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Institute of Mathematics, Polish Academy of Sciences
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Conference

COMPLEX DIFFERENTIAL AND DIFFERENCE EQUATIONS II

August 27 – September 2, 2023

Mathematical Research and Conference Center, Będlewo

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Galina Filipuk (Warsaw University, Poland),
Yoshishige Haraoka (Josai University, Japan),
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Yoshitsugu Takei (Doshisha University, Japan).

Program of the Conference
Monday, August 28

8.15 Breakfast

9.00–9.10 OPENING CEREMONY

Morning Session

9.10–10.10 Yoshishige Haraoka (Josai University, JAPAN)
Shift operators and Riemann's problem

10.20–11.20 Toshiyuki Mano (University of the Ryukyus, JAPAN)
On a geometric notion associated with linear differential equations of Okubo normal form

11.20–11.40 Coffee break

11.40–12.40 Tsvetana Stoyanova (Sofia University, BULGARIA)
Non-integrability of the Sasano system of type $D_5^{(1)}$

13.00 Lunch

Afternoon Session I

15.00–15.40 Shunya Adachi (Chiba University, JAPAN)
Unitary monodromies of second order linear ordinary differential equations

15.50–16.30 Haru Negami (Chiba University, JAPAN)
Multiplicative middle convolution for KZ-type equations and construction of representations of braid groups

16.30–17.00 Coffee break

Afternoon Session II

17.00–17.40 Hidehito Nagao (Akashi College, JAPAN)
A multivariable generalization of the additive difference Painlevé equation with affine Weyl group symmetry type $D_4^{(1)}$

17.50–18.30 Huan Dai (University of Lille, FRANCE)
Analytical and asymptotic properties of solutions of a non-homogeneous functional differential equation

18.30 Dinner

19.10–19.50 Zinelaabidine Latreuch
(National Higher School of Mathematics, ALGERIA)
On the periodicity of entire functions and their differential polynomials

Tuesday, August 29

8.15 Breakfast

Morning Session

9.00–10.00 Yoshitsugu Takei (Doshisha University, JAPAN)
On the exact WKB analysis for difference equations

10.10–11.10 Sampei Hirose (Shibaura Institute of Technology, JAPAN)
On WKB solutions for a differential system satisfied by an oscillatory integral

11.10–11.40 Coffee break

11.40–12.40 Shofu Uchida (Kindai University, JAPAN)
An elementary approach of the connection formula for WKB solutions to the Pearcey system with a large parameter

13.00 Lunch

Afternoon Session I

15.00–15.40 Takahiro Shigaki (Kwansei Gakuin University, JAPAN)
Exact WKB analysis of nonlinear eigenvalue problems for a certain first order equation

15.50–16.30 Yumiko Takei (National Institute of Technology, Ibaraki College, JAPAN)
Voros coefficients and the topological recursion for the hypergeometric differential equations

16.30–17.00 Coffee break

Afternoon Session II

17.00–17.40 Paweł Wójcicki (Warsaw University of Technology, POLAND)
Reproducing kernel Hilbert spaces generated by some elliptic operators

17.50–18.30 Hubert Grzebuła (Cardinal Stefan Wyszyński University, POLAND)
Spherical polyharmonics

18.30 Conference Dinner

Wednesday, August 30

8.15 Breakfast

Morning Session

9.00–10.00 Ilpo Laine (University of Eastern Finland, FINLAND)
Meromorphic solutions of delay-differential equations

10.10–11.10 Yasunori Okada (Chiba University, JAPAN)
Differential operator representations of continuous homomorphisms in mixed cases for entire functions

11.10–11.40 Coffee break

11.40–12.40 Toshio Oshima (Josai University, JAPAN)
Integral transformations of hypergeometric functions with several variables

13.00 Lunch

14.00–18.30 Excursion

18.30 Dinner

Thursday, August 31

8.15 Breakfast

Morning Session

9.00–10.00 SŁAWOMR MICHALIK (Cardinal Stefan Wyszyński University, POLAND)
On sequences preserving summability

