Entire solutions of linear systems of moment differential equations and related asymptotic growth at infinity

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Outline of the talk

Introduction and motivation

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In [1], D. Matignon describes the stability properties of linear systems of fractional differential equations of the form

$$\partial_z^\alpha y(z) = Ay(z),$$

where $A \in \mathbb{C}^{n\times n}$ is a constant matrix,
∂_z^α is a fractional derivative of order $\alpha > 0$,
$y(z) = (y_1, \ldots, y_n)^T$ stands for a vector of unknown functions.

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The solutions of the previous system turn out to be asymptotically stable, i.e.

$$\lim_{t\to\infty, t\in\mathbb{R}} \|y(t)\| = 0,$$

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$$\lim_{t\to\infty, t\in\mathbb{R}} \|y(t)\| = 0,$$

iff $|\arg(\lambda)| > \alpha\frac{\pi}{2}$ for every eigenvalue λ of A.
In this situation, the solutions decay to 0 at infinity like $t^{-\alpha}$, $t \in \mathbb{R}_+$.

Introduction and motivation

Observe that in the case $\alpha = 1$ one recovers the classical problem

$$y' = Ay,$$

whose general solution is written as a linear combination of functions

$$z^p e^{\lambda z} v$$

for $\lambda \in \mathrm{spec}(A)$, v an associated eigenvector, and for all $p = 0, 1, \dots$

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The solutions are asymptotically stable iff $|\arg(\lambda)| > \frac{\pi}{2}$ for $\lambda \in \mathrm{spec}(A)$:

For $t > 0$,

$$|t^p e^{\lambda t}| = t^p \exp(|\lambda| t \cos(\arg(\lambda))).$$

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Let $A \in \mathbb{C}^{n\times n}$ and let $m = (m(p))_{p\geq 0}$ denote some sequence of positive numbers under certain assumptions.

We consider the linear system of moment differential equations with constant coefficients

$$\partial_m y = Ay$$

$y = (y_1, \ldots, y_n)^T$ is a vector of unknown functions,
∂_m stands for the moment differential operator associated to m.

Introduction and motivation

Given a sequence of positive real numbers $(m_p)_{p\geq 0}$ one can (formally) define the operator

$$\partial_m \left(\sum_{p\geq 0} \frac{a_p}{m_p} z^p \right) = \sum_{p\geq 0} \frac{a_{p+1}}{m_p} z^p,$$

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acting on any formal power series with complex coefficients.

∂_m is known as the moment derivative operator.

∂_m was firstly defined by W. Balser and M. Yoshino in [2].

[2] W. Balser, M. Yoshino, Gevrey order of formal power series solutions of inhomogeneous partial differential equations with constant coefficients. Funkcial. Ekvac. 53, 411–434 (2010)

Introduction and motivation

It holds that

$$\partial_m(x^p) = \begin{cases} 0 & p = 0 \\ \frac{m_p}{m_{p-1}} x^{p-1} & p \geq 1 \end{cases}$$

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Some of the most important situations in applications are those in which $m = (m_p)_{p \geq 0}$ is a sequence of moments associated to some measure.

- Case $m = (p!)_{p \geq 0}$. ∂_m coincides with usual derivation.

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$$p! = \int_0^\infty x^p e^{-x} dx, \qquad p \geq 0.$$

- Case $m = (\Gamma(1+\alpha p))_{p\geq 0}$, for fixed $\alpha > 0$. ∂_m is directly linked with Caputo fractional derivative.

$$\Gamma(1+\alpha p) = \int_0^\infty x^p \frac{1}{\alpha} x^{\frac{1}{\alpha}} e^{-x^{\frac{1}{\alpha}}} dx.$$

The formal differential operator satisfies

$$(\partial_m f)(z^\alpha) = {}^C D_z^\alpha (f(z^\alpha)), \qquad f \in \mathbb{C}[[z]],$$

where ${}^C D_z^\alpha$ stands for Caputo fractional derivative.

