# MEROMORPHIC SOLUTIONS OF DELAY-DIFFERENTIAL EQUATIONS,

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## 1. Difference Nevanlinna background

In this presentation, we assume that the audience is familiar with the basic notions and key results of the Nevanlinna theory of meromorphic functions $f$, such as the proximity function $m(r,f)$, counting function $N(r,f)$ and characteristic function $T(r,f)$, see e.g. [4]. We also assume that the audience knows the notions of order, hyper-order, defects and the exponents of convergence. Clunie lemma is the only standard result, we would like to offer separately here, due to its importance in considerations of complex differential equations:

**Lemma 1.1.** *Suppose $f$ is a transcendental meromorphic solution to $f^nP(z,f) = Q(z,f)$, where $P,Q$ are differential polynomials in $f$ with (proximity) small coefficients $\alpha$ in the sense of $m(r,\alpha) = S(r,f) = o(T(r,f))$, where $S(r,f)$ stands for a possible exceptional set of finite linear measure. Then $m(r,P(z,f)) = S(r,f)$.*

The first observation towards difference Nevanlinna theory had been made by Dugué [2] iin 1947. He constructed an example to show that Nevanlinna results may depend on the origin. In fact, his example

$$f(z) = \frac{e^{2\pi i e^z} - 1}{e^{2\pi i e^{-z}} - 1}$$

pointed out that $f(z), f(z-1)$ may have different defects. Observe that its hyper-order $\rho_2(f) = 1$; this is standard assumption in difference Nevanlinna theory. For a more detailed analysis of shift-invariance of defects, see [4].

Actual difference Nevanlinna theory had been started, independently, by Halburd and Korhonen [5] and Chiang and Feng [1], see also Halburd and Korhonen [6]. The key result here is the counterpart to the logarithmic derivative lemma, typically called as *logarithmic difference lemma* saying that

$$m\left(\frac{f(z+c)}{f(z)}\right) = S(r,f),$$

provided $\rho_2(f) < 1$. This restriction to the hyper-order of $f$ is typical in difference Nevanlinna theory; in this connection the possible exceptional set means a set of finite logarithmic measure. Some extensions to this lemma are due to, e.g., Korhonen and Zheng. Observe, moreover, that logarithmic difference lemma easily extends to the delay-difference setting, see e.g. the book [10]:

$$m\left(\frac{f^{(k)}(z+c)}{f(z)}\right) = S(r,f).$$

In particular again, difference counterpart to the Clunie lemma [9] deserves to be separately recalled:

**Lemma 1.2.** . *Suppose $f$ is a transcendental meromorphic solution to $H(z,f)P(z,f) = Q(z,f)$, where $H,P,Q$ are difference polynomials in $f$ with small coefficients such that* $\deg H(z,f) \geq \deg Q(z,f)$*, both in $f(z)$ and* *its shifts $f(z+c)$. If $H(z,f)$ contains just one term of maximal degree (relative to $f$ and its shifts), then $m(r,P(z,f))=S(r,f)$.*

Clunie lemma also easily extends to the delay-differential case as well.

## 2. Shift results

During the last fifteen years or so, difference Nevanlinna theory prompted a large number of results, dealing with (1) properties of difference, resp. delay-differential, polynomials and with (2) meromorphic solutions to difference, resp. delay-differential, equations. As an example, recall a result in [7]:

**Theorem 2.1.** *Let $f$ be transcendental meromorphic of finite order $\rho(f)$ with a few poles in the sense of $N(r,f)=S(r,f)$, let*

$$g(f):=\sum_{j=1}^{k} b_j(z)f^{(k_j)}(z+c_j)\not\equiv 0$$

*be a delay-differential polynomial with small coefficients, and let $b(z)$ be small. If $n\geq 2, s\geq 1$, then $F:=f^n g(f)^s-b$ has many zeros in the sense of its exponent of convergence: $\lambda(F)=\rho(f)$.*

Many of the recent results in difference, resp. delay-difference, setting appear to be correct as such, although often being not best possible. Towards this end, let me recall [12]:

**Theorem 2.2.** *Suppose $f$ is a transcendental meromorphic solution to*

$$f(z+1)+f(z)+f(z-1)=\frac{a(z)}{f(z)}+c(z),$$

*where $a(z)\not\equiv c(z)$ are small. Then $f$ has infinitely any zeros and poles.*