10.10–11.10 ALBERTO LASTRA (University of Alcalá, SPAIN)
Entire solutions of linear systems of moment differential equations and related asymptotic growth at infinity

11.10–11.40 Coffee break

11.40–12.40 JAVIER SANZ (University of Valladolid, SPAIN)
Optimal flat funtions and local right inverses for the asymptotic Borel map in ultraholomorphic classes

13.00 Lunch

Afternoon Session I

15.00–15.40 IGNACIO MIGUEL-CANTERO (University of Valladolid, SPAIN)
Stability properties for ultraholomorphic classes defined in unbounded sectors

15.50–16.30 JAVIER JIMÉNEZ-GARRIDO (University of Cantabria, SPAIN)
On generalized definitions of ultradifferentiable and ultraholomorphic classes

16.30–17.00 Coffee break

Afternoon Session II

17.00–17.40 MARIA SUWIŃSKA (Cardinal Stefan Wyszyński University, POLAND)
On Gevrey regularity of solutions for inhomogeneous nonlinear moment partial differential equations

17.50–18.30 HIROSHI OGAWARA (Josai University, JAPAN)
Differential transcendence of solutions for second order linear q-difference equations

18.30 BBQ/Bonfire

Friday, September 1

8.15 Breakfast

Morning Session

9.00–10.00 Stephane Malek (University of Lille, FRANCE)
Gevrey asymptotics for logarithmic type solutions to singularly perturbed problems with nonlocal nonlinearities

10.10–11.10 Masafumi Yoshino (Hiroshima University, JAPAN)
Summability and global property of transseries solution of Hamiltonian system

11.10–11.40 Coffee break

11.40–12.40 Grzegorz Łysik (Jan Kochanowski University in Kielce, POLAND)
Smoothness of the Dunkl analytic functions

13.00 Lunch

15.00 Walk to the lake

18.30 Dinner

Complex Differential and Difference Equations II

August 27 – September 2, 2023

Abstracts of talks

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Shunya Adachi
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Unitary monodromies of second order linear ordinary differential equations

In this talk, I will discuss about the unitarity of monodromy representations of second order Fuchsian differential equations (or Fuchsian systems of rank two) of SL type.

To study the unitarity of monodromies, it is natural and useful to consider the character variety, which is a moduli space of monodromy representations. In the talk, after introducing the notion of the character variety, I will give a constructive characterization of unitary monodromies in terms of the character variety. Then, the signatures of unitary monodromies can be classified.

If time permits, I will discuss my ongoing work related to the case when the equation has irregular singular points.

References

[1] S. Adachi, *Monodromy invariant Hermitian forms for second order Fuchsian differential equations with four singularities*, Opusc. Math. **42** (2022), no. 3, 361–391.
[2] S. Adachi, *Unitary monodromies for rank two Fuchsian systems with* $(n+1)$ *singularities*, arXiv:2210.14729, submitted.

♢ ♢ ♢

Huan Dai (joint work with C. Zhang and G. Chen)
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Analytical and asymptotic properties of solutions of a non-homogeneous functional differential equation

Consider a nonhomogeneous functional differential equation

$$y'(x) = ay(qx) + by(x) + g(x),$$

where the nonhomogeneous term g is a rational function, which can be discussed in the following three cases: polynomials, fractions of singularities at 0, and fractions of singularities at a nonconstant constant. We investigate the existence, analytic and asymptotic properties of solutions in terms of these three cases respectively.

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HUBERT GRZEBUŁA (joint work with S. MICHALIK)
Cardinal Stefan Wyszyński University, POLAND
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Spherical polyharmonics

We introduce and develop the notion of spherical polyharmonics, which are a natural generalisation of spherical harmonics. In particular we study the theory of zonal polyharmonics, which allows us, analogously to zonal harmonics, to construct Poisson kernels for polyharmonic functions on the union of rotated balls. We find the representation of Poisson kernels and zonal polyharmonics in terms of the Gegenbauer polynomials. In this talk we also show the connection between the classical Poisson kernel for harmonic functions on the ball, Poisson kernels for polyharmonic functions on the union of rotated balls, and the Cauchy-Hua kernel for holomorphic functions on the Lie ball.