Introduction and motivation

- Case $m = ([p]_q!)_{p\geq 0}$, where $[p]_q!$ stands for the $q-$factorial defined by $[0]_q! := 1$ and $[p]_q! := [p]_q \cdot [p-1]_q \cdot \ldots \cdot [1]_q$ for any positive integer $p \geq 1$. Here, $q \in \mathbb{R}_+ \setminus \{1\}$, and we write $[p]_q = 1 + q + \ldots + q^{p-1}$.

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  The $q-$derivative is defined by

$$D_{q,z}f(z) = \frac{f(qz) - f(z)}{qz - z},$$

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  This last sequence is quite related (in terms of the growth of their coefficients) to the sequence of moments $(q^{p(p-1)/2})_{p\geq 0}$ associated to

  $$\sqrt{2\pi \ln(q)} \exp(\frac{\ln^2(\sqrt{q}x)}{2\ln(q)}) \quad \text{and also} \quad \ln(q)/\Theta_{1/q}(x),$$

  where $\Theta_{1/q}$ stands for Jacobi Theta function

  $$\Theta_{1/q}(z) = \sum_{p\in\mathbb{Z}} q^{-\frac{p(p-1)}{2}} z^p.$$

Introduction and motivation

We are mainly working with m being a sequence of moderate growth. The growth of $(q^{p(p-1)/2})_{p\geq 0}$ and $([p]_q!)_{p\geq 0}$ is not of moderate growth. Therefore, the results presented do not apply in that setting, in principle.

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Some remarks on this situation:

- The construction of the entire solutions explained in the talk can be adapted when considering an adequate kernel function.

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Some remarks on this situation:

- The construction of the entire solutions explained in the talk can be adapted when considering an adequate kernel function.

- The asymptotic study of the entire solutions can also be adapted. However, this is well-known also in a more general framework [3].

[3] J. P. Ramis, About the growth of entire functions solutions of linear algebraic $q-$difference equations, Annales de la faculté des sciences de Toulouse, 1, 53–94 (1992)

Introduction and motivation

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A coherent definition of moment derivatives has successfully been developed for other families of functions.

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- Moment derivation at $z_0 \in \mathbb{C}$ of holomorphic functions defined on some open set containing z_0 (Taylor expansion)
- Holomorphic functions defined on sectors of the complex plane which are sums of formal power series [4],

  [4] A. L., S. Michalik, M. Suwińska, Summability of formal solutions for some generalized moment partial differential equations. Result. Math. 76, No. 1 (2021) Paper No. 22.

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- Holomorphic functions defined on sectors of the complex plane which are multisums of formal power series [5].

  [5] A. L., S. Michalik, M. Suwińska, Multisummability of formal solutions for a family of generalized singularly perturbed moment differential equations. Result. Math. 78, No. 2, Paper No. 49, 31 p. (2023).

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The procedure is linked to the summability theory of formal solutions of functional equations.

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An adaptation of the classical Borel-Laplace procedure has been put forward in this more general framework [6,7].

[6] J. Sanz, Flat functions in Carleman ultraholomorphic classes via proximate orders. J. Math. Anal. Appl. 415(2), 623–643 (2014)

[7] A. L., S. Malek, J. Sanz, Summability in general Carleman ultraholomorphic classes. J. Math. Anal. Appl. 430, 1175–1206 (2015)

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The solution of the main problem is based on the construction of the so-called pair of kernel functions for generalized summability, (e, E), associated to a strongly regular sequence $\mathbb{M}$.

Strongly regular sequences

Strongly regular sequences

Let $\mathbb{M} = (M_p)_{p\geq 0}$ be a sequence of positive real numbers. We assume $M_0 = 1$ and such that:

(lc) $M_p^2 \leq M_{p-1}M_{p+1}$, for all $p \geq 1$.

(mg) There exists $A_1 > 0$ such that $M_{p+q} \leq A_1^{p+q} M_p M_q$ for any $(p,q) \in \mathbb{N}_0^2$.

