On sequences preserving summability

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Introduction

1. We introduce sequences preserving summability and describe their properties.
2. We introduce moment differentiation and operators of order zero.
3. We prove the characterisation of sequences preserving summability.
4. We show the sequence $([n]_q!)_{n\geq 0}$ preserves summability for every $q \in [0, 1)$.
5. As an application we characterise summable formal power series solutions of linear q-difference-differential equations with constant coefficients

$$\begin{cases} P(D_{q,t}, \partial_z)u = 0, \\ D^j_{q,t}u(0, z) = \varphi_j(z) \in \mathcal{O}(D) \text{ for } j = 0, \dots, p-1 \end{cases}$$

   in terms of analytic continuation properties and growth estimates of the Cauchy data $\varphi_j(z)$, $j = 0, \dots, p-1$.

Functions of exponential growth

Definition (Function of exponential growth)

Let $\mathbb{E}$ be a Banach space and $\hat{S}_d = S_d \cup D$ be an infinity disc-sector in a direction $d \in \mathbb{R}$. A function $u \in \mathcal{O}(\hat{S}_d, \mathbb{E})$ is of exponential growth of order at most $k \in \mathbb{R}$ as $x \to \infty$ in $\hat{S}_d$ if for any subsector $\tilde{S} \prec \hat{S}_d$ there exist $A, B > 0$ such that

$$\|u(x)\|_{\mathbb{E}} < Ae^{B|x|^k} \quad \text{for every} \quad x \in \tilde{S}.$$

The space of such functions is denoted by $\mathcal{O}^k(\hat{S}_d, \mathbb{E})$.

Borel operator

Definition (Borel operator)

For a fixed sequence $m = (m(n))_{n\geq 0}$ of positive numbers with $m(0) = 1$, a linear operator $\mathcal{B}_{m,t} \colon \mathbb{E}[[t]] \to \mathbb{E}[[t]]$ defined by

$$(\mathcal{B}_{m,t}\hat{u})(t) := \sum_{n=0}^{\infty} \frac{a_n}{m(n)} t^n \quad \text{for} \quad \hat{u}(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{E}[[t]]$$

is called an *m*-Borel operator with respect to *t*.

Remark

Observe that for a given sequence $m = (m(n))_{n\geq 0}$ of positive numbers with $m(0) = 1$, an inverse *m*-Borel operator $\mathcal{B}^{-1}_{m,t} \colon \mathbb{E}[[t]] \to \mathbb{E}[[t]]$, called sometimes an *m*-Laplace operator, is given by $\mathcal{B}^{-1}_{m,t} = \mathcal{B}_{m^{-1},t}$ on $\mathbb{E}[[t]]$, where $m^{-1} = (m(n)^{-1})_{n\geq 0}$. Hence an *m*-Borel operator $\mathcal{B}_{m,t}$ is a linear automorphism on the space of formal power series $\mathbb{E}[[t]]$.

Gevrey order and summability

Let $k > 0$ and $\Gamma_{1/k} := (\Gamma(1 + n/k))_{n\geq 0}$, where $\Gamma(\cdot)$ denotes the Gamma function.

Definition (Gevrey order)

A series $\hat{u} \in \mathbb{E}[[t]]$ is called a formal power series of Gevrey order $1/k$ if there exists a disc $D \subseteq \mathbb{C}$ with centre at the origin such that $\mathcal{B}_{\Gamma_{1/k},t}\hat{u} \in \mathcal{O}(D, \mathbb{E})$.
The space of formal power series of Gevrey order $1/k$ is denoted by $\mathbb{E}[[t]]_{1/k}$.

Definition (k-summability)

Let $d \in \mathbb{R}$. A series $\hat{u} \in \mathbb{E}[[t]]$ is called k-summable in a direction d if there exists a disc-sector $\hat{S}_d$ in a direction d such that $\mathcal{B}_{\Gamma_{1/k},t}\hat{u} \in \mathcal{O}^k(\hat{S}_d, \mathbb{E})$.
The space of k-summable formal power series in a direction d is denoted by $\mathbb{E}\{t\}_{k,d}$.

Sequences preserving summability

Definition (Sequence preserving summability)

Let $m = (m(n))_{n\geq 0}$ be a sequence of positive numbers with $m(0) = 1$. We say that a sequence m preserves summability if for any $k > 0$, $d \in \mathbb{R}$ and any $\hat{u} \in \mathbb{E}[[t]]$ the following equivalence holds:

$$\hat{u} \in \mathbb{E}\{t\}_{k,d} \quad \text{if and only if} \quad \mathcal{B}_{m,t}\hat{u} \in \mathbb{E}\{t\}_{k,d}.$$

Example

1. The sequence $\mathbf{1} = (1)_{n\geq 0}$ preserves summability in a trivial way.
2. More generally, if $a > 0$ and $\mathbf{a} := (a^n)_{n\geq 0}$ then the sequence $\mathbf{a}$ preserves summability, because

$$\mathcal{B}_{\mathbf{a},t}\hat{u}(t) = \hat{u}(t/a) \quad \text{for every} \quad \hat{u} \in \mathbb{E}[[t]].$$

Sequences preserving Gevrey order

Definition (Sequence preserving Gevrey order)

Let $m = (m(n))_{n\geq 0}$ be a sequence of positive numbers with $m(0) = 1$. We say that a sequence m preserves Gevrey order if for any $k > 0$ and any $\hat{u} \in \mathbb{E}[[t]]$ the following equivalence holds:

$$\hat{u} \in \mathbb{E}[[t]]_{1/k} \quad \text{if and only if} \quad \mathcal{B}_{m,t}\hat{u} \in \mathbb{E}[[t]]_{1/k}.$$

Proposition

A sequence $m = (m(n))_{n\geq 0}$ preserves Gevrey order if and only if m is a sequence of order zero
(i.e. there exists $a, A > 0$ such that $a^n \leq m(n) \leq A^n$ for every $n \in \mathbb{N}_0$).