Immediate questions looking at this result are: (1) Why only three terms on the left hand side? (2) Why not more general shifts ? (3) Why not more general right hand side ? All these questions have been replied in [8]. To attack these problems, consider

$$L(z,f):=\sum_{j=1}^{n}\beta_j(z)f^{(k_j)}(z+c_j),$$

where the shifts $c_j\in\mathbb{C}$, $k_j\in\mathbb{N}\bigcup\{0\}$, and the coefficients $\beta_j$ are small. Then we obtained, see [8]:

**Theorem 2.3.** *Suppose $f$ is a transcendental meromorphic solution to*

$$L(z,f)=\frac{P(z,f)}{Q(z,f)}=\frac{\alpha_0(z)+\alpha_1(z)f+\cdots+\alpha_p(z)f^p}{\gamma_0(z)+\gamma_1(z)f+\cdots+\gamma_q(z)f^q},$$

*where $P,Q$ are relatively prime polynomials in $f$ with small coefficients. Then, among other conclusions, (1) if $d=\max\{p,q\}\geq 2$ or if $d=q=1$, then $\lambda_2(1/f)=\rho_2(f)$ and (2) if $\alpha_0(z)\not\equiv 0$, then $\lambda_2(f)=\rho_2(f)$.*

An example $f'(z)+f(z+1)+f(z-1)=\frac{-\pi 1}{2}(f^2-1)+2/f$ solved by $f(z)=\frac{e^{\pi iz}+1}{e^{\pi iz}-1}$. Here $\lambda(f)=\lambda(1/f)=\rho(f)=1$.

Instead of delay-differential sum, we may consider delay-differential product as well:

$$M(z,f):=\Pi_{j=1}^{n} f^{(k_j)}(z+c_j)=\frac{P(z,f)}{Q(z,f)}=\frac{\alpha_0(z)+\alpha_1(z)f+\cdots+\alpha_p(z)f^p}{\gamma_0(z)+\gamma_1(z)f+\cdots+\gamma_q(z)f^q}.$$

Given now a meromorphic solution, if (1) $d\geq n+1$ or $d=q\geq 1$, then $\lambda_2(1/f)=\rho_2(f)$ and (2) if $\alpha_0(z)\not\equiv 0$, then $\lambda_2(f)=\rho_2(f)$.

## 3. Fermat type delay-differential equations

Most of the existing material for delay-differential equations is to show non-existence, resp. concrete form, of entire solutions to delay-differential equations of Fermat type. An easy example follows:

**Theorem 3.1.** *No transcendental entire solutions of finite order $\rho$ to $(f^{(k)}(z))^n+f(z+c)^m=1$ exist, provided $m\neq n$.*

*Proof.* (sketch) **A:** $m>n$.

$$mT(r,f(z+c))=mT(r,f(z))+S(r,f)=nT(r,f^{(k)}(z))+S(r,f)=nT(r,f(z))+S(r,f),$$

hence $T(r,f)=S(r,f)$, a contradiction.

**B:** $n>m\geq 2$. By second main theorem,

$$nT(r,f^{(k)}(z))=T(r,(f^{(k)}(z))^n)\leq \overline{N}(r,1/f^{(k)}(z))+\overline{N}(r,\frac{1}{(f^{(k)}(z))^n-1})+S(r,f)$$

$$\leq T(r,f^{(k)}(z))+\overline{N}(r,\frac{1}{(f^{(k)}(z))^n-1})+S(r,f).$$

This implies

$$(n-1)T(r,f^{(k)}(z))\leq \overline{N}(r,\frac{1}{(f^{(k)}(z))^n-1})+S(r,f)=\overline{N}(r,1/f(z+c)^m)+S(r,f)$$

$$=\overline{N}(r,1/f(z+c))+S(r,f)\leq T(r,f(z))+S(r,f)=T(r,f)+S(r,f).$$

Thus,

$$2T(r,f(z))=2T(r,f(z+c))+S(r,f)\leq mT(r,f(z+c))+S(r,f)=nT(r,f^{(k)}(z))+S(r,f)$$

$$\leq \frac{n}{n-1}T(r.f(z))+S(r,f)\leq \frac{3}{2}T(r,f(z))+S(r,f),$$

hence $T(r,f)=S(r,f)$, a contradiction.

**C:** $n>m=1$. By differentiation, $nf'(z)^{n-1}f''(z)=-f'(z+c)$. By Clunie argument, $T(r,f)=S(r,f)$, a contradiction again. ☐

Now, immediate extensions could be considered. Indeed, similar results might perhaps easily follow, whenever (1) entire solutions of hyper-order $< 1$ are treated, or (2) meromorphic solutions are considered, at least when $N(r,f) = S(r,f)$.

