On the Periodicity of Entire Functions and their Differential Polynomials

Zinelaabidine LATREUCH

Jointly with: M-A. Zemirni and I. Laine

National Higher School of Mathematics (Algeria)

August 28, 2023

CDDE II, Bedlewo

NATIONAL HIGHER SCHOOL
OF MATHEMATICS
المدرسة الوطنية
العليا في الرياضيات

Introduction and Motivation

Let's consider the functional equation

$$\boxed{Q(f(z)) = g(z),} \tag{1}$$

where $f(z)$, $g(z)$ are entire functions and $Q(z)$ is a non-constant polynomial.

Introduction and Motivation

Let's consider the functional equation

$$\boxed{Q(f(z)) = g(z),} \tag{1}$$

where $f(z)$, $g(z)$ are entire functions and $Q(z)$ is a non-constant polynomial.

Question 1

Given that $g(z)$ is periodic function of period c, what can be said about the solutions $f(z)$ of (1)?

Introduction and Motivation

Let's consider the functional equation

$$\boxed{Q(f(z)) = g(z),} \tag{1}$$

where $f(z)$, $g(z)$ are entire functions and $Q(z)$ is a non-constant polynomial.

Question 1

Given that $g(z)$ is periodic function of period c, what can be said about the solutions $f(z)$ of (1)?

$$\boxed{Q(f(z+c)) = Q(f(z)).}$$

Introduction and Motivation

In 1965, Alfréd and Catherine Rényi gave an answer to this question.

Theorem (Rényi & Rényi, 1965)

Let $Q(z)$ be an non-constant polynomial and $f(z)$ be an entire function. If $Q(f(z))$ is a periodic function, then $f(z)$ must be periodic.

A. Rényi and C. Rényi, *Some remarks on periodic entire functions*. J. Anal. Math. **14**(1) (1965), 303–310.

Introduction and Motivation

Consider now the differential equation

$$\boxed{f(z)f''(z) = -\sin^2(z).} \tag{2}$$

Introduction and Motivation

Consider now the differential equation

$$f(z)f''(z) = -\sin^2(z). \tag{2}$$

Theorem (Titchmarsh, 1939)

The differential equation (2) has no real entire solutions of finite order other than $f(z) = \pm\sin(z)$.

Titchmarsh, E. C., *The Theory of Functions*, second edition. Oxford University Press, Oxford, 1939.

Introduction and Motivation

Consider now the differential equation

$$\boxed{f(z)f''(z) = -\sin^2(z).} \tag{2}$$

Theorem (Titchmarsh, 1939)

The differential equation (2) has no real entire solutions of finite order other than $f(z) = \pm\sin(z)$.

Titchmarsh, E. C., *The Theory of Functions*, second edition. Oxford University Press, Oxford, 1939.

Theorem (Li, Lü, Yang, 2019)

The differential equation (2) has no entire solutions other than $f(z) = \pm\sin(z)$.

Introduction and Motivation

$$\boxed{f(z)f''(z) = -\sin^2(z) \implies f(z) = \pm\sin(z).}$$

Introduction and Motivation

$$\boxed{f(z)f''(z) = -\sin^2(z) \implies f(z) = \pm\sin(z).}$$

Yang's Conjecture

Let $f(z)$ be a transcendental entire function and k be a positive integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

Introduction and Motivation

$$f(z)f''(z) = -\sin^2(z) \implies f(z) = \pm\sin(z).$$

Yang's Conjecture

Let $f(z)$ be a transcendental entire function and k be a positive integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

Remark

Obviously, Yang's Conjecture is also related to the difference equation

$$f(z)f^{(k)}(z) = f(z+c)f^{(k)}(z+c),$$

Phenomenon of periodicity

Example

The periodic function $f(z) = e^{z/4} + e^{-z/4}$ satisfies the differential equation

$$f(z)^4 - 64f(z)f''(z) + 2 = e^z + e^{-z}.$$

Phenomenon of periodicity

Example

The periodic function $f(z) = e^{z/4} + e^{-z/4}$ satisfies the differential equation

$$f(z)^4 - 64f(z)f''(z) + 2 = e^z + e^{-z}.$$

Question 2

Can we replace the polynomial $Q(z)$ in Rényi & Rényi's results and $f(z)f^{(k)}(z)$ in Yang's conjecture with a general differential polynomial

$$P(z,f) = \sum_{j=1}^{l} a_j(z) f^{n_{0j}} (f')^{n_{1j}} \ldots (f^{(k)})^{n_{1j}} ?$$

Too good to be true

Example

(1) The function $f(z) = \exp(e^{2\pi iz} - z)$ is not periodic whereas the polynomial

$$P(z, f) := e^{2z} f(z)^2 + e^z f(z)$$

is periodic.

Too good to be true

Example

(1) The function $f(z) = \exp(e^{2\pi iz} - z)$ is not periodic whereas the polynomial

$$P(z,f) := e^{2z}f(z)^2 + e^z f(z)$$

is periodic.

(2) The function $f(z) = ze^z$ is not periodic whereas the differential polynomial

$$P(z,f) := (f'(z))^2 - f(z)f''(z) = e^{2z}$$

is periodic.

Let's Revise the Question

Therefore, the natural way to deal with the aforementioned question is to consider the following problem, instead.

Problem

Under what conditions the periodicity of a differential polynomial $P(z, f)$ implies that of $f(z)$?