Remark

Since for every $k > 0$ and $d \in \mathbb{R}$ we have $\mathbb{E}\{t\}_{k,d} \subset \mathbb{E}[[t]]_{1/k}$, we see that if a sequence $m = (m(n))_{n\geq 0}$ preserves summability then m also preserves Gevrey order, or equivalently m is a sequence of order 0.

Example

We show that not every sequence of order 0 preserves summability:

Example

Let $m(n) := \begin{cases} 1 & n \text{ is even} \\ 2^{-1} & n \text{ is odd} \end{cases}$. The series $\hat{x}(t) = \sum_{n=0}^{\infty} n! t^n$ is 1-summable in any direction $d \neq 0 \mod 2\pi$, because for $m_1(n) = n!$ and for any $d \neq 0 \mod 2\pi$

$$\mathcal{B}_{m_1,t}\hat{x}(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \in \mathcal{O}^1(\hat{S}_d).$$

On the other hand the series
$\hat{y}(t) = \mathcal{B}_{m,t}\hat{x}(t) = \sum_{n=0}^{\infty} \frac{n!}{m(n)} t^n = \sum_{k=0}^{\infty} (2k)! t^{2k} + \sum_{k=0}^{\infty} 2(2k+1)! t^{2k+1}$ is 1-summable only for directions $d \neq 0 \mod \pi$, because the function

$$\mathcal{B}_{m_1,t}\hat{y}(t) = \sum_{k=0}^{\infty} t^{2k} + \sum_{k=0}^{\infty} 2t^{2k+1} = \frac{1}{1-t^2} + \frac{2t}{1-t^2} = \frac{1+2t}{1-t^2} \in \mathcal{O}^1(\hat{S}_d),\ d \neq 0 \mod \pi$$

has a simple pole not only at $t = 1$, but also at $t = -1$.
Hence $\hat{x}(t) \in \mathbb{C}\{t\}_{1,\pi}$, but $\hat{y}(t) = \mathcal{B}_{m,t}\hat{x}(t) \notin \mathbb{C}\{t\}_{1,\pi}$.

Moment functions

Definition (Kernel functions and corresponding moment function)

A pair of functions $e_{\mathfrak{m}}$ and $E_{\mathfrak{m}}$ is said to be kernel functions of order k ($k > 1/2$) if they have the following properties:

1. $e_{\mathfrak{m}} \in \mathcal{O}(S_0(\pi/k))$, $e_{\mathfrak{m}}(z)/z$ is integrable at the origin, $e_{\mathfrak{m}}(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e_{\mathfrak{m}}$ is exponentially flat of order k in $S_0(\pi/k)$ (i.e. for every $\varepsilon > 0$ there exists $A, B > 0$ such that $|e_{\mathfrak{m}}(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S_0(\pi/k - \varepsilon)$).
2. $E_{\mathfrak{m}} \in \mathcal{O}^k(\mathbb{C})$ and $E_{\mathfrak{m}}(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.
3. The connection between $e_{\mathfrak{m}}$ and $E_{\mathfrak{m}}$ is given by the corresponding moment function $\mathfrak{m}$ of order $1/k$ as follows. The function $\mathfrak{m}$ is defined in terms of $e_{\mathfrak{m}}$ by

$$\mathfrak{m}(u) := \int_0^\infty x^{u-1} e_{\mathfrak{m}}(x)dx \quad \text{for} \quad \mathrm{Re}\, u \geq 0$$

   and the kernel function $E_{\mathfrak{m}}$ has the power series expansion

$$E_{\mathfrak{m}}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\mathfrak{m}(n)} \quad \text{for} \quad z \in \mathbb{C}.$$

4. Additionally we assume that $\mathfrak{m}(u)$ satisfies the normalization condition $\mathfrak{m}(0) = 1$.

Moment functions

Remark

The integral representation for the reciprocal moment function $\mathfrak{m}$ is given by

$$\frac{1}{\mathfrak{m}(u)} = \frac{1}{2\pi i}\int_\gamma E_\mathfrak{m}(w) w^{-u-1} dw$$

with γ as in Hankel's formula of the reciprocal Gamma function [Balser, 2000, p. 228].

Example

The canonical examples of kernel functions $e_\mathfrak{m}$ and $E_\mathfrak{m}$ of order $k > 0$ and the corresponding moment function $\mathfrak{m}$, which are used in the classical theory of k-summability, are given by

- $e_\mathfrak{m}(z) = kz^k e^{-z^k}$,
- $\mathfrak{m}(u) = \Gamma(1 + u/k)$,
- $E_\mathfrak{m}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(1+j/k)} =: \mathbf{E}_{1/k}(z)$, where $\mathbf{E}_{1/k}$ is the Mittag-Leffler function of index $1/k$.