## 4. Malmquist idea and delay-differential equations

The first observation in this realm had been pointed out by Malmquist in 1913: Suppose $f$ is a transcendental meromorphic solution to $f' = R(z,f)$, where $R(z,f)$ is rational in both arguments. Then the equation reduces to the Riccati differential equation $f' = a_0(z) + a_1(z)f + a_2(z)f^2$.

This result has later on treated, including more detailed observations, by Yosida, the present author and Steinmetz, among others. To look at the Malmquist idea in extenso, one may indeed go to looking at differential, resp. difference, resp. delay-difference, equations with small (relative to a solution) coefficients. If $f$ is meromorphic, then the equation reduces to a relatively simple form. As the first result in the difference setting was due to Yanagihara in 1980: If $f$ is a transcendental meromorphic solution to $f(z+1) = R(z,f)$, $R(z,f)$ rational in both arguments, then $\deg(R) = 1$.

Later on (2000), Ablowitz, Halburd and Herbst treated a slightly more general case with two terms on the left hand size: If $f$ is a meromorphic solution to $f(z+1) + f(z-1) = R(z,f)$, resp. $f(z+1)f(z-1) = R(z,f)$, then $\deg(R) \leq 2$. Moreoever, in fact, if the left hand side has $n$ similar terms (sum or product), then $\deg(R) \leq n$.

As a more general, recent result, the following one may perhaps deserve to be separately recalled:

Halburd, Korhonen (2017): $f$ transcendental mero solution of hyper-order $< 1$ to

$$f(z+1) - f(z-1) + a(z)\frac{f'(z)}{f(z)} = R(z,f) = \frac{P(z,f}{Q(z,f)},$$

where $a(z)$ rational, $P(z,f)$ a poly in $f$ with rational coefficients and $Q(z,f)$ a poly in $f$ with coefficients that are nonzero rational and not roots of $P(z,f)$, Then either $\deg(Q) + 1 = \deg(P) \leq 3$ or $\deg(R) \leq 1$.

As for more general cases, see [8], let us consider

$$\Omega(z, f_1, \ldots, f_t) = \sum_{\mu=1}^{n} M_\mu(z, f_1, \ldots, f_t),$$

where

$$M_\mu(z, f_1, \ldots, f_t) = a_\mu(z)\Pi_{j=1}^{t} f_j^{\alpha_{j,0}^{\mu}} (f_j')^{\alpha_{j,1}^{\mu}} \cdots (f_j^{k_j})^{\alpha_{j,k_j}^{\mu}}.$$

Here $\alpha$:s are non-negative integers and $a_\mu$:s are small coefficients. Moreover, denote $f_j := f(z + c_j)$. Finally, we may consider $\Phi(z, f_1, \ldots, f_t) = \Omega_1(z, f_1, \ldots, f_t)/\Omega_2(z, f_1, \ldots, f_t)$. Recalling now standard notions of degree, weight and hyper-weight defined for $M_\mu$, say, as

$$\gamma_j(M_\mu) := \Sigma_{s=1}^{k_j}\alpha_{j,s}^{\mu}, \Gamma_j(M_\mu) := \Sigma_{s=1}^{k_j} s\alpha_{j,s}^{\mu}, \Delta_j(M_\mu) := \Sigma_{s=1}^{k_j}(s+1)\alpha_{j,s}^{\mu}$$

and then use, similarly, for sums, resp. for quotients, to take maxima of d, w, h-w, by going through all terms under consideration. Look now at

$$\Phi(z, f(z+c_1), \dots, f(z+c_t)) = \frac{P(z,f)}{Q(z,f)} = \frac{\alpha_0(z) + \cdots + \alpha_p(z)f^p}{\beta_0 + \cdots + \beta_q f^q}, \quad (4.1)$$

$P, Q$ being relatively prime polynomials with small meromorphic coefficients. Now one easily gets

**Theorem 4.1.** *Suppose $f$ is a meromorphic solution to (4.1) of hyper-order $< 1$. Then*

$$d = \max\{p, q\} \leq \min\{\Delta_\Phi; \gamma_\Phi + \Gamma_\Phi \overline{N}(r, f)\}.$$

Considering this type of results, one might immediately ask about there sharpness. Indeed, one should always give example(s) to show this: The result above may fail whenever $\rho_2(f) \geq 1$: $f(z) = (e^{e^z} - 1)^{-1}$ solves

$$f'(z+c) = \frac{-2e^z f^2(f+1)^2}{(2f+1)^2},$$

provided $e^c = 2$. Indeed, in this case $d = 4$, while $\Delta \leq 2$.