Nevanlinna's theory

- For every $r \geq 0$, let $n(r, f)$ be the number of poles, counting multiplicity, of f in the disc $|z| \leq r$. Then define the integrated counting function by

Nevanlinna's theory

- For every $r \geq 0$, let $n(r, f)$ be the number of poles, counting multiplicity, of f in the disc $|z| \leq r$. Then define the integrated counting function by

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r.$$

Nevanlinna's theory

- For every $r \geq 0$, let $n(r,f)$ be the number of poles, counting multiplicity, of f in the disc $|z| \leq r$. Then define the integrated counting function by

$$N(r,f) = \int_0^r (n(t,f) - n(0,f))\frac{dt}{t} + n(0,f)\log r.$$

- Similarly,

$$N(r,1/f) = \int_0^r (n(t,1/f) - n(0,1/f))\frac{dt}{t} + n(0,1/f)\log r$$

where $n(r,1/f)$ is the number of zeros, counting multiplicity, of f in the disc $|z| \leq r$.

Nevanlinna's theory

- Let $\log^+ x = \max\{\log x, 0\}$. Then, the proximity function is defined by

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta.$$

Nevanlinna's theory

- Let $\log^+ x = \max\{\log x, 0\}$. Then, the proximity function is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta.$$

- Finally, define the Nevanlinna characteristic function by

Nevanlinna's theory

- Let $\log^+ x = \max\{\log x, 0\}$. Then, the proximity function is defined by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta.$$

- Finally, define the Nevanlinna characteristic function by

$$T(r,f) = m(r,f) + N(r,f).$$

Nevanlinna's theory

- Let $\log^+ x = \max\{\log x, 0\}$. Then, the proximity function is defined by

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta.$$

- Finally, define the Nevanlinna characteristic function by

$$T(r,f) = m(r,f) + N(r,f).$$

- We say that $a(z)$ is small function of f if $T(r,a) = S(r,f)$, where

$$S(r,f) = o(T(r,f)), \quad r \to \infty$$

outside of a possible exceptional set of finite linear measure.

Nevanlinna's theory

First Main Theorem of Nevanlinna

For an arbitrary meromorphic function $f(z)$ and for an arbitrary $a \in \mathbb{C}$,

$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1).$$

Nevanlinna's theory

First Main Theorem of Nevanlinna

For an arbitrary meromorphic function $f(z)$ and for an arbitrary $a \in \mathbb{C}$,

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1).$$

Second Main Theorem of Nevanlinna

Let $f(z)$ be a non-constant meromorphic function, let $q \geq 2$, and let $a_1, \ldots, a_q$ be distinct complex constants. Then

$$(q-1)T(r,f) \leq N(r,f) + \sum_{k=1}^{q} N\left(r, \frac{1}{f-a_k}\right) + S(r,f).$$

Nevanlinna's theory

- The order and the hyper-order of a meromorphic function $f(z)$ are defined, respectively, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

Nevanlinna's theory

- The order and the hyper-order of a meromorphic function $f(z)$ are defined, respectively, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

- The convergence exponent of a-points of f is defined as

$$\lambda(f-a) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{f-a}\right)}{\log r}.$$

Nevanlinna's theory

- The order and the hyper-order of a meromorphic function $f(z)$ are defined, respectively, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

- The convergence exponent of a-points of f is defined as

$$\lambda(f-a) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{f-a}\right)}{\log r}.$$

- If $\rho_2(f) < 1$, then

$$T(r, f(z+c)) = T(r, f(z)) + S(r,f),$$

Some Previous Results

- $P(z,f) = (f^n(z))^{(k)}$

Theorem (Wang-Hu & Liu, 2019)

Let $f(z)$ be a transcendental entire function and k be a positive integer. If $(f^n(z))^{(k)}$ is a periodic function, then $f(z)$ is also a periodic function.

Some Previous Results

- $P(z, f) = (f^n(z))^{(k)}$

Theorem (Wang-Hu & Liu, 2019)

Let $f(z)$ be a transcendental entire function and k be a positive integer. If $(f^n(z))^{(k)}$ is a periodic function, then $f(z)$ is also a periodic function.

- $P(z, f) = (Q(f))^{(k)}$, where $Q(z)$ is a polynomial

Theorem (Wei, Liu & Liu, 2020)

Let $f(z)$ be a transcendental entire function and k be a positive integer. If $(Q(f(z)))^{(k)}$ is a periodic function and $n \geq 2$, then f is also a periodic function.

Some previous results

- $P(z,f) = f^n + L(z,f)$, where $L(z,f)$ is a linear differential polynomial

Let

$$\boxed{L(z,f) = a_1 f'(z) + \cdots + a_k f^{(k)}(z),}$$

where $a_1, \cdots, a_k$ are constants.

Theorem (Lü & Zhang, 2020)

Let $f(z)$ be a transcendental entire function. If $f(z)^n + L(z,f)$ is a periodic function with period c, and if one of the following conditions holds

(1) $n = 2$ *or* $n \geq 4$,

(2) $n = 3$ *and* $\rho_2(f) < 1$,

then $f(z)$ is periodic of period c or nc.

Some previous results

- $P(z, f) = f^n f^{(k)}$

Picard exceptional value

We say $a \in \mathbb{C}$ is a Picard exceptional value of the entire function $f(z)$ if the $f(z) - a$ doesn't have zeros.