Moment functions

Remark

By [Balser, 2000, Theorems 31 and 32], if $\mathfrak{m}_1(u)$ and $\mathfrak{m}_2(u)$ are moment functions of positive orders $1/k_1$ and $1/k_2$ respectively, then

1. $\mathfrak{m}(u) = \mathfrak{m}_1(u)\mathfrak{m}_2(u)$ is a moment function of order $1/k_1 + 1/k_2$,
2. $\mathfrak{m}(u) = \mathfrak{m}_1(u)/\mathfrak{m}_2(u)$ is a moment function of order $1/k_1 - 1/k_2$ under condition that $1/k_1 > 1/k_2$.

Using the above remark we may extend the definition of moment functions to real order:

Definition

We say that $\mathfrak{m}(u)$ is a moment function of order $1/k < 0$ if $1/\mathfrak{m}(u)$ is a moment function of order $-1/k > 0$.
Moreover, $\mathfrak{m}(u)$ is called a moment function of order 0 if there exist moment functions $\mathfrak{m}_1(u)$ and $\mathfrak{m}_2(u)$ of the same order $1/k > 0$ such that $\mathfrak{m}(u) = \mathfrak{m}_1(u)/\mathfrak{m}_2(u)$.

Moment functions

Remark

Observe that by the definitions any moment function $\mathfrak{m}(u)$ of order $s \in \mathbb{R}$ satisfies conditions

- $\mathfrak{m}(u) > 0$ for every $u \geq 0$,
- $\mathfrak{m}(0) = 1$.

Remark

By the general method of summability (see [Balser, 2000, Section 6.5 and Theorem 38]), in the definition of k-summability one can replace the sequence $\Gamma_{1/k} = (\Gamma(1 + n/k))_{n\geq 0}$ by any sequence $\mathfrak{m} = (\mathfrak{m}(n))_{n\geq 0}$, where $\mathfrak{m}(u)$ is a moment function of order $1/k$.

Hence

Example

For any moment function $\mathfrak{m}(u)$ of order zero, the sequence $(\mathfrak{m}(n))_{n\geq 0}$ preserves summability.

Moment functions

In particular we get

Example

Suppose that

$$\mathfrak{m}(n) = \frac{\Gamma(1+a_1 n)\cdots\Gamma(1+a_k n)}{\Gamma(1+b_1 n)\cdots\Gamma(1+b_l n)} \quad \text{for} \quad n \in \mathbb{N}_0,$$

where $a_1, \dots, a_k$ and $b_1, \dots b_l$ are positive numbers satisfying

$$a_1 + \cdots + a_k = b_1 + \cdots + b_l.$$

Then the sequence $(\mathfrak{m}(n))_{n\geq 0}$ preserves summability.

The group of sequences preserving summability

Remark

The set of sequences preserving summability forms a group $\mathcal{M}$ with a group operation given by the multiplication:

1. If $m_1 = (m_1(n))_{n\geq 0}$ and $m_2 = (m_2(n))_{n\geq 0}$ preserve summability then also their product $m = m_1 \cdot m_2$ (i.e. $m = (m(n))_{n\geq 0}$, where $m(n) = m_1(n) \cdot m_2(n)$ for any $n \in \mathbb{N}_0$) preserves summability.
2. If $m = (m(n))_{n\geq 0}$ preserves summability then also its inverse element $m^{-1} = (m(n)^{-1})_{n\geq 0}$ preserves summability.
3. The identity element $\mathbf{1} = (1)_{n\geq 0}$ preserves summability.

Remark

The set

$$\mathfrak{M} = \left\{ (\mathfrak{m}(n))_{n\geq 0} : \mathfrak{m}(u) \text{ is a moment function of order zero} \right\}$$

forms a subgroup of the group $\mathcal{M}$ of sequences preserving summability.

Moment differentiation

We extend the notion of m-moment differentiation introduced by [Balser, Yoshino, 2010] originally for the sequence of moments $m = (\mathfrak{m}(n))_{n\geq 0}$ inherited from a moment function $\mathfrak{m}(u)$.

Definition (Moment differentiation)

For a given sequence $m = (m(n))_{n\geq 0}$ of positive numbers with $m(0) = 1$, an operator $\partial_{m,t} \colon \mathbb{E}[[t]] \to \mathbb{E}[[t]]$ defined by

$$\partial_{m,t}\Big(\sum_{n=0}^{\infty} \frac{u_n}{m(n)} t^n\Big) := \sum_{n=0}^{\infty} \frac{u_{n+1}}{m(n)} t^n$$

is called an *m-moment differentiation*. If additionally m is a sequence of order 0 then $\partial_{m,t}$ is called an *m-moment differentiation of order 0* or an *operator of order 0* for short.

Remark

Notice that the operator $\partial_{m,t} \colon \mathbb{E}[[t]] \to \mathbb{E}[[t]]$ can be equivalently defined as

$$\partial_{m,t}\Big(\sum_{n=0}^{\infty} a_n t^n\Big) := \sum_{n=0}^{\infty} \frac{m(n+1)}{m(n)} a_{n+1} t^n.$$

Moment differentiation

Remark

Observe that in the most important case $m = (n!)_{n\geq 0}$, the operator $\partial_{m,t}$ is the m-moment differentiation, which coincides with the usual differentiation ∂_t.