REFERENCES

[1] H. Grzebuła, S. Michalik, *Spherical polyharmonics and Poisson kernels for polyharmonic functions*, Complex Var. Elliptic Equ. **64** (2019), 420–442.

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YOSHISHIGE HARAOKA
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Shift operators and Riemann's problem

For a Fuchsian system of ordinary differential equations, a shift operator is a gauge transformation that sends the system to another Fuchsian system with the same monodromy and with the local exponents shifted by integers. For the rigid case, the existence of shift operators is shown by Oshima [2, Chapter 11]. By calculating a non-rigid example, I find, in the case, that a shift operator always exists for any shift and it gives a birational transformation of the accessory parameters. We are going to generalize this result for any Fuchsian systems. The existence of shift operators is equivalent to the existence of shifted Fuchsian systems, and hence is closely related to Riemann's problem. We find that the theory of Lappo-Danilevsky [1] works to solve this problem. I would like to report what we have done.

REFERENCES

[1] J. A. Lappo-Danilevsky, Mémoires sur la théorie des systèmes des équations différentielles linéaires, Chelsea Publishing Company, 1953.
[2] T. Oshima, Fractional calculus of Weyl algebra and Fuchsian differential equations, MSJ Memoirs, **28**, 2012.

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Sampei Hirose
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On WKB solutions for a differential system satisfied by an oscillatory integral

This talk considers WKB solutions for the differential system satisfied by the oscillatory integral of the form

$$\psi(x, \eta) = \int e^{\eta F(t,x)} dt, \quad dt = dt_1 \wedge \cdots \wedge dt_m$$

where $F(t, x)$ is a polynomial. Under suitable assumptions for $F(t, x)$, by considering the relationship between the WKB solution and the oscillatory integral, we show that the Borel summability of WKB solutions can be described by a semi-algebraic set. We also discuss the connection formula and resurgence property of WKB solutions, and, if time permits, the representation using WKB solutions of higher residue pairings, which play an important role in the theory of primitive forms.

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Javier Jiménez-Garrido (joint work with D. N. Nenning and G. Schindl)
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On generalized definitions of ultradifferentiable and ultraholomorphic classes

In the past decade and simultaneously, two ways to generalize Denjoy-Carleman ultradifferentiable classes have appeared. We show that the ultradifferentiable-like classes of smooth functions introduced and studied by S. Pilipović, N. Teofanov and F. Tomić are special cases of the general framework of ultradifferentiable (and ultraholomorphic) spaces of functions defined in terms of weight matrices in the sense of A. Rainer and G. Schindl. We study classes “beyond geometric growth factors” defined in terms of a weight sequence and an exponent sequence and we prove that these new types admit a weight matrix representation. Thanks to this representation, we can transfer known results from one context to another, which will allow us to address the concepts of asymptotic expansion and summability in a natural way.

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Ilpo Laine
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Meromorphic solutions of delay-differential equations

In this talk, we shortly describe value distribution of meromorphic solutions of various types of delay-differential equations, such as Fermat, Riccati and Malmquist types of equations. In particular, we consider zeros and poles of meromorphic solutions.

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Alberto Lastra
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Entire solutions of linear systems of moment differential equations and related asymptotic growth at infinity

We study the general entire solution to a linear system of moment differential equations in terms of a moment kernel function for generalized summability, and the Jordan decomposition of the matrix defining the problem.

We also describe the growth at infinity of a solution of the system from different points of view.

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Zinelaabidine Latreuch (joint work with M. A. Zemirni and I. Laine)
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On the periodicity of entire functions and their differential polynomials

We obtain some results regarding the problem of the periodicity of entire functions $f(z)$ when differential polynomials $P(z, f)$ with constant coefficients generated by $f(z)$ are periodic. We provide some sufficient conditions that ensure the periodicity of $f(z)$, and we discuss some properties of periodic functions. Our results generalize and improve some earlier ones and have an importance concerning entire solutions of differential equations of the form $P(z, f) = h(z)$, where $h(z)$ is a periodic function.