Some previous results

- $P(z, f) = f^n f^{(k)}$

Picard exceptional value

We say $a \in \mathbb{C}$ is a Picard exceptional value of the entire function $f(z)$ if the $f(z) - a$ doesn't have zeros.

Theorem (Liu et al, 2019; Latreuch & Zemirni, 2022)

Let $f(z)$ be a transcendental entire function and n, k be positive integers. Suppose that $f(z)^n f^{(k)}(z)$ is a periodic function with period c, and one of the following holds:

(i) *$f(z)$ has the value 0 as a Picard exceptional value, and $\rho_2(f) < \infty$.*

(ii) *$f(z)$ has a nonzero Picard exceptional value.*

Then $f(z)$ is periodic of period c or $(n+1)\, c$.

Some Previous Results

Take

$$f(z) = (e^z - 1)e^{e^z} + d$$

Some Previous Results

Take

$$f(z) = (e^z - 1)e^{e^z} + d$$

$$\lambda(f - d) = \lambda(e^z - 1) = 1 < \rho(f) = \infty.$$

Some Previous Results

Take

$$f(z) = (e^z - 1)e^{e^z} + d$$

$$\lambda(f - d) = \lambda(e^z - 1) = 1 < \rho(f) = \infty.$$

Theorem (Lü & Zhang, 2020)

Let $f(z)$ be a transcendental entire function with $\rho_2(f) < 1$ and n, k be positive integers. If there is a constant d such that

$$\lambda(f - d) < \rho(f) \leq \infty$$

and $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is a periodic function as well.

$P(z, f)$ is a differential monomial

A differential monomial $M(z, f)$ is defined by

$$\boxed{M(z, f) = f(z)^{\lambda_0} \left(f'(z)\right)^{\lambda_1} \cdots \left(f^{(n)}(z)\right)^{\lambda_n},}$$

where $\lambda_0, \ldots, \lambda_n$ are non-negative integers.

$P(z, f)$ is a differential monomial

A differential monomial $M(z, f)$ is defined by

$$\boxed{M(z,f) = f(z)^{\lambda_0}\left(f'(z)\right)^{\lambda_1}\cdots\left(f^{(n)}(z)\right)^{\lambda_n},}$$

where $\lambda_0, \ldots, \lambda_n$ are non-negative integers. The quantities

$$\gamma_M := \lambda_0 + \cdots + \lambda_n \quad \text{and} \quad \Gamma_M := \lambda_1 + 2\lambda_2 + \cdots + n\lambda_n$$

are called the **degree** and the **weight** of $M(z, f)$, respectively.

$P(z, f)$ is a differential monomial

Theorem (Zemirni, Laine & Latreuch, 2023)

Let $f(z)$ be a transcendental entire function with $\lambda(f) < \rho(f) \leq \infty$. If $M(z, f)$ is a periodic function with period c, then the following holds:

1. *If $\rho_2(f) < 1$, then $f(z) = e^{az+b}$, where $a, b \in \mathbb{C} \setminus \{0\}$ and $e^{\gamma_M ac} = 1$.*
2. *If $1 \leq \rho_2(f) < \infty$ and $\lambda(f) < \rho_2(f)$, then $f(z)$ is c-periodic.*

$P(z, f)$ is a differential monomial

Theorem (Zemirni, Laine & Latreuch, 2023)

Let $f(z)$ be a transcendental entire function with $\lambda(f) < \rho(f) \leq \infty$. If $M(z, f)$ is a periodic function with period c, then the following holds:

1. *If $\rho_2(f) < 1$, then $f(z) = e^{az+b}$, where $a, b \in \mathbb{C} \setminus \{0\}$ and $e^{\gamma_M ac} = 1$.*
2. *If $1 \leq \rho_2(f) < \infty$ and $\lambda(f) < \rho_2(f)$, then $f(z)$ is c-periodic.*

Theorem (Zemirni, Laine & Latreuch, 2023)

Let $f(z)$ be a transcendental entire function, and suppose there exists a constant $d \neq 0$ such that $\lambda(f - d) < \rho(f) \leq \infty$. If $M(z, f)$ is a periodic function with period c and $\lambda_0 > 0$, then $f(z)$ is c-periodic.

Improving Rényi's result once more

Let

$$Q(f) = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f(z)^{\nu_s}, \quad l \geq 2,\ \nu_1 < \cdots < \nu_l,$$

where $\alpha_{\nu_s}(z)$ are non-vanishing small functions.

Improving Rényi's result once more

Let

$$Q(f) = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f(z)^{\nu_s}, \quad l \geq 2,\ \nu_1 < \cdots < \nu_l,$$

where $\alpha_{\nu_s}(z)$ are non-vanishing small functions.

Theorem

Let $f(z)$ be a transcendental entire function with $N(r, 1/f) = S(r, f)$. If $Q(f)$ is a periodic function of period c, then

1. *The terms $\alpha_{\nu_s}(z) f(z)^{\nu_s}$ are periodic of period c.*

Improving Rényi's result once more

Let

$$Q(f) = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f(z)^{\nu_s}, \quad l \geq 2,\ \nu_1 < \cdots < \nu_l,$$

where $\alpha_{\nu_s}(z)$ are non-vanishing small functions.