By the direct calculation we get

Proposition

Let $m_1 = (m_1(n))_{n\geq 0}$ and $m_2 = (m_2(n))_{n\geq 0}$ be sequences of positive numbers. Then the operators $\mathcal{B}_{m_1,t}, \partial_{m_2,t} \colon \mathbb{E}[[t]] \to \mathbb{E}[[t]]$ commute in a such way that

$$\mathcal{B}_{m_1,t}\partial_{m_2,t} = \partial_{m_1 m_2,t}\mathcal{B}_{m_1,t}.$$

Proposition (The moment Taylor formula)

Let $\hat{\varphi} \in \mathbb{E}[[t]]$ and $m = (m(n))_{n\geq 0}$ be a sequence of positive numbers with $m(0) = 1$. Then

$$\hat{\varphi}(t) = \sum_{n=0}^{\infty} \frac{\partial_{m,t}^n \hat{\varphi}(0)}{m(n)} t^n.$$

Operators of order zero

Examples of operators of order 0:

Example

1. If $\mathbf{1} = (1)_{n\geq 0}$ then $\partial_{\mathbf{1},t}\hat{u}(t) = \frac{\hat{u}(t)-\hat{u}(0)}{t}$ for $\hat{u}(t) \in \mathbb{E}[[t]]$.
   More generally, for every $n \in \mathbb{N}$ we get
   $$\partial^n_{\mathbf{1},t}\hat{u}(t) = \frac{\hat{u}(t) - \sum_{k=0}^{n-1} \frac{\partial_t^k \hat{u}(0)}{k!} t^k}{t^n} \quad \text{for} \quad \hat{u}(t) \in \mathbb{E}[[t]].$$
   Hence we may write the usual Taylor's theorem as
   $$\hat{u}(t) = \sum_{k=0}^{n-1} \frac{\partial_t^k \hat{u}(0)}{k!} t^k + R_n(t),$$
   where the reminder term $R_n(t)$ of the Taylor polynomial is given by $R_n(t) = t^n \partial^n_{\mathbf{1},t}\hat{u}(t)$.
2. If $a > 0$ and $\mathbf{a} = (a^n)_{n\geq 0}$ then $\partial_{\mathbf{a},t}\hat{u}(t) = \frac{a(\hat{u}(t)-\hat{u}(0))}{t} = a\partial_{\mathbf{1},t}\hat{u}(t)$ for $\hat{u}(t) \in \mathbb{E}[[t]]$.
3. If $\mathfrak{m}(u)$ is a moment function of order 0 then $\mathfrak{m} = (\mathfrak{m}(n))_{n\geq 0}$ is a sequence of order 0. Hence $\partial_{\mathfrak{m},t}$ is an operator of order 0.

Operators of order zero

Example

4. Fix $q \in [0,1)$. Let $D_{q,t}$ be the q-difference operator defined by

$$D_{q,t}\hat{u}(t) := \frac{\hat{u}(qt)-\hat{u}(t)}{qt-t} \quad \text{for} \quad \hat{u}(t) \in \mathbb{E}[[t]].$$

Observe that $D_{q,t}t^n = [n]_q t^{n-1}$, where $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$.
It means that

$$D_{q,t}\hat{u}(t) = \partial_{m,t}\hat{u}(t) \quad \text{for} \quad \hat{u}(t) \in \mathbb{E}[[t]],$$

where $m = ([n]_q!)_{n\geq 0}$ and $[n]_q! = [1]_q \cdots [n]_q$.
Since $q \in [0,1)$ we get $1 \leq [n]_q \leq \frac{1}{1-q}$ for every $n \in \mathbb{N}_0$ and we conclude that

$$1 \leq [n]_q! \leq \left(\frac{1}{1-q}\right)^n \quad \text{for every } n \in \mathbb{N}_0.$$

Hence the q-difference operator $D_{q,t}$ is an m-moment differentiation of order 0.
Observe also that in the special case $q = 0$ we get

$$D_{0,t}\hat{u}(t) = \partial_{\mathbf{1},t}\hat{u}(t) = \frac{\hat{u}(t)-\hat{u}(0)}{t}.$$

Sequences preserving summability

Theorem ([Ichinobe, Michalik, 2023])

A sequence $m = (m(n))_{n\geq 0}$ preserves summability if and only if for every $k > 0$ and for every $\theta \neq 0 \mod 2\pi$ there exists a disc-sector $\hat{S}_\theta$ such that

$$\mathcal{B}_{m,t}\Big(\sum_{n=0}^{\infty} t^n\Big) \in \mathcal{O}^k(\hat{S}_\theta) \quad \textit{and} \quad \mathcal{B}_{m^{-1},t}\Big(\sum_{n=0}^{\infty} t^n\Big) \in \mathcal{O}^k(\hat{S}_\theta).$$

Proof.