References

[1] M. A. Zemirni, I. Laine, Z. Latreuch *New findings on the periodicity of entire functions and their differential polynomials*, Mediterr. J. Math. (2023) https://link.springer.com/article/10.1007/s00009-023-02351-z.
[2] P. Li, W. R. Lü, C. C. Yang, *Entire solutions of certain types of nonlinear differential equations*, Houston J. Math. **45** (2019), no. 2, 431–437.

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Grzegorz Łysik
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Smoothness of the Dunkl analytic functions

For the reflection group W associated with a finite root system and a W-invariant weight function ω_κ Dunkl introduced in [1] a differential-difference operators T_j, $j = 1, \ldots .n$, and the Dunkl Laplacian $\Delta_\kappa = \sum_{j=1}^n T_j^2$. A continuous function on a W-invariant set Ω is called

Dunkl analytic if its mean value function over balls in Ω of radius R with respect to the measure $\omega_\kappa(x)dx$ is convergent for small $R > 0$. During the talk we shall show that Dunkl analytic functions are smooth.

References

[1] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.

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Stephane Malek
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Gevrey asymptotics for logarithmic type solutions to singularly perturbed problems with nonlocal nonlinearities

We investigate a family of nonlinear partial differential equations which are singularly perturbed in a complex parameter and singular in a complex time variable at the origin. These equations combine differential operators of Fuchsian type in time and space derivatives on horizontal strips in the complex plane with a nonlocal operator acting on the perturbation parameter known as the formal monodromy around 0. Their coefficients and forcing terms comprise polynomial and logarithmic type functions in time and are bounded holomorphic in space. A set of logarithmic type solutions are shaped by means of Laplace transforms relatively to time and the perturbation parameter and Fourier integrals in space. Furthermore, a formal logarithmic type solution is modeled which represents the common asymptotic expansion of Gevrey type of the genuine solutions with respect to the perturbation parameter on bounded sectors at the origin.

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Toshiyuki Mano
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On a geometric notion associated with linear differential equations of Okubo normal form

In the talk, we would like to introduce a geometric notion called "flat structure (without metric)" associated with linear differential equations of Okubo normal form. Our formulation is based on two key ingredients: "discriminant locus" and "space of Okubo-Saito potentials". Also, we would like to explain how the flat structure is related to the global behavior (i.e. monodromy data) of solutions to a linear differential equation taking the Gauss hypergeometric equation (and transcendental solutions to the Painlevé VI equation if time permits) for examples.

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Sławomr Michalik (joint work with K. Ichinobe)
Cardinal Stefan Wyszynski University, POLAND
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On sequences preserving summability

In this talk we introduce and characterise sequences preserving summability. We also explain their importance for the study of summable formal power series solutions of moment differential equations.

In particular we show that the sequence $([n]_q!)_{n\geq 0}$ preserves summability for every $q \in [0,1)$. This allows us to find summable solutions of linear q-difference-differential equations with constant coefficients.

References

[1] K. Ichinobe, S. Michalik, *On the summability and convergence of formal solutions of linear q-difference-differential equations with constant coefficients*, Math. Ann. (2023), https://doi.org/10.1007/s00208-023-02672-0.

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Ignacio Miguel-Cantero (joint work with J. Jiménez-Garrido, J. Sanz and G. Schindl)
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Stability properties for ultraholomorphic classes defined in unbounded sectors

We present a characterization of some stability properties for Carleman-Roumieu ultraholomorphic classes, defined on sectors and in terms of a weight matrix. We generalize the result of J. A. Siddiqi and M. Ider [1], for such classes defined on sectors not wider than a half plane and in terms of a single sequence which controls (except for a constant and a geometric factor) the growth of the complex derivatives. More precisely, we generalize in three directions:

(i) We give the proof in the general weight matrix setting and get, in particular, the corresponding theorem for the sequence case.

(ii) We extend the list of stability properties and consider sectors of unrestricted opening. This generalization rests of the construction, under suitable assumptions, of the characteristic functions in arbitrary sectors.