Another simple example might be $f(z) = \tan\frac{\pi}{4}z$ solving

$$\Omega := f'(z+1) + f'(z-1) = \pi((f^2+1)/(f^2-1))^2$$

This example offers equality: Indeed, we have $d = 4, \Delta_\Omega = 4, \gamma_\Omega = 2, \Gamma_\Omega = 2, \Theta(\infty, f) = 0$.

## 5. Exponential polynomial case

Consider next difference equations of type

$$f(z)^n + q(z)e^{Q(z)}f(z+c) = P(z), \quad (5.1)$$

where $q, Q, P$ are polynomials, $n \geq 2$ and $c \neq 0$, see [3]. It is not difficult to see that all meromorphic solutions of hyper-order $< 1$ are necessarily entire. Suppose now that $q$ is non-vanishing and $Q$ is non-constant. Then it follows:

(1) Every mero solution to (5.1) is of finite order $\rho(f) = \deg Q$.
(2) Every mero solution to (5.1) satisfies $\lambda(f) = \rho(f)$ iff $P$ is non-vanishing.
(3) Every mero solution to (5.1) is of the form $f(z) = e^{\alpha(z)}$, $\alpha$ poly iff $P$ vanishes.
(4) If $f$ is an exponential polynomial solution of the form $f(z) = \Sigma_{j=1}^{k} P_j(z)e^{Q_j(z)}$, then $f$ takes the form $f(z) = e^{\alpha(z)} + d$, $\alpha$ being a non-constant polynomial and $d$ constant. Moreover, if $d \neq 0$, then $\rho(f) = 1$.

Similar results to (5.1) easily follow with the left hand side being of delay-differential type instead of being of difference type only. Indeed, considering

$$f(z)^n + q(z)e^{Q(z)}f^{(k)}(z+c) = P(z).$$

and comparing to 5.1, essentially the same observations hold with using $p(z)e^{\alpha(z)}$ instead of $e^{\alpha(z)}$, and $\tilde{p}(z)$ instead of $d$; $p, \tilde{p}$ being now polynomials.

In immediate question is now what happens if the right hand side of (5.1) is an exponential polynomial instead of being a polynomial ? In fact, a number of (relatively) recent papers appear, where the right hand side is, e.g. $p_1(z)e^{\lambda_1 z} + p_2(z)e^{\lambda_2 z}$, or $p_1(z)e^{\lambda_1 z} + p_2(z)e^{\lambda_2 z} + p_3(z)e^{\lambda_3 z}$ etc.

Proceeding to the general case, where the right hand side is an exponential polynomial, a natural device to attack the situation is to apply value distribution results due to Steinmetz [11]: Let

$$w(z) := H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}$$

be an exponential polynomial so that $H_j(z)$ are either exponential polynomials of degree $< q$ or ordinary polynomials. Let for $W \subset \mathbb{C}$, $co(W)$ be the intersection of all convex sets containing $W$, and $C(co(W))$ be the length of the perimeter of $co(W)$. Consider now $W := \{\overline{\omega_1}, \ldots, \overline{\omega_m}\}$ and $W_0 := \{0, \overline{\omega_1}, \ldots, \overline{\omega_m}\}$. Then

$$T(r, w) = C(co(W_0))\frac{r^q}{2\pi},$$

Moreover,

$$m\left(r, \frac{1}{w}\right) = o(r^q),$$

whenever $H_0(z) \equiv 0$ and

$$N\left(r, \frac{1}{w}\right) = C(co(W))\frac{r^q}{2\pi} + o(r^q),$$

whenever $H_0(z) \not\equiv 0$.

As an example on may consider equation of type

$$f(z)^n + P(z)f^{(k)}(z+c) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \quad (5.2)$$

where $P, H_j$ are entire functions of order $< q$. Suppose now $f$ is a meromorphic solution to (5.2) with $N(r, f) = S(r, f)$. Then, provided $n \geq 3$,

(1) if $H_0$ is non-vanishing, then $\lambda(f) = \rho(f) = q$ and $n \leq m + 4$ and

(2) if $H_0$ vanishes, and whenever $\lambda(f) < \rho(f) = q$ or $n > m + 3$, then $m = 2$ and, essentially, $f(z) = g(z)e^{\omega_2 z^q}$, where $g(z)$ is a small exponential.