($\Rightarrow$) Take any $k > 0$ and $\theta \neq 0 \mod 2\pi$. Let $\hat{u}(t) := \sum_{n=0}^{\infty} \Gamma(1+n/k)t^n$. Since $\mathcal{B}_{\Gamma_{1/k},t}\hat{u}(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t} \in \mathcal{O}^k(\hat{S}_\theta)$, we see that $\hat{u}$ is k-summable in a direction θ. It means that also $\mathcal{B}_{m,t}\hat{u}(t)$ and $\mathcal{B}_{m^{-1},t}\hat{u}(t)$ are k-summable in a direction θ for any sequence m preserving summability. Hence we conclude that

$$\mathcal{B}_{m,t}\Big(\sum_{n=0}^{\infty} t^n\Big) = \mathcal{B}_{\Gamma_{1/k},t}(\mathcal{B}_{m,t}\hat{u}) \in \mathcal{O}^k(\hat{S}_\theta), \quad \mathcal{B}_{m^{-1},t}\Big(\sum_{n=0}^{\infty} t^n\Big) = \mathcal{B}_{\Gamma_{1/k},t}(\mathcal{B}_{m^{-1},t}\hat{u}) \in \mathcal{O}^k(\hat{S}_\theta).$$

□

Sequences preserving summability

Proof.

($\Leftarrow$) Take any $k>0$ and $d\in\mathbb{R}$. Assume that $\hat{x}(t)=\sum_{n=0}^{\infty}x_nt^n\in\mathbb{E}[[t]]$ is k-summable in a direction d. It is sufficient to show that also $\mathcal{B}_{m,t}\hat{x}(t)$ and $\mathcal{B}_{m^{-1},t}\hat{x}(t)$ are k-summable in the same direction d.

Since $\hat{x}(t)\in\mathbb{E}\{t\}_{k,d}$, we see that the function $\varphi(t):=\mathcal{B}_{\Gamma_{1/k},t}\hat{x}(t)$ belongs to the space $\mathcal{O}^k(\hat{S}_d,\mathbb{E})$. Let $\hat{u}(t,z)$ be a solution of the Cauchy problem

$$\begin{cases} (\partial_{\tilde{m},t}-\partial_z)u=0\\ u(0,z)=\varphi(z)\in\mathcal{O}^k(\hat{S}_d,\mathbb{E}),\end{cases}$$

where $\tilde{m}=(m(n)n!)_{n\geq 0}$. Then since $\hat{u}(t,z)=\sum_{n\geq 0}\frac{\varphi^{(n)}(z)}{\tilde{m}(n)}t^n$, we have

$$\hat{u}(t,0)=\sum_{n=0}^{\infty}\frac{\varphi^{(n)}(0)}{m(n)n!}t^n=\sum_{n=0}^{\infty}\frac{x_n}{\Gamma(1+n/k)m(n)}t^n=\mathcal{B}_{\Gamma_{1/k},t}(\mathcal{B}_{m,t}\hat{x}(t)),$$

which is convergent at $t=0$. To prove that $\mathcal{B}_{m,t}\hat{x}(t)$ is k-summable in the direction d, it is sufficient to show that $u(t,0)\in\mathcal{O}^k(\hat{S}_d,\mathbb{E})$. □

Sequences preserving summability

Proof.

To this end observe that using the integral representation of u we get

$$u(t,0) = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{\varphi(\zeta)}{\zeta}\Big(\sum_{n=0}^{\infty}\frac{(t/\zeta)^n}{m(n)}\Big)\,d\zeta = \frac{1}{2\pi i}\oint_{|\zeta|=\rho}\frac{\varphi(\zeta)}{\zeta}\psi(t/\zeta)\,d\zeta$$

for sufficiently small $\rho > 0$, where the kernel ψ is defined as

$$\psi(t) := \sum_{n=0}^{\infty}\frac{t^n}{m(n)} = \mathcal{B}_{m,t}\Big(\sum_{n=0}^{\infty} t^n\Big).$$

Observe that $\psi \in \mathcal{O}^k(\hat{S}_\theta)$ for every $\theta \neq 0 \mod 2\pi$ by the assumption. In particular the power series ψ has a positive radius of convergence $r > 0$, i.e. $\psi \in \mathcal{O}(D_r)$.
Since $\varphi \in \mathcal{O}(\hat{S}_d, \mathbb{E})$ and $\psi \in \mathcal{O}(\hat{S}_\theta)$ for every $\theta \neq 0 \mod 2\pi$, we may deform the path of integration from $\zeta \in \partial D_\rho$ to $\zeta \in \Gamma(R) := \partial(\hat{S}_d \cap D_R)$ for some positive R. Then for every fixed $t \in \hat{S}_d$ we may find sufficiently large R such that $|\frac{t}{\zeta}| < r$ for $\zeta \in \Gamma(R)$ such that $\arg\zeta = \arg t$. Hence $u(t,0) \in \mathcal{O}(\hat{S}_d, \mathbb{E})$. □

Sequences preserving summability

Proof.