(iii) By applying these results to the weight matrix $\mathcal{M}_\omega$ associated to the weight function ω, we characterize the stability for the ultraholomorphic class associated with ω under suitable conditions.

References

[1] M. Ider and J. A. Siddiqi, *A symbolic calculus for analytic Carleman classes*, Proc. Amer. Math. Soc. (1987), https://doi.org/10.2307/2046638.

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Hidehito Nagao
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A multivariable generalization of the additive difference Painlevé equation with affine Weyl group symmetry type $D_4^{(1)}$

We construct a multivariable generalization of the additive difference Painlevé equation with the affine Weyl group symmetry of type $D_4^{(1)}$ and give three representations of the Lax forms and generalized hypergeometric special solutions. Also, we show some relations to additive difference Painlevé equations [1] and the q-Garnier system [3] (given as the multivariable generalization of the q-Painlevé equation with the affine Weyl group symmetry of type $D_5^{(1)}$) and Ormerod-Rains' additive difference system [2]. In this talk, we present the results.

References

[1] Nagao, H.: *The Padé interpolation method applied to additive difference Painlevé equations*. Lett. Math. Phys. **111** (2021), no. 135, 28 pages.
[2] Ormerod, C. M., and Rains, E. M.: *Commutation Relations and Discrete Garnier Systems.* SIGMA **12** (2016) 110, 50 pages.
[3] Sakai H., *A q-analog of the Garnier system*, Funkcialaj Ekvacioj **48** (2005), 273–297.

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Haru Negami (joint work with K. Hiroe)
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Multiplicative middle convolution for KZ-type equations and construction of representations of braid groups

There are various ways to define braid groups B_n. One is to view it as the fundamental group of the configuration space of unordered n-points on the complex plane, and another is to view it as the mapping class group of a disk with n-points, and so on. The monodromy representation for KZ-type equations is the anti-representation of the pure braid group P_n through the former view. In Paper [1], Haraoka obtained a method to construct a new anti-representation of the P_n from any given anti-representation of the P_n through multiplicative middle convolution of the KZ-type equation.

In this talk, we will apply the Katz-Long-Moody construction, a construction method of representations of braid groups mentioned in [2], to the case of P_n and discuss the correspondence with Haraoka's construction method. Then, we will describe some properties shown by the correspondence.

References

[1] Y. Haraoka, *Multiplicative middle convolution for KZ equations*, Mathematische Zeitschrift (2020)
[2] K. Hiroe and H. Negami, *Long-Moody construction of braid representations and Katz middle convolution*, https://arxiv.org/pdf/2303.05770.pdf.

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Hiroshi Ogawara
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Differential transcendence of solutions for second order linear q-difference equations

We consider a second order linear q-difference equation,

$$a_2(x)y(q^2x) + a_1(x)y(qx) + a_0(x)y(x) = 0, \tag{1}$$

where $a_0(x), a_1(x), a_2(x) \in \mathbb{C}[x]$.

Nishioka gave a criterion of differential transcendence of solutions to second order linear difference equations [1]. By using Nishioka's criterion, the author showed differential transcendence of solutions to the q-difference equation of the Ramanujan function [2]. In this talk, we simplify the assumptions of Nishioka's criterion for (1) and apply the simplified criterion to linear q-difference equations of the hypergeometric type.

References

[1] S. Nishioka, *Differential transcendence of solutions of difference Riccati equations and Tietze's treatment*, J. Algebra **511** (2018), 16–40.

[2] H. Ogawara, *Differential transcendence of solutions for q-difference equation of Ramanujan function*, in Recent Trends in Formal and Analytic Solutions of Diff. Equations, Contemporary Mathematics, vol. **782**, Amer. Math. Soc., Providence, RI, 2023, 143–153.

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Yasunori Okada (joint work with T. Aoki and R. Ishimura)
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Differential operator representations of continuous homomorphisms in mixed cases for entire functions

The problem of continuity theorems and differential operator representations on endomorphisms of some spaces of entire functions arises from a context of superoscillations, and enlarges the applicability of the symbol calculus of operators of infinite order. Such problems for the spaces of entire functions with growth given by an order and by a proximate order are solved in [2] and [3].