Theorem

Let $f(z)$ be a transcendental entire function with $N(r, 1/f) = S(r, f)$. If $Q(f)$ is a periodic function of period c, then

1. *The terms $\alpha_{\nu_s}(z) f(z)^{\nu_s}$ are periodic of period c.*
2. *For any distinct $m, n \in \{1, \ldots, l\}$ for which $\nu_m \nu_n > 0$, the functions*

$$F_{m,n}(z) := \frac{\alpha_{\nu_m}(z)^{\nu_n}}{\alpha_{\nu_n}(z)^{\nu_m}}$$

are periodic of period c.

Improving Rényi's result once more

Proof.

- Since $Q(f)$ is periodic of period c, it follows that

$$\sum_{s=1}^{l} \alpha_{\nu_s}(z+c) f_c^{\nu_s} = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f^{\nu_s}, \quad f_c := f(z+c). \tag{3}$$

Improving Rényi's result once more

Proof.

- Since $Q(f)$ is periodic of period c, it follows that

$$\sum_{s=1}^{l} \alpha_{\nu_s}(z+c) f_c^{\nu_s} = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f^{\nu_s}, \quad f_c := f(z+c). \tag{3}$$

- $T(r, f_c) \sim T(r, f)$ as $r \to \infty$.

Improving Rényi's result once more

Proof.

- Since $Q(f)$ is periodic of period c, it follows that

$$\sum_{s=1}^{l} \alpha_{\nu_s}(z+c) f_c^{\nu_s} = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f^{\nu_s}, \quad f_c := f(z+c). \tag{3}$$

- $T(r, f_c) \sim T(r, f)$ as $r \to \infty$.
- Dividing both sides of (3) by $\alpha_{\nu_m}(z) f(z)^{\nu_m}$ yields

$$\sum_{s=1}^{l} \frac{\alpha_{\nu_s}(z+c)}{\alpha_{\nu_m}(z)} \frac{f_c^{\nu_s}}{f^{\nu_m}} - \sum_{\substack{s=1 \\ s \neq m}}^{l} \frac{\alpha_{\nu_s}(z)}{\alpha_{\nu_m}(z)} f^{\nu_s - \nu_m} = 1.$$

Improving Rényi's result once more

Proof.

- Since $Q(f)$ is periodic of period c, it follows that

$$\sum_{s=1}^{l} \alpha_{\nu_s}(z+c) f_c^{\nu_s} = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f^{\nu_s}, \quad f_c := f(z+c). \tag{3}$$

- $T(r, f_c) \sim T(r, f)$ as $r \to \infty$.
- Dividing both sides of (3) by $\alpha_{\nu_m}(z) f(z)^{\nu_m}$ yields

$$\sum_{s=1}^{l} \frac{\alpha_{\nu_s}(z+c)}{\alpha_{\nu_m}(z)} \frac{f_c^{\nu_s}}{f^{\nu_m}} - \sum_{\substack{s=1 \\ s \neq m}}^{l} \frac{\alpha_{\nu_s}(z)}{\alpha_{\nu_m}(z)} f^{\nu_s - \nu_m} = 1.$$

- Using Nevanlinna's reasoning

$$\frac{\alpha_{\nu_m}(z+c)}{\alpha_{\nu_m}(z)} \left(\frac{f_c}{f} \right)^{\nu_m} \equiv 1, \quad m \in \{1, \dots, l\}.$$

Improving Rényi's result once more

Corollary

Let $f(z)$ be a transcendental entire function with $N(r, 1/f) = S(r, f)$. If $Q(f) = \sum_{s=1}^{l} \alpha_{\nu_s}(z) f(z)^{\nu_s}$ is a periodic function of period c, then the statements below are equivalent:

(i) One coefficient $\alpha_{\nu_s}(z)$ with $\nu_s > 0$ is c-periodic;

(ii) All the coefficients are c-periodic;

(iii) $f(z)$ is c-periodic.

Another Key Lemma

Lemma

Let $v(z) \not\equiv 0$ be a meromorphic function of order $\rho(v) < \infty$, and $g(z)$ be a non-constant entire function. If $F(z) = v(z)e^{g(z)}$ is a periodic function of period τ, then either

- $\rho(g) \geq 1$, or

Another Key Lemma

Lemma

Let $v(z) \not\equiv 0$ be a meromorphic function of order $\rho(v) < \infty$, and $g(z)$ be a non-constant entire function. If $F(z) = v(z)e^{g(z)}$ is a periodic function of period τ, then either

- $\rho(g) \geq 1$, or
- $g(z)$ is polynomial with

$$\begin{cases} \rho(v) \geq \deg(g), & \text{if} \quad \deg(g) \geq 2; \\ v(z+\tau)/v(z) \text{ is constant}, & \text{if} \quad \deg(g) = 1. \end{cases}$$

Preparing for the proofs

- Let $f(z)$ be a transcendental entire function with $\lambda(f-d) < \rho(f)$.

Preparing for the proofs

- Let $f(z)$ be a transcendental entire function with $\lambda(f-d) < \rho(f)$.
- Then,

$$f(z) = \pi(z)e^{h(z)} + d$$

where $h(z)$ is an entire function and $\pi(z)$ is the canonical product of zeros of $f(z) - d$ with $\rho(\pi) < \rho(f)$.

Preparing for the proofs

- Let $f(z)$ be a transcendental entire function with $\lambda(f-d) < \rho(f)$.
- Then,

$$f(z) = \pi(z)e^{h(z)} + d$$

where $h(z)$ is an entire function and $\pi(z)$ is the canonical product of zeros of $f(z) - d$ with $\rho(\pi) < \rho(f)$.