To estimate $\|u(t,0)\|_{\mathbb{E}}$ for $t \in \hat{S}_d$, $|t| \to \infty$, we split the contour $\Gamma(R)$ into 2 arcs $\Gamma_1(R) := \Gamma(R) \cap (\partial D_R)$ and $\Gamma_2(R) := \Gamma(R) \cap D_R$. Then we get

$$u(t,0) = \frac{1}{2\pi i} \int_{\Gamma_1(R)} \frac{\varphi(\zeta)}{\zeta} \psi(t/\zeta)\, d\zeta + \frac{1}{2\pi i} \int_{\Gamma_2(R)} \frac{\varphi(\zeta)}{\zeta} \psi(t/\zeta)\, d\zeta.$$

If $\zeta \in \Gamma_1(R)$ then $|\zeta| = R$ and $\zeta \in \hat{S}_d$. Taking $R = 2|t|/r$, where the constant $r > 0$ is the radius of convergence of ψ, we see that R and t both go to infinity together and that the function $t \mapsto \psi(t/\zeta)$ is bounded. Since moreover $\varphi \in \mathcal{O}^k(\hat{S}_d, \mathbb{E})$, we conclude that the first integral has exponential growth of order k as $|t| \to \infty$ in $\hat{S}_d$.
To estimate the second integral, observe that if $\zeta \in \Gamma_2(R)$ then $\arg \zeta \neq d \mod 2\pi$. It means that the function $t \mapsto \psi(t/\zeta)$ has exponential growth of order k as $|t| \to \infty$ in $\hat{S}_d$. Since moreover $\varphi \in \mathcal{O}^k(\hat{S}_d, \mathbb{E})$, in this case we also conclude that the second integral has exponential growth of order k as $|t| \to \infty$ in $\hat{S}_d$.
Hence the function $t \mapsto u(t,0)$ has also exponential growth of order k as $|t| \to \infty$ in $\hat{S}_d$ and $\mathcal{B}_{m,t}\hat{x}(t)$ is k-summable in the direction d.
Replacing m by m^{-1} and repeating the above proof we conclude that $\mathcal{B}_{m^{-1},t}\hat{x}(t)$ is also k-summable in the same direction d. □

The sequence $([n]_q!)_{n\geq 0}$ preserves summability

Theorem ([Ichinobe, Michalik, 2023])

If $q \in [0,1)$ then the sequence $([n]_q!)_{n\geq 0}$ preserves summability.

Lemma 1

For every $k > 0$ and every $\theta \neq 0 \mod 2\pi$ there exists a disc-sector $\hat{S}_\theta$ such that

$$x(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \in \mathcal{O}^k(\hat{S}_\theta).$$

Proof of Lemma 1.

Observe that $x(t)$ coincides with the q-exponential function $\exp_q(t)$. By the properties of $\exp_q(t)$ we get the assertion. □

The sequence $([n]_q!)_{n\geq 0}$ preserves summability

Lemma 2

For every $k>0$ and every $\theta \neq 0 \mod 2\pi$ there exists a disc-sector $\hat{S}_\theta$ such that

$$y(t)=\sum_{n=0}^{\infty}[n]_q!t^n \in \mathcal{O}^k(\hat{S}_\theta).$$

Proof of Lemma 2.

We consider the initial value problem

$$\begin{cases} (\partial_t - \partial_{\tilde{m},z})u = 0 \\ u(0,z) = \varphi(z) := \frac{1}{1-z} \end{cases}, \quad \text{where} \quad \tilde{m} = (n![n]_q!)_{n\geq 0}.$$

The formal power series solution is given by $\hat{u}(t,z)=\sum_{n=0}^{\infty}\frac{\partial^n_{\tilde{m},z}\varphi(z)}{n!}z^n$.
By the moment Taylor formula for $\varphi(z)$ we see that

$$\frac{\partial^n_{\tilde{m},z}\varphi(0)}{n![n]_q!}=\frac{\partial^n_{\tilde{m},z}\varphi(0)}{\tilde{m}(n)}=\frac{\varphi^{(n)}(0)}{n!} \quad \text{for every} \quad n\in\mathbb{N}_0.$$

□

The sequence $([n]_q!)_{n\geq 0}$ preserves summability

Proof of Lemma 2.

Hence we conclude that

$$u(t,0)=\sum_{n=0}^{\infty}\frac{\partial^n_{\tilde{m},t}\varphi(0)}{n!}t^n=\sum_{n=0}^{\infty}\frac{[n]_q!\varphi^{(n)}(0)}{n!}t^n=y(t).$$

Since $[n]_q!=\frac{(q;q)_n}{(1-q)^n}$, where $(a;q)_n=(1-q)\cdots(1-q^n)$, by the Cauchy integral formula, we see that

$$y(t)=u(t,0)=\frac{1}{2\pi i}\oint_{|\eta|=\rho}\frac{\varphi(\eta)}{\eta}\sum_{n=0}^{\infty}(q;q)_n\Big(\frac{t}{(1-q)\eta}\Big)^n\,d\eta$$

for sufficiently small $|t|$.
We will follow the proof of necessity in [Ichinobe, Adachi, 2020, Theorem 3.1] with $\kappa=\nu=1$ and $x_0=0$. By Heine's transformation formula we obtain

$$\sum_{n=0}^{\infty}(q;q)_n\Big(\frac{t}{(1-q)\eta}\Big)^n=\frac{\big(q,q\frac{t}{(1-q)\eta};q\big)_\infty}{\big(\frac{t}{(1-q)\eta};q\big)_\infty}\sum_{j=0}^{\infty}\frac{\big(\frac{t}{(1-q)\eta};q\big)_j}{\big(q\frac{t}{(1-q)\eta},q;q\big)_j}q^j.$$

□

The sequence $([n]_q!)_{n\geq 0}$ preserves summability

Proof of Lemma 2.