In [1], we studied continuous homomorphisms between spaces of entire functions with growth given by proximate orders and gave their differential operator representations both in Roumieu cases and Beurling cases. Today, we report our recent study on such representations for mixed cases. We also introduce the concept of the topological intersection/union theorems, and clarify the merit of studying homomorphisms between spaces given by proximate orders.

References

[1] T. Aoki, R. Ishimura, and Y. Okada, *A differential operator representation of continuous homomorphisms between the spaces of entire functions of given proximate orders*, Complex Anal. Oper. Theory **14** (2020), Paper No. 75, 22.

[2] T. Aoki, R. Ishimura, Y. Okada, D. C. Struppa, and S. Uchida, *Characterization of continuous endomorphisms of the space of entire functions of a given order*, Complex Var. Elliptic Equ. **66** (2021), 1439–1450.

[3] X. Jin, *The spaces of formal power series of class M of Roumieu type and of Beurling type*, Hiroshima Math. J. **50** (2020), 117–135.

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Toshio Oshima
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Integral transformations of hypergeometric functions with several variables

As a generalization of Riemann-Liouville integral, we introduce integral transformations of convergent power series which can be applied to hypergeometric functions with several variables.

References

[1] T. Oshima, *Integral transformations of hypergeometric functions with several variables*, preprint, https://www.ms.u-tokyo.ac.jp/~oshima/paper/ihgorg.pdf.

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Javier Sanz (joint work with J. Jiménez-Garrido, I. Miguel-Cantero and G. Schindl)
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Optimal flat funtions and local right inverses for the asymptotic Borel map in ultraholomorphic classes

We prove the existence of optimal flat functions in Carleman-Roumieu ultraholomorphic classes, defined by general strongly nonquasianalytic weight sequences and in sectors of suitably restricted opening. The key fact is the interpretation of a condition of M. Langenbruch, recently recovered by D. N. Nenning, A. Rainer and G. Schindl in a mixed setting, in terms of a property of regular variation related to the classical conditions (M3) of H. Komatsu.

If the defining sequence is regular in the sense of Dyn'kin, from these optimal flat functions one may obtain local right inverses for the asymptotic Borel map, that interpolate in Banach spaces of asymptotic power series with a control of the type. Finally, we discuss some examples (including the well-known q-Gevrey case) where such optimal flat functions can be obtained in a more explicit way.

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Takahiro Shigaki
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Exact WKB analysis of nonlinear eigenvalue problems for a certain first order equation

Bender, Fring and Komijani studied a certain first order differential equation as a typical example of nonlinear eigenvalue problems. In their result, an asymptotic behavior of the eigenvalues is obtained.

We try to apply exact WKB analysis to this problem. In our talk, we describe the construction of so-called 0-parameter solution and the Borel summability in some Stokes regions. We also introduce that the eigenfunctions are written by the Borel sum of a 0-parameter solution.

References

[Sh] T. Shigaki, *Toward exact WKB analysis of nonlinear eigenvalue problems*, RIMS-Kôkyûroku Bessatsu **B75** (2019), 177-201.

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Non-integrability of the Sasano system of type $D_5^{(1)}$

In [1] Yusuke Sasano introduced higher order Painlevé system with affine Weyl group symmetry of type $D_l^{(1)}$. In this talk we study the integrability of the Sasano system which is invariant under the extended affine Weyl group $\widetilde{W}(D_5^{(1)})$. This system is expressed as the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}$$

with the Hamiltonian

$$H = H_V(x, y, t; \alpha_2+\alpha_5, \alpha_1, \alpha_2+2\alpha_3+\alpha_4) + H_V(z, w, t; \alpha_5, \alpha_3, \alpha_4) + \frac{2y\,z\,((z-1)\,w+\alpha_3)}{t},$$

where by $H_V(q, p, t; \gamma_1, \gamma_2, \gamma_3)$ is denoted the Hamiltonian associated with the Painlevé V equation, i.e.