(snq) There exists $A_2 > 0$ such that

$$\sum_{q\geq p} \frac{M_q}{(q+1)M_{q+1}} \leq A_2 \frac{M_p}{M_{p+1}}, \qquad p \geq 0.$$

$\mathbb{M}$ is said to be a strongly regular sequence [8].

[8] V. Thilliez, Division by flat ultradifferentiable functions and sectorial extensions. Results. Math. 44, 169–188 (2003).

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The previous properties determine good properties of ultraholomorphic and ultradifferentiable function spaces such as closure with respect to product and composition, and the existence of flat functions.

Kernel functions for generalized summability

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Let $M : [0,\infty) \to [0,\infty)$ be $M(0) = 0$ and

$$M(x) := \sup_{p\geq 0} \log\left(\frac{x^p}{M_p}\right), \qquad x > 0,$$

and also $\omega(\mathbb{M}) > 0$ with

$$\omega(\mathbb{M}) := \left(\limsup_{r\to\infty} \max\left\{0, \frac{\log(M(r))}{\log(r)}\right\}\right)^{-1}.$$

✓ $\omega(\mathbb{M}) < 2$.

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- ✓ $\omega(\mathbb{M}) < 2$.
- ✓ $\mathbb{M}$ admits a nonzero proximate order. (prof. Okada's talk)

Example: $\left(p!^{\alpha} \prod_{m=0}^{p} \log^{\beta}(e+m)\right)_{p\geq 0}$ for every choice of $0 < \alpha < 2$ and $\beta \in \mathbb{R}$, modifying the first terms in case $\beta < 0$.

Kernel functions for generalized summability

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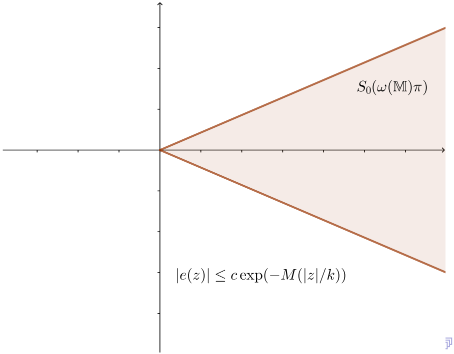
Let $\mathbb{M} = (M_p)_{p \geq 0}$ be a sequence of positive real numbers which is a strongly regular sequence admitting a proximate order, and with $\omega(\mathbb{M}) < 2$.

Then, following [6], there exists a pair of kernel functions for generalized sumability (e, E) such that

- $e \in \mathcal{O}(S_0(\omega(\mathbb{M})\pi))$, and $e(z)/z$ is locally uniformly integrable at the origin. Moreover, for every $\epsilon > 0$ there exist $c, k > 0$ such that

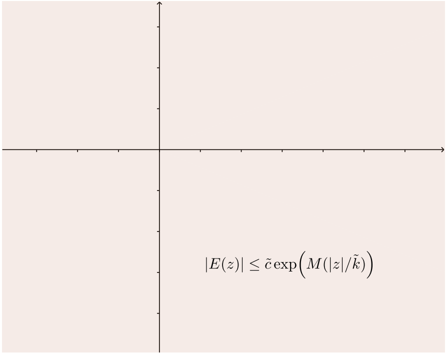
$$|e(z)| \leq c \exp\left(-M(|z|/k)\right), \qquad z \in S_0(\omega(\mathbb{M})\pi - \epsilon),$$



Kernel functions for generalized summability

- $E \in \mathcal{O}(\mathbb{C})$, and there exist $\tilde{c}, \tilde{k} > 0$ such that

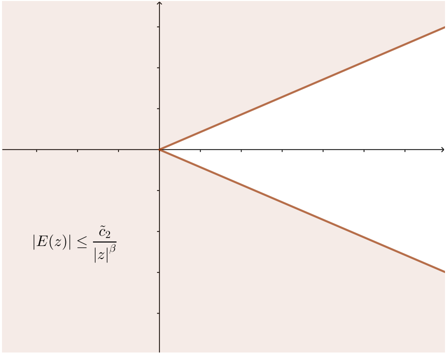
$$|E(z)| \leq \tilde{c} \exp\left(M(|z|/\tilde{k})\right), \qquad z \in \mathbb{C}.$$