For fixed $t \neq 0$, the function

$$\eta \longmapsto \frac{1}{\left(\frac{t}{(1-q)\eta};q\right)_\infty} = \prod_{n=0}^{\infty} \frac{\eta}{\eta - (1-q)^{-1}tq^n}$$

is meromorphic on $\mathbb{C}$ with simple poles at

$$\eta = \eta_n(t) := (1-q)^{-1}tq^n \quad \text{for} \quad n \in \mathbb{N}_0.$$

Using the residue theorem we see that

$$y(t) = (q;q)_\infty \sum_{n=0}^{\infty} \varphi\Big(\frac{tq^n}{1-q}\Big) \operatorname*{Res}_{\eta=\eta_n(t)} \frac{1}{\left(\frac{t}{(1-q)\eta};q\right)_\infty} \frac{1}{\eta} (q^{1-n};q)_\infty \sum_{j=0}^{\infty} \frac{(q^{-n};q)_j}{(q^{1-n},q;q)_j} q^j.$$

Since $(q^{-n},q)_j = 0$ for $j > n$ and $\frac{(q^{1-n};q)_\infty}{(q^{1-n};q)_j} = 0$ for $j < n$, we get

$$(q^{1-n};q)_\infty \sum_{j=0}^{\infty} \frac{(q^{-n};q)_j}{(q^{1-n},q;q)_j} q^j = (q^{1-n};q)_\infty \frac{(q^{-n};q)_n}{(q^{1-n},q;q)_n} q^n = (q;q)_\infty \frac{(q^{-n};q)_n}{(q;q)_n} q^n.$$

□

The sequence $([n]_q!)_{n\geq 0}$ preserves summability

Proof of Lemma 2.

Moreover

$$\operatorname*{Res}_{\eta=\eta_n(t)} \frac{1}{\left(\frac{t}{(1-q)\eta};q\right)_\infty}\frac{1}{\eta} = \frac{1}{(q^{-n};q)_n(q;q)_\infty}.$$

Hence

$$y(t) = (q;q)_\infty \sum_{n=0}^{\infty} \varphi\left(\frac{tq^n}{1-q}\right)\frac{q^n}{(q;q)_n}.$$

Since there exist $A, B < \infty$ such that $|\varphi(z)| \leq Ae^{B|z|^k}$ for every $z \in \hat{S}_\theta$, we conclude that

$$|y(t)| \leq Ae^{B(1-q)^{-k}|t|^k}\sum_{n=0}^{\infty}\frac{(q;q)_\infty}{(q;q)_n}q^n \leq Ae^{\tilde{B}|t|^k}\sum_{n=0}^{\infty}q^n \leq \tilde{A}e^{\tilde{B}|t|^k}$$

for some positive constants $\tilde{A}, \tilde{B} < \infty$ and for every $t \in \hat{S}_\theta$. It means that $y(t) \in \mathcal{O}^k(\hat{S}_\theta)$. □

The sequence $([n]_q!)_{n\geq 0}$ preserves summability

Remark

In the similar way one can show that the sequence $([an]_q!)_{n\geq 0}$ preserves summability for every $a \in \mathbb{N}_0$

Example

Suppose that

$$m(n) = \frac{[a_1 n]_q! \cdots [a_k n]_q!}{[b_1 n]_q! \cdots [b_l n]_q!} \quad \text{for} \quad n \in \mathbb{N}_0,$$

where $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$ are natural numbers.
Then the sequence $(m(n))_{n\geq 0}$ preserves summability.

The Cauchy problem for moment operators of order 0

We consider the Cauchy problem for the linear equations $P(\partial_{m,t}, \partial_z)u = 0$ with constant coefficients, where $\partial_{m,t}$ is an operator of order 0. We will show that if additionally a sequence m preserves summability then summable solutions are characterised in the same way as for the solutions of $P(\partial_{\mathbf{1},t}, \partial_z)u = 0$, which is a special case of the equation $P(\partial_{\mathfrak{m}_1,t}, \partial_{\mathfrak{m}_2,z})u = 0$ already studied in [Michalik, 2013] under condition that $\mathfrak{m}_1(u)$ and $\mathfrak{m}_2(u)$ are moment functions of real orders. By the main result of the paper it allows us to characterise summable solutions of general linear q-difference-differential equations $P(D_{q,t}, \partial_z)u = 0$ with constant coefficients. It gives a far greater generalisation of the results from [Ichinobe, Adachi, 2020].

We assume that $P(\lambda, \zeta)$ is a general polynomial of two variables of order p with respect to λ and $\varphi_j(z) \in \mathcal{O}(D)$ for $j = 0, \ldots, p-1$.