$$H_V(q, p, t; \gamma_1, \gamma_2, \gamma_3) = \frac{q\,(q-1)\,p\,(p+t) - (\gamma_1+\gamma_3)\,q\,p + \gamma_1\,p + \gamma_2\,t\,q}{t}.$$

The complex parameters $\alpha_j, 0 \le j \le 5$ satisfy the relation

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1.$$

In this talk we will present a rigorous proof that when

$$\alpha_1 = \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = 1, \quad \alpha_0 = -\alpha_5$$

the Sasano system of type $D_5^{(1)}$ is not integrable in the sense of the Hamiltonian dynamics by meromorphic first integrals which are rational functions in t. To obtain this result we utilize the Morales-Ramis-Simó theory of non-integrability of analytic Hamiltonian systems. In addition, with the aid of the Bäcklund transformations of the Sasano system we extend this non-integrable result to the entire orbit of the parameters $\alpha_j, 0 \le j \le 5$.

This talk is partially supported by Grant KP-06-N 62/5 of Bulgarian Fond "Scientific Research".

References

[1] Y. Sasano, *Higher order Painlevé equations of type* $D_5^{(1)}$, RIMS Kokyuroku **1473** (2006), pp. 143–163.

♢ ♢ ♢

Maria Suwińska (joint work with P. Remy)
Cardinal Stefan Wyszynski University in Warsaw, POLAND
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On Gevrey regularity of solutions for inhomogeneous nonlinear moment partial differential equations

We study the Gevrey regularity of formal solutions for a certain class of inhomogeneous nonlinear moment differential equations of the form

$$(2)\qquad \begin{cases} \partial_{m_0;t}^{\kappa} u - P(t,x,(\partial_{m_0;t}^{i}\partial_{m;x}^{q}u)_{(i,q)\in\Lambda}) = \widetilde{f}(t,x) \\ \qquad\qquad \partial_{m_0;t}^{j} u(t,x)_{|t=0} = \varphi_j(x) \text{ for } 0 \le j < \kappa, \end{cases},$$

where P is a polynomial with coefficients analytic at a certain neighbourhood of the origin, the initial conditions are also analytic at a neighbourhood of the origin and the inhomogeneity $\widetilde{f}(t,x)$ is σ-Gevrey for some $\sigma \ge 0$. Our aim is to show the connection between the Gevrey order of $\widetilde{f}(t,x)$ and the shape of the Newton polygon for Eq. (2), and the Gevrey order of the unique formal solution of (2). The study effectively connects the methods and results from previous works on nonlinear partial differential equations and linear moment partial differential equations.

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On the exact WKB analysis for difference equations

In this talk we consider the exact WKB analysis for difference equations. Taking some concrete examples which arise from (confluent) hypergeometric equations, we study their WKB-type formal solutions, Stokes curves and connection formulas. If time permits, we also discuss some problems related to difference equations.

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Voros coefficients and the topological recursion for the hypergeometric differential equations

In this talk, we prove that the Voros coefficients for hypergeometric differential equations are described by the generating functions of free energies defined in terms of Eynard-Orantin's topological recursion. Furthermore, as its applications we show the following objects can be explicitly computed for hypergeometric equations: (i) three-term difference equations that the generating function of free energies satisfies, (ii) explicit form of the free energies, and (iii) explicit form of Voros coefficients ([1, 2]).

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An elementary approach of the connection formula for WKB solutions to the Pearcey system with a large parameter

We investigate the system of partial differential equations which characterizes the Pearcey integral from the viewpoint of the exact WKB analysis. In this talk, we discribe that the Borel transform of the WKB solutions to the system can be expressed as a linear combination of the branches of an algebraic function of degree 4. As a corollary, we see that the WKB solutions are resurgent functions. Furthermore, we provide several connection formulas for the system under suitable assumptions.

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Reproducing kernel Hilbert spaces generated by some elliptic operators

In this talk, reproducing kernel Hilbert spaces generated by some elliptic operators will be defined.