- Notice that

$$\left(\pi(z)e^{h(z)}\right)^{(k)} = \left(\pi(z)h'(z)^k + \mathcal{Q}_k(\pi, h')\right) e^{h(z)}, \quad k \in \mathbb{N},$$

where $\mathcal{Q}_k(\pi, h')$ is a differential polynomial in $\pi(z)$ and $h'(z)$ with constant coefficients.

Preparing for the proofs

- Substituting $f(z) = \pi(z)e^{h(z)} + d$ into $M(z, f)$ yields

$$M(z, f) = H(z)\left(1 + \frac{d}{\pi(z)}e^{-h(z)}\right)^{\lambda_0} e^{\gamma_M h(z)}.$$

Preparing for the proofs

- Substituting $f(z) = \pi(z)e^{h(z)} + d$ into $M(z, f)$ yields

$$M(z, f) = H(z) \left(1 + \frac{d}{\pi(z)} e^{-h(z)}\right)^{\lambda_0} e^{\gamma_M h(z)}.$$

$$H(z) = \pi(z)^{\lambda_0} \prod_{k=1}^{n} \mathcal{L}_k(\pi, h')^{\lambda_k}$$

and $\mathcal{L}_k(\pi, h') = \pi(z)\, h'(z)^k + \mathcal{Q}_k(\pi, h')$.

Case $d = 0$

- Here we have $d = 0$, and therefore $M(z, f) = H(z)e^{\gamma_M h(z)}$,

Case $d = 0$

- Here we have $d = 0$, and therefore $M(z, f) = H(z)e^{\gamma_M h(z)}$,
- If $\rho_2(f) < 1$, then $h(z) = az + b$, $\rho(\pi) < 1$ and

$$H(z) = \pi(z)^{\lambda_0} \prod_{k=1}^{n} \mathcal{L}_k(\pi, h')^{\lambda_k} \equiv Cst$$

Case $d = 0$

- Here we have $d = 0$, and therefore $M(z, f) = H(z)e^{\gamma_M h(z)}$,
- If $\rho_2(f) < 1$, then $h(z) = az + b$, $\rho(\pi) < 1$ and

$$H(z) = \pi(z)^{\lambda_0} \prod_{k=1}^{n} \mathcal{L}_k(\pi, h')^{\lambda_k} \equiv Cst$$

- Notice that

$$\mathcal{L}_k(\pi, h') := \pi^{(k)}(z) + c_{k-1}\pi^{(k-1)}(z) + \ldots + c_1\pi(z),$$

and $\rho(\mathcal{L}_k) \leq \rho(\pi) < 1$.

Case $d = 0$

- Here we have $d = 0$, and therefore $M(z, f) = H(z)e^{\gamma_M h(z)}$,
- If $\rho_2(f) < 1$, then $h(z) = az + b$, $\rho(\pi) < 1$ and

$$H(z) = \pi(z)^{\lambda_0} \prod_{k=1}^{n} \mathcal{L}_k(\pi, h')^{\lambda_k} \equiv Cst$$

- Notice that

$$\mathcal{L}_k(\pi, h') := \pi^{(k)}(z) + c_{k-1}\pi^{(k-1)}(z) + \ldots + c_1\pi(z),$$

and $\rho(\mathcal{L}_k) \leq \rho(\pi) < 1$.

- If now $\lambda_0 > 0$, then $\pi(z)$ is a constant. Otherwise,

$$\pi^{(k+1)}(z) + c_{k-1}\pi^{(k)}(z) + \ldots + c_1\pi'(z) = 0.$$

Case $d = 0$

- Suppose now that $1 \leq \rho_2(f) < \infty$ and $\lambda(f) < \rho_2(f)$. Then $\rho(\pi) < \infty$, and hence $\rho(H) < \infty$.

$$e^{\gamma_M q(z)} = \frac{H(z)}{H(z+c)}, \quad q(z) = h(z+c) - h(z). \tag{4}$$

Case $d = 0$

- Suppose now that $1 \leq \rho_2(f) < \infty$ and $\lambda(f) < \rho_2(f)$. Then $\rho(\pi) < \infty$, and hence $\rho(H) < \infty$.

$$e^{\gamma_M q(z)} = \frac{H(z)}{H(z+c)}, \quad q(z) = h(z+c) - h(z). \tag{4}$$

- $q(z)$ is polynomial. If $\deg(q) = t \geq 1$, then

$$\pi_c(z)^{\gamma_M} = e^{-\gamma_M q(z)} \pi(z)^{\gamma_M},$$

which implies that $f_c(z)^{\gamma_M} = f(z)^{\gamma_M}$. Hence, $f(z)$ is periodic of period c or $\gamma_M c$.

Case $d = 0$

- Suppose now that $1 \leq \rho_2(f) < \infty$ and $\lambda(f) < \rho_2(f)$. Then $\rho(\pi) < \infty$, and hence $\rho(H) < \infty$.

$$e^{\gamma_M q(z)} = \frac{H(z)}{H(z+c)}, \quad q(z) = h(z+c) - h(z). \tag{4}$$

- $q(z)$ is polynomial. If $\deg(q) = t \geq 1$, then

$$\pi_c(z)^{\gamma_M} = e^{-\gamma_M q(z)} \pi(z)^{\gamma_M},$$

which implies that $f_c(z)^{\gamma_M} = f(z)^{\gamma_M}$. Hence, $f(z)$ is periodic of period c or $\gamma_M c$.

