unbounded connected sets in the universal covering). We show that these disks are not uncommon, and are related to important dynamical phenomena. We also study the dynamics on these disks under certain generic conditions. Our first main result implies that for the torus (or for arbitrary surfaces, with an additional condition) if the rotation set of a map has nonempty interior and is not locally constant, then the map is C^r-accumulated by diffeomorphisms exhibiting periodic homotopically unbounded disks. Our second result shows that C^r-generically, if the rotation set has nonempty interior (plus an additional hypothesis if the genus of S is greater than 1) a maximal periodic disk which is unbounded and has a rational prime ends rotation number must be the basin of some compact attractor or repeller contained in the disk. As a byproduct we obtain results describing certain periodic components of the complement of the closure of stable or unstable manifolds of a periodic orbit in the C^r-generic setting. This is a joint work with Andres Koropecki, from UFF, Niteroi, Brazil.

Newhouse phenomenon in the complex Hénon family
Zhiyuan Zhang

In a work in progress with Avila and Lyubich, we show that there are maps in the complex Hénon family with a stable homoclinic tangency. This is due to a new mechanism on the stable intersections between two dynamical Cantor sets generated by two classes of conformal IFSs on the complex plane.