Kernel functions for generalized summability

- There exists $\beta > 0$ such that for all $0 < \tilde{\theta} < 2\pi - \omega(\mathbb{M})\pi$ and $M_E > 0$, there exist $\tilde{c}_2 > 0$ with

$$|E(z)| \leq \tilde{c}_2/|z|^{\beta}, \qquad z \in S_{\pi}(\tilde{\theta}) \setminus D(0, M_E).$$



Kernel functions for generalized summability

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They are the cornerstone for the construction of Laplace-like operators involved in the mentioned generalized theory of summability of formal solutions to functional equations in the complex domain, showing good properties with respect to moment differential operators.

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Mellin transform of e determines the moment function

$$m(z) := \int_0^\infty t^{z-1} e(x) dx,$$

with $m \in \mathcal{O}(\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\})$ and continuous up to its boundary.

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The kernel function E is represented in the form

$$E(z) = \sum_{p \geq 0} \frac{z^p}{m(p)}, \qquad z \in \mathbb{C}.$$

Kernel functions for generalized summability

Classical Kernels	Generalized Kernels
e^{-z}	$e(z)$
$p! = \int_0^\infty x^p e^{-x} dx$	$m(p) = \int_0^\infty x^p e(x) dx$
$e^z = \sum_{p\geq 0} \frac{z^p}{p!}$	$E(z) = \sum_{p\geq 0} \frac{z^p}{m(p)}$
∂_z	∂_m

Some words on $q-$Gevrey settings

Although the sequences $(q^{\frac{n(n-1)}{2}})_{p\geq 0}$ and $([p]_q!)_{p\geq 0}$ are not strongly regular sequences, the construction of kernel functions in the $q-$Gevrey settings can follow the parallel theory of $q-$Gevrey asymptotic expansions by means of a $q-$Laplace transform in the sense of [9].

[9] C. Zhang, Transformations de $q-$Borel-Laplace au moyen de la fonction thêta de Jacobi. C. R. Acad. Sci. Paris, t. 331, Série 1, 31–34 (2000)

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$$q^{\frac{n(n-1)}{2}} = \frac{q}{\ln(q)} \int_0^{\infty(d)} \frac{t^n}{\Theta_{1/q}(qt)} dt.$$

The problem

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It is natural to ask the following questions:

- What is the form of its solutions?
- How do solutions behave?

The solution

The solution

Lemma

Let $(z_0, y_0) \in \mathbb{C}^{1+n}$. The Cauchy problem

$$\begin{cases} \partial_m y &= Ay \\ y(z_0) &= y_0 \end{cases}$$

admits a unique solution which is a vector of entire functions.

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The set of solutions to $\partial_m y = Ay$ is a subspace of $(\mathcal{O}(\mathbb{C}))^n$ of dimension n.

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Lemma

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$ with associated eigenvector $v \in \mathbb{C}^n$. The following properties hold:

- If $\lambda \neq 0$, then the function $y(z) = E(\lambda z)v$ is an entire solution of $\partial_m y = Ay$.
- If $\lambda = 0$, then the constant function $y(z) = v$ is a solution of $\partial_m y = Ay$.

The solution

Theorem

Let $A \in \mathbb{C}^{n \times n}$ be a diagonalizable matrix.

Let $\{\lambda_j\}_{1 \leq j \leq k}$, for some $1 \leq k \leq n$ be the set of eigenvalues of A, and let $\{v_{j,1}, \ldots, v_{j,\ell_j}\}$ be a basis of $\mathrm{Ker}(A - \lambda_j I_n)$ for every $1 \leq j \leq k$, and some $\ell_j \geq 1$.