- If $\deg(q) = t = 0$ we apply he same arguments as for the case $t \geq 1$.

Case $d \neq 0$

- Since $\lambda_0 > 0$, it follows that

$$M(z,f) = H(z) \sum_{i=0}^{\lambda_0} \binom{\lambda_0}{i} \left(\frac{d}{\pi(z)} \right)^i e^{(\gamma_M - i)h(z)}. \quad (5)$$

Case $d \neq 0$

- Since $\lambda_0 > 0$, it follows that

$$M(z,f) = H(z) \sum_{i=0}^{\lambda_0} \binom{\lambda_0}{i} \left(\frac{d}{\pi(z)} \right)^i e^{(\gamma_M - i)h(z)}. \quad (5)$$

- $T(r,\pi) = S(r,e^h)$ and consequently $T(r,H) = S(r,e^h)$.

Case $d \neq 0$

- Since $\lambda_0 > 0$, it follows that

$$M(z,f) = H(z) \sum_{i=0}^{\lambda_0} \binom{\lambda_0}{i} \left(\frac{d}{\pi(z)} \right)^i e^{(\gamma_M - i)h(z)}. \tag{5}$$

- $T(r,\pi) = S(r,e^h)$ and consequently $T(r,H) = S(r,e^h)$.
- Thus, $M(z,f)$ can be regarded as a polynomial in $e^{h(z)}$ with small coefficients. Hence

$$H(z+c)e^{\gamma_M h_c(z)} = H(z)e^{\gamma_M h(z)}$$

and

$$\frac{H(z+c)}{\pi_c(z)} e^{(\gamma_M - 1)h_c(z)} = \frac{H(z)}{\pi(z)} e^{(\gamma_M - 1)h(z)}.$$

Case $d \neq 0$

- Since $\lambda_0 > 0$, it follows that

$$M(z,f) = H(z) \sum_{i=0}^{\lambda_0} \binom{\lambda_0}{i} \left(\frac{d}{\pi(z)} \right)^i e^{(\gamma_M - i)h(z)}. \tag{5}$$

- $T(r,\pi) = S(r,e^h)$ and consequently $T(r,H) = S(r,e^h)$.
- Thus, $M(z,f)$ can be regarded as a polynomial in $e^{h(z)}$ with small coefficients. Hence

$$H(z+c)e^{\gamma_M h_c(z)} = H(z)e^{\gamma_M h(z)}$$

and

$$\frac{H(z+c)}{\pi_c(z)} e^{(\gamma_M - 1)h_c(z)} = \frac{H(z)}{\pi(z)} e^{(\gamma_M - 1)h(z)}.$$

- we get that $\pi(z)e^{h(z)}$ is periodic of period c. Thus $f(z)$ is c-periodic.

$P(z,f)$ is a differential polynomial with at least two terms

$$P(z,f)=\sum_{j=1}^{m}\alpha_j M_j(z,f).$$

$$M_j(z,f)=f(z)^{\lambda_{0j}}\left(f'(z)\right)^{\lambda_{1j}}\cdots\left(f^{(n)}(z)\right)^{\lambda_{nj}},\quad \lambda_{0j},\ldots,\lambda_{nj}\in\mathbb{N}.$$

The total degree γ_P and total weight Γ_P of $P(z,f)$ are defined by

$$\gamma_P=\max_{1\leq j\leq m}\gamma_j \quad \text{and} \quad \Gamma_P=\max_{1\leq j\leq m}\Gamma_j.$$

More Terms, more Challenges

Example

The function $f(z) = e^{z^2}$ is not periodic and

$$P(z, f) = f'(z)^2 f(z) - f''(z) f(z)^2 + 2f(z)^3 \equiv 0.$$

More Terms, more Challenges

Example

The function $f(z) = e^{z^2}$ is not periodic and

$$P(z,f) = f'(z)^2 f(z) - f''(z) f(z)^2 + 2f(z)^3 \equiv 0.$$

Example

The function $f(z) = ze^z + d$, where d is a constant, is not periodic whereas the differential polynomial

$$P(z,f) := (f'(z))^2 - f(z)f''(z) + df(z) = (e^z - d)^2$$

is periodic.

New Notations

Define the sequence of positive integers $\delta_1, \delta_2, \ldots, \delta_l$ as follows:

$$\delta_1 = \min_j \gamma_j,$$

$$\delta_2 = \min_j \{\gamma_j : \gamma_j \neq \delta_1\},$$

$$\vdots$$

$$\delta_l = \min_j \{\gamma_j : \gamma_j \neq \delta_i, i = 1, \ldots, l-1\} = \gamma_P.$$

New Notations

Define the sequence of positive integers $\delta_1, \delta_2, \ldots, \delta_l$ as follows:

$$\delta_1 = \min_j \gamma_j,$$

$$\delta_2 = \min_j \{\gamma_j : \gamma_j \neq \delta_1\},$$

$$\vdots$$

$$\delta_l = \min_j \{\gamma_j : \gamma_j \neq \delta_i, i = 1, \ldots, l-1\} = \gamma_P.$$

We denote by $\Lambda(\delta_i)$ the set that contains the indices of the terms in

$$P(z,f) = \sum_{j=1}^{m} \alpha_j M_j(z,f).$$

with the highest weights among those of degree δ_i.