What about more general cases with the left-hand side being a delay-differential polynomial in $f$ and the right-hand side of being as in (5.2):

$$f(z)^n + g(f(z)) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q},$$

$$g(f(z)) := \sum_{j=0}^{k} b_j(z)f^{(k_j)}(z + c_j),$$

with small coefficients. In general, as far as I know, this situation remains open. While making use of Steinmetz' results, this means reasoning that would be partially analytic and partially geometric. As an immediate observation is that whenever $co(W)$ is outside of the origin, then we clearly have $C(co(W)) < C(co(W_0)$. Let us assume this situation, and let us consider

$$F(z) := f(z)^n g(f(z)) - b(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \quad (5.3)$$

assuming that $N(r, f) = S(r, f)$. Recall that $\lambda(F) = \lambda(LHS) = \rho(LHS) = \rho(F) = \rho(RHS) = q$. Moreover, provided that $H_0(z)$ is non-vanishing, then

$$C(co(W)) = tC(co(W_0)) < C(co(W_0)), \qquad t \in (0, 1).$$

Then

$$N(r, 1/F) = N(r, 1/LHS) = N(r, 1/RHS) = C(co(W))\frac{r^q}{2\pi} + o(r^q)$$

$$= tC(co(W_0))\frac{r^q}{2\pi} + o(r^q) = (t+o(1))T(r, RHS) = (t+o(1))T(r, LHS) = (t+o(1))T(r, F).$$

This implies $\delta(0, F) = 1 - t \in (0, 1)$. Moreover, as pointed out before, we have

$$\lambda(F(z)) = \rho(F(z)), \qquad \delta(0, F(z)) = t\rho(F(z)), \qquad t < 1.$$

Consider next

$$F(z + c) = f(z + c)^n g(f(z + c)) - b(z + c)$$

$$= H_0(z + c) + H_1(z + c)e^{\omega_1 (z+c)^q} + \cdots + H_m(z + c)e^{\omega_m (z+c)^q}$$

$$= H_{0,c}(z) + H_{1,c}(z)e^{\omega_1 z^q} + \cdots + H_{m,c}(z)e^{\omega_m z^q}.$$

Therefore,

$$\lambda(F(z + c)) = \rho(F(z + c)) = \rho(F(z)),$$

$$\delta(0, F(z + c)) = t\rho(F(z + c)) = t\rho(F(z))$$

and so $\delta(0, F(z + c)) = \delta(0, F(z))$.

More analysis to delay-differential equations of type

$$F(z) := f(z)^n g(f(z)) - b(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}.$$

is definitely needed. Another similar equation, not yet analyzed so far I know, is

$$f(z)^n + g(f(z)) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}.$$

## 6. Example

Let me close this presentation by looking at the following simple example, call it $(*)$:

$$f' = (1 + e^{iz} + e^{2z}) - f^2.$$

Suppose a solution $f$ to $(*)$ is meromorphic. Observe that for $A(z) := 1 + e^{iz} + e^{2z})$, we have

$$T(r, A) = (3 + \sqrt{5})\frac{r}{2\pi}, \qquad N(r, 1/A) = 2\sqrt{5}\frac{r}{2\pi}$$

and

$$\delta(0, A) = \frac{3 - \sqrt{5}}{3 + \sqrt{5}}.$$

Although these observations are immediate to obtain, to solve $(*)$ is by no means clear. In fact, mathematical software cannot do this. To look at the situation a bit further, it is immediate to see that all poles of $f$ are simple, of residue $= -1$. Then there exists an entire function $g$ such that $f = -g'/g$ and

$$g'' + (1 + e^{iz} + e^{2z})g = 0.$$

Since $1 + e^{iz} + e^{2z}$ is an entire function, it is well known that there exists two linearly independent solutions $g_1, g_2$ and then $f_{1,2} = -g'_{1,2}/g_{1,2}$ are meromorphic solutions to $(*)$. Does there exist more meromorphic solutions to $(*)$? If so, it is well known that all meromorphic solutions to $(*)$ form a one-parameter family. To obtain some idea of what might happen, let us look at

$$g'' + (H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q})g = 0$$

in general. Clearly, $g$ cannot be of finite order. Indeed, if so, looking at

$$g''/g = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q},$$

we immediately have

$$O(\log r) = m(r, LHS) = m(r, RHS) = T(r, RHS) = C(co(W_0))\frac{r^q}{2\pi},$$

a contradiction. What about trying

$$g(z) = \exp(\mathcal{H}_0(z) + \mathcal{H}_1(z)e^{\omega_1 z^q} + \cdots + \mathcal{H}_m(z)e^{\omega_m z^q}) \qquad ?$$

Then $g''/g$ becomes an exponential polynomial as well. Trying to compare the left- and right-hand sides by making use of the Steinmetz results follows, as mentioned before, a reasoning to apply both analysis and geometry.

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