Then, the general solution of $\partial_m y = Ay$ is given by

$$y(z) = \sum_{j=1}^{k} \sum_{p=1}^{\ell_j} C_{j,p} E(\lambda_j z) v_{j,p},$$

with $C_{j,p}$ being arbitrary constants.

The solution

Theorem

Let $\{\lambda_j\}_{1\leq j\leq k}$ for some $1 \leq k \leq n$ be the set of eigenvalues of A.
Assume that λ_j is an eigenvalue of algebraic multiplicity $m_j \geq 1$, for every $1 \leq j \leq k$. Then, the general solution of $\partial_m y = Ay$ can be written in the form

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- $y_{j,p}$ are entire functions determined from the Jordan decomposition of A.

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where $C_{j,p}$ are arbitrary constants.

- $y_{j,p}$ are entire functions determined from the Jordan decomposition of A.
- In this procedure, the entire functions

$$\Delta_h E(\lambda, z) = \sum_{p\geq h} \binom{p}{h} \frac{\lambda^{p-h} z^p}{m(p)}$$

appear in their construction. These functions satisfy

$$(\partial_m - \lambda)(\Delta_h E(\lambda, z)) = \Delta_{h-1} E(\lambda, z).$$

The solution

In the classical case, $m = (p!)_{p \geq 0}$, one has that

$$\Delta_h E(\lambda, z) = \frac{z^h}{h!} \exp(\lambda z),$$

recovering the classical construction of the solution in terms of generalized eigenvectors of A and the Jordan decomposition of A.

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In the case of Caputo fractional derivatives, $m = (\Gamma(1 + \frac{p}{k}))_{p \geq 0}$, one has that

$$h! \Delta_h E(\lambda, t^{1/k}) = t^{h/k} \left(\frac{d^h}{dz^h} E_{1/k} \right) (\lambda t^{1/k}),$$

for all $h \geq 0$, where $E_{1/k}(z)$ is Mittag-Leffler function

$$E_{1/k}(z) = \sum_{p \geq 0} \frac{z^p}{\Gamma(1 + p/k)}.$$

Asymptotic study of the solution

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Based on the relation between growth of an entire function and growth rate of its Taylor coefficients, [10], one has the following result:

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Theorem

Let $y(z) = (y_1(z), \ldots, y_n(z))$ be the solution of any Cauchy problem associated to equation $\partial_m y = Ay$. Then,

- There exist $C_1, C_2 > 0$ such that

$$|y_j(z)| \leq C_1 \exp(M(C_2|z|)), \qquad 1 \leq j \leq n, \quad z \in \mathbb{C}.$$

- If $\lambda = 0$ is the only eigenvalue of A, then $y(z)$ has a polynomial growth at infinity. In addition to this, there exists $C > 0$ such that

$$|y_j(z)| \leq C|z|^{n-1}, \qquad 1 \leq j \leq n, \quad z \in \mathbb{C}.$$

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The order of f:

$$\rho = \rho_f := \limsup_{r\to\infty} \frac{\ln^+(\ln^+(M_f(r)))}{\ln(r)}.$$

Given $f \in \mathcal{O}(\mathbb{C})$ of order $\rho \in \mathbb{R}$, the type of f is defined by

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For $f(z) = \exp(\sigma z^\rho)$, $\rho, \sigma > 0$,

$$M_f(r) = \max\{\exp(\sigma r^\rho \cos(\rho\theta)) : \theta \in \mathbb{R}\} = \exp(\sigma r^\rho),$$

$$\rho_f = \lim_{r\to\infty} \frac{\ln^+(\sigma r^\rho)}{\ln(r)} = \rho, \qquad \sigma_f = \lim_{r\to\infty} \frac{\sigma r^\rho}{r^\rho} = \sigma.$$

Asymptotic study of the solution

Theorem

Let $\mathbb{M}$ be a strongly regular sequence which admits a nonzero proximate order, say $\rho(t) \to \rho > 0$, for $t \to \infty$. Let $y = y(z)$ be a solution of $\partial_m y = Ay$. Then,

- if A admits a nonzero eigenvalue, then y is an entire function of order ρ and type upper bounded by $\sigma := \max\{|\lambda|^\rho : \lambda \in \mathrm{spec}(A)\}$, or an entire function of order 0.
- if 0 is the only eigenvalue of A, then y is a polynomial. Therefore, its order is zero.