New Notations

Define the sequence of positive integers $\delta_1, \delta_2, \ldots, \delta_l$ as follows:

$$\delta_1 = \min_j \gamma_j,$$

$$\delta_2 = \min_j \{\gamma_j : \gamma_j \neq \delta_1\},$$

$$\vdots$$

$$\delta_l = \min_j \{\gamma_j : \gamma_j \neq \delta_i, i = 1, \ldots, l-1\} = \gamma_P.$$

We denote by $\Lambda(\delta_i)$ the set that contains the indices of the terms in

$$P(z,f) = \sum_{j=1}^{m} \alpha_j M_j(z,f).$$

with the highest weights among those of degree δ_i.

$$\Lambda_P = \left\{ \delta_i : \sum_{k \in \Lambda(\delta_i)} \alpha_k \neq 0 \right\}$$

New Notations

Define the sequence of positive integers $\delta_1, \delta_2, \ldots, \delta_l$ as follows:

$$\delta_1 = \min_j \gamma_j,$$

$$\delta_2 = \min_j \{\gamma_j : \gamma_j \neq \delta_1\},$$

$$\vdots$$

$$\delta_l = \min_j \{\gamma_j : \gamma_j \neq \delta_i, i = 1, \ldots, l-1\} = \gamma_P.$$

We denote by $\Lambda(\delta_i)$ the set that contains the indices of the terms in

$$P(z,f) = \sum_{j=1}^{m} \alpha_j M_j(z,f).$$

with the highest weights among those of degree δ_i.

$$\Lambda_P = \left\{ \delta_i : \sum_{k \in \Lambda(\delta_i)} \alpha_k \neq 0 \right\} \subset \{\delta_1, \delta_2, \ldots, \delta_l\}.$$

New notations

Example

Let $P(z,f)=\sum_{i=0}^{n} f^{(i)}(z)$.

New notations

Example

Let $P(z,f)=\sum_{i=0}^{n} f^{(i)}(z)$. That is,

$$P(z,f)=\underbrace{f(z)}_{(\gamma=1,\Gamma=0)}+\underbrace{f'(z)}_{(\gamma=1,\Gamma=1)}+\underbrace{f''(z)}_{(\gamma=1,\Gamma=2)}+\ldots+\underbrace{f^{(n)}(z)}_{(\gamma=1,\Gamma=n)}.$$

New notations

Example

Let $P(z,f)=\sum_{i=0}^{n} f^{(i)}(z)$. That is,

$$P(z,f)=\underbrace{f(z)}_{(\gamma=1,\Gamma=0)}+\underbrace{f'(z)}_{(\gamma=1,\Gamma=1)}+\underbrace{f''(z)}_{(\gamma=1,\Gamma=2)}+\ldots+\underbrace{f^{(n)}(z)}_{(\gamma=1,\Gamma=n)}.$$

$$\gamma_1=\gamma_P=\delta_1=1 \quad \text{and} \quad \Gamma_i=i \implies \Lambda(\delta_1)=\{n\} \implies \sum_{k\in\Lambda(\delta_1)}\alpha_k=1.$$

New notations

Example

Let $P(z,f)=\sum_{i=0}^{n} f^{(i)}(z)$. That is,

$$P(z,f)=\underbrace{f(z)}_{(\gamma=1,\Gamma=0)}+\underbrace{f'(z)}_{(\gamma=1,\Gamma=1)}+\underbrace{f''(z)}_{(\gamma=1,\Gamma=2)}+\ldots+\underbrace{f^{(n)}(z)}_{(\gamma=1,\Gamma=n)}.$$

$$\gamma_1=\gamma_P=\delta_1=1 \quad \text{and} \quad \Gamma_i=i \implies \Lambda(\delta_1)=\{n\} \implies \sum_{k\in\Lambda(\delta_1)}\alpha_k=1.$$

$$\Lambda_P=\{\delta_1\}=\{1\}.$$

New notations

Example

Consider now

$$P(z,f) = \underbrace{f''(z)f(z)^2}_{(\gamma=3,\Gamma=2)} - \underbrace{2(f'(z))^2 f(z)}_{(\gamma=3,\Gamma=2)} - \underbrace{f'(z)f(z)^2}_{(\gamma=3,\Gamma=1)}$$

$$+\underbrace{f''(z)f(z)}_{(\gamma=2,\Gamma=2)} - \underbrace{(f'(z))^2}_{(\gamma=2,\Gamma=2)} + \underbrace{f''(z)}_{(\gamma=1,\Gamma=2)} + \underbrace{f'(z)}_{(\gamma=1,\Gamma=1)},$$

New notations

Example

Consider now

$$P(z,f) = \underbrace{f''(z)f(z)^2}_{(\gamma=3,\Gamma=2)} - \underbrace{2(f'(z))^2 f(z)}_{(\gamma=3,\Gamma=2)} - \underbrace{f'(z)f(z)^2}_{(\gamma=3,\Gamma=1)}$$

$$+\underbrace{f''(z)f(z)}_{(\gamma=2,\Gamma=2)} - \underbrace{(f'(z))^2}_{(\gamma=2,\Gamma=2)} + \underbrace{f''(z)}_{(\gamma=1,\Gamma=2)} + \underbrace{f'(z)}_{(\gamma=1,\Gamma=1)},$$

where $\delta_1 = 1$, $\delta_2 = 2$ and $\delta_3 = \gamma_P = 3$.