The problem of existence of a corresponding reproducing kernel will be refered to the regularity of a considered elliptic operator.

Connections between reproducing kernels of considered Hilbert spaces and Green's functions of their corresponding elliptic operators will be described.

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Summability and global property of transseries solution of Hamiltonian system

Let H_0 and H_1 be given, respectively, by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{3}$$

$$H_1 = \sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \tag{4}$$

where $B_j(q_1, s, q)$'s are holomorphic at the origin, λ_j's are constants and σ is a positive integer. Define $H := H_0 + H_1$. Consider the Hamitonian system with n degrees of freedom

$$\dot{q}_j = \nabla_{p_j} H, \quad \dot{p}_j = -\nabla_{q_j} H, \quad j = 1, 2, \ldots, n. \tag{5}$$

Assume

$$B_\nu \equiv B_\nu(q_1, q_1^{2\sigma} p_1, q) = B_{\nu,0}(q_1, q) + q_1^{2\sigma} p_1 B_{\nu,1}(q_1, q), \ \ \nu = 2, \ldots, \mathrm{n}, \tag{6}$$

where $B_{\nu,0}$ and $B_{\nu,1}$ are analytic at $(q_1, q) = (0, 0)$. Suppose that the Poincaré condition holds

$$Re\, \lambda_j > 0, \quad j = 2, 3, \ldots, n. \tag{7}$$

Assume the nonresonance condition

$$\sum_{\nu=2}^{n} \lambda_\nu k_\nu - \lambda_j \neq 0, \quad \forall\, \mathrm{k}_\nu \in \mathbf{Z}_+, \ \nu = 2, \ldots, \mathrm{n}, \ \mathrm{j} = 2, \ldots, \mathrm{n}. \tag{8}$$

Set $\lambda = (\lambda_2, \ldots, \lambda_n)$. The formal transseries is defined by

$$\sum_{k \geq k_0,\, \ell \geq \ell_0} c_{k,\ell}\, t^{-\frac{\ell}{2\sigma-1}} e^{\lambda k t}, \tag{9}$$

where $k = (k_2, \ldots, k_n)$, $\lambda k = \lambda_2 k_2 + \cdots + \lambda_n k_n$. Here $c_{k,\ell}$'s are complex constants, k_0 is a multi-integer and $\ell_0 \geq 0$ is an integer. Consider the formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$. Then we have

Proposition Suppose that (6),(7) and (8) are satisfied. Then there exists a formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ of (5) in the domain $\{t|\ \mathrm{Re}(\lambda_j t) < 0,\ j = 2, \ldots, n\}$.

Theorem Suppose that (6), (7) and (8) are satisfied. Then the formal transseries solution $(q_1(t), p_1(t), q(t), p(t))$ is $(2\sigma-1)$-Borel summable in every direction in $\{t|\ \mathrm{Re}(\lambda_j t) < 0,\ j = 2, \ldots, n\}$. There exists a neighborhood of $t = 0$, Ω_1 such that the Borel sum is the analytic transseries solution of (5) in the domain $\{t|\ \mathrm{Re}(\lambda_j t) < 0,\ j = 2, \ldots, n\} \cap \Omega_1$.

Example. The following Hamiltonian is the local counterpart of the Hamiltonian studied by Taimanov related with the nonintegrability of a geodesic flow

$$H_1 := c q_1^{4\sigma} p_1^2 + \sum_{j=2}^{n} B_j(q_1) p_j^2, \tag{10}$$

where c is a constant and $B_j(q_1)$ is an analytic function in some neighborhood of $q_1 = 0$. For H_0 in (3), we define $H := H_0 + H_1$. We see that χ_H is not C^ω-Liouville integrable at the origin under a certain condition, while it is C^∞-Liouville integrable at the origin.

Analytic continuation and connection problem of transseries solution.
We discuss the analytic continuation of the summed transseries solution. We use first integrals in order to study the global property of the summed transseries solution. The idea is closely related with the notion of a semi-formal solution defined by Balser. (cf. [1]).

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The Banach Center Conference

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