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A result on the radial growth of the solution in the case of a diagonalizable matrix A has also been achieved [11].

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A general study remains open.

Asymptotic study of the solution

Proposition

Let $\mathbb{M}$ be a strongly regular sequence which admits a nonzero proximate order, say $\rho(t) \to \rho > 0$, for $t \to \infty$. Let $y = y(z)$ be a solution of $\partial_m y = Ay$, where A is a diagonalizable matrix. Then,

$$\left\| y(re^{i\theta}) \right\| \leq \frac{C}{r^{\beta}}, \quad r \geq R_0,$$

for some $C, \beta, R_0 > 0$, provided that θ belongs to the set

$$\bigcap_{j=1}^{k} \left\{ \theta \in \mathbb{R} : \frac{\omega(\mathbb{M})\pi}{2} < \theta + \arg(\lambda_j) < 2\pi - \frac{\omega(\mathbb{M})\pi}{2} \right\},$$

whenever this set is not empty.

Asymptotic study of the solution

Let the indicator of $f \in \mathcal{O}(\mathbb{C})$ be $h_f(\theta) = \limsup_{r\to\infty} \frac{\ln|f(re^{i\theta})|}{r^{\rho(r)}}$.

Proposition

Under the previous assumptions, one has that

$$h_y(\theta) \leq \max\{|\lambda_j|^\rho h_E(\theta + \arg(\lambda_j)) : \lambda_j \in \mathrm{spec}(A)\}.$$

Further advances

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Moment matrix exponential

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Given $m = (m(p))_{p\geq 0}$ as before, one can formally define the m-exponential of the matrix A by

$$E(A) := E_m(A) = \sum_{p\geq 0} \frac{1}{m(p)} A^p \in \mathbb{C}[[A]].$$

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It holds that

- If $C \in \mathbb{C}^{n\times n}$ is invertible, and $A = CBC^{-1}$, then $E(A) = CE(B)C^{-1}$.
- $E(A+B) \neq E(A)E(B)$, in general. It is also false when A and B commute.
- $E(\text{diag}(\lambda_1, \dots, \lambda_n)) = \text{diag}(E(\lambda_1), \dots, E(\lambda_n))$.
- $\partial_m E(Az) = AE(Az)$.

Further advances

If

$$\lim_{p\to\infty} \inf (m(p))^{1/p} = +\infty$$

holds, then $E(Az)$ defines an entire function (with values in the Banach space $\mathbb{C}^{n\times n}$).

Further advances

If

$$\lim_{p\to\infty} \inf (m(p))^{1/p} = +\infty$$

holds, then $E(Az)$ defines an entire function (with values in the Banach space $\mathbb{C}^{n\times n}$). Also, $\partial_m E(Az)$ determines an entire function.

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holds, then $E(Az)$ defines an entire function (with values in the Banach space $\mathbb{C}^{n\times n}$). Also, $\partial_m E(Az)$ determines an entire function.

This is a natural assumption as any (lc) sequence with

$$sup_{p\geq 1} m_p^{1/p} < \infty$$

determines a class of ultradifferentiable functions which is contained in the class of analytic functions.

Further advances

Theorem

Let $A \in \mathbb{C}^{n \times n}$. The general solution of $\partial_m y = Ay$ is

$$y(z) = E(Az)c,$$

where c is an n-dimensional constant column vector.

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As a consequence, given any fundamental matrix $X(z) \in (\mathcal{O}(\mathbb{C}))^{n \times n}$ associated to $\partial_m y = Ay$, then

$$E(Az) = X(z)X(0)^{-1}.$$