$$\sum_{k\in\Lambda(1)} \alpha_k = 1, \quad \sum_{k\in\Lambda(2)} \alpha_k = 1 - 1 = 0 \quad \text{and} \quad \sum_{k\in\Lambda(3)} \alpha_k = 1 - 2 = -1.$$

New notations

Example

Consider now

$$P(z,f) = \underbrace{f''(z)f(z)^2}_{(\gamma=3,\Gamma=2)} - \underbrace{2(f'(z))^2 f(z)}_{(\gamma=3,\Gamma=2)} - \underbrace{f'(z)f(z)^2}_{(\gamma=3,\Gamma=1)}$$

$$+\underbrace{f''(z)f(z)}_{(\gamma=2,\Gamma=2)} - \underbrace{(f'(z))^2}_{(\gamma=2,\Gamma=2)} + \underbrace{f''(z)}_{(\gamma=1,\Gamma=2)} + \underbrace{f'(z)}_{(\gamma=1,\Gamma=1)},$$

where $\delta_1 = 1$, $\delta_2 = 2$ and $\delta_3 = \gamma_P = 3$.

$$\sum_{k\in\Lambda(1)} \alpha_k = 1, \quad \sum_{k\in\Lambda(2)} \alpha_k = 1 - 1 = 0 \quad \text{and} \quad \sum_{k\in\Lambda(3)} \alpha_k = 1 - 2 = -1.$$

Thus, we have

$$\Lambda_P = \{1,3\}.$$

Why Λ_P?

The function $f(z) = ze^z + 1$ is not periodic whereas the differential polynomial

$$P(z, f) = (f'(z))^2 - f(z)f''(z) + f(z) = (e^z - 1)^2$$

is periodic.

Why Λ_P?

The function $f(z) = ze^z + 1$ is not periodic whereas the differential polynomial

$$P(z, f) = (f'(z))^2 - f(z)f''(z) + f(z) = (e^z - 1)^2$$

is periodic. Note that

$$P(z, f) = \underbrace{(f'(z))^2}_{(\gamma=2,\Gamma=2)} - \underbrace{f(z)f''(z)}_{(\gamma=2,\Gamma=2)} + \underbrace{f(z)}_{(\gamma=1,\Gamma=0)},$$

where $\delta_1 = 1$ and $\delta_2 = \gamma_P = 2$.

$$\boxed{\gamma_P \notin \Lambda_P = \{\delta_1\}}$$

Our results

$$P(z,f)=\sum_{j=1}^{m}\alpha_j f(z)^{\lambda_{0j}}\left(f'(z)\right)^{\lambda_{1j}}\cdots\left(f^{(n)}(z)\right)^{\lambda_{nj}}.$$

Theorem (Zemirni, Laine & Latreuch, 2023)

Let $f(z)$ be a transcendental entire function with $\rho_2(f)<1$, and suppose that there exists $d\in\mathbb{C}$ such that $\lambda(f-d)<\rho(f)\leq\infty$. Suppose that $P(z,f)\not\equiv 0$ is periodic with period c, $\lambda_{0j}>0$ for every $j\in\{1,\ldots,m\}$ and one of the following holds

(i) $d=0$,

(ii) $d\neq 0$ *and* $\lambda_{01}=\cdots=\lambda_{0m}=\lambda>0$,

(iii) $d\neq 0$ *and* $\gamma_P\in\Lambda_P$.

Then $f(z)$ is c-periodic.

Our Results

Corollary

Let $f(z)$ be a transcendental entire function with a finite Picard exceptional value d and $\rho_2(f) < 1$. Suppose that $P(z, f) \not\equiv 0$ is a periodic function with period c. Then $f(z)$ is c-periodic.

Our Results

Theorem (Zemirni, Laine & Latreuch, 2023)

Let $f(z)$ be a transcendental entire function with $1 \leq \rho_2(f) < \infty$, and suppose that there exists $d \in \mathbb{C}$ such that $\lambda(f - d) < \rho_2(f)$. Suppose that $P(z, f)$ is a periodic function with period c, $\Lambda_P \neq \emptyset$ and one of the following holds:

(i) $d = 0$;

(ii) $d \neq 0$ and $\gamma_P \in \Lambda_P$.

Then $f(z)$ is c-periodic.

Our Results

Theorem (Zemirni, Laine & Latreuch, 2023)

Let $f(z)$ be a transcendental entire function with $1 \leq \rho_2(f) < \infty$, and suppose that there exists $d \in \mathbb{C}$ such that $\lambda(f-d) < \rho_2(f)$. Suppose that $P(z,f)$ is a periodic function with period c, $\Lambda_P \neq \emptyset$ and one of the following holds:

(i) $d = 0$;

(ii) $d \neq 0$ *and* $\gamma_P \in \Lambda_P$.

Then $f(z)$ is c-periodic.

Remark

The function $f(z) = e^{\sin z}$ satisfies $\lambda(f) = 0 < 1 = \rho_2(f)$ and

$$P(z,f) = (f'(z))^2 - f''(z)f(z) = e^{2\sin z}\sin z.$$

Here, $P(z,f)$ and $f(z)$ are both periodic while $\Lambda_P = \emptyset$.

Thank You & Good Night!