The Cauchy problem for moment operators of order 0

We study the relation between the solution $\hat{u}(t,z) \in \mathcal{O}(D)[[t]]$ of the Cauchy problem

(1) $$\begin{cases} P(\partial_{m,t}, \partial_z)u = 0 \\ \partial_{m,t}^j u(0,z) = \varphi_j(z), \ j = 0, \dots, p-1, \end{cases}$$

and the solution $\hat{v}(t,z) \in \mathcal{O}(D)[[t]]$ of the similar initial value problem

(2) $$\begin{cases} P(\partial_{\mathbf{1},t}, \partial_z)v = 0 \\ \partial_{\mathbf{1},t}^j v(0,z) = \varphi_j(z), \ j = 0, \dots, p-1. \end{cases}$$

First, let us observe that

Proposition

A formal power series $\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{u_n(z)}{m(n)} t^n$ is a solution of (1) if and only if $\hat{v}(t,z) = \sum_{n=0}^{\infty} u_n(z) t^n$ is a formal power series solution of (2).

The Cauchy problem for moment operators of order 0

Proposition

A formal power series $\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{u_n(z)}{m(n)} t^n$ is a solution of (1) if and only if $\hat{v}(t,z) = \sum_{n=0}^{\infty} u_n(z) t^n$ is a formal power series solution of (2).

Proof.

($\Rightarrow$) Let $\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{u_n(z)}{m(n)} t^n$ be a formal solution of (1). Using the commutation formula

$$\mathcal{B}_{m^{-1},t}\partial_{m,t} = \partial_{\mathbf{1},t}\mathcal{B}_{m^{-1},t} \quad \text{with} \quad m^{-1} = (m(n)^{-1})_{n\geq 0}$$

and applying the Borel transform $\mathcal{B}_{m^{-1},t}$ to the Cauchy problem (1) we conclude that $\hat{v}(t,z) = \mathcal{B}_{m^{-1},t}\hat{u}(t,z)$ is a formal solution of (2).

($\Leftarrow$) The proof is analogous. It is sufficient to apply the Borel transform $\mathcal{B}_{m,t}$ to the Cauchy problem (2) and to observe that $\hat{u}(t,z) = \mathcal{B}_{m,t}\hat{v}(t,z)$. □

The Cauchy problem for moment operators of order 0

Proposition

Let $P(\lambda, \zeta)$ be a polynomial of two variables of order p with respect to λ, $k > 0$ and $d \in \mathbb{R}$. We also assume that a sequence $m = (m(n))_{n \geq 0}$ preserves summability. Then a formal power series solution $\hat{u}(t, z) \in \mathcal{O}(D)[[t]]$ of the Cauchy problem (1) is k-summable in a direction d if and only if a power series solution $\hat{v}(t, z) = \mathcal{B}_{m^{-1},t}\hat{u}(t, z)$ of the Cauchy problem (2) is k-summable in the same direction.

In the case $m = ([n]_q!)_{n \geq 0}$ for $q \in [0, 1)$ we can rewrite (1) as the Cauchy problem for the general homogeneous linear q-difference-differential equation with constant coefficients

$$(3) \qquad \begin{cases} P(D_{q,t}, \partial_z)u = 0 \\ D^j_{q,t}u(0, z) = \varphi_j(z) \in \mathcal{O}(D), \; j = 0, \dots, p-1, \end{cases}$$

The Cauchy problem for moment operators of order 0

Since the sequence $([n]_q!)_{n\geq 0}$ preserves summability we get

Theorem ([Ichinobe, Michalik, 2023])

Let $P(\lambda, \zeta)$ be a polynomial of two variables of order p with respect to λ and $q \in [0, 1)$. We also assume that $\hat{u}(t, z) = \sum_{n=0}^{\infty} \frac{u_n(z)}{[n]_q!} t^n$ and $\hat{v}(t, z) = \sum_{n=0}^{\infty} u_n(z) t^n$ are formal power series belonging to the space $\mathcal{O}(D)[[t]]$. Then the following equivalences hold:

1. *$\hat{u}(t, z)$ is a formal power series solution of (3) if and only if $\hat{v}(t, z)$ is a formal power series solution of (2).*
2. *Fix $k > 0$. $\hat{u}(t, z)$ is a formal power series solution of (3) of Gevrey order $1/k$ if and only if $\hat{v}(t, z)$ is a formal power series solution of (2) of the same Gevrey order $1/k$.*
3. *Fix $k > 0$ and $d \in \mathbb{R}$. $\hat{u}(t, z)$ is a formal power series solution of (3) that is k-summable in a direction d if and only if $\hat{v}(t, z)$ is a formal power series solution of (2) that is k-summable in the same direction.*

Final remarks

1. In the similar way one can define the sequences preserving q-asymptotic expansions (for $q > 1$). In this case one can get also the characterisation of such sequences:

Theorem ([Lastra, Michalik, 2023])

A sequence $m = (m_n)_{n\geq 0}$ preserves q-Gevrey asymptotic expansions if and only if for every $s > 0$ and every $\theta \neq 0 \mod 2\pi$, $\mathcal{B}_{m,t}\left(\sum_{n\geq 0} t^n\right)$ and $\mathcal{B}_{m^{-1},t}\left(\sum_{n\geq 0} t^n\right)$ belong to $\mathbb{C}\{t\}$ and each of them can be extended to an infinite sector of bisecting direction θ with q-exponential growth of order $1/s$.

We hope that such sequences will be useful in the study of q-summable solutions of some q-difference equations.

2. It seems be also interesting to find another characterisation of sequences preserving summability, which is given more directly in terms of these sequences.

Thank you for your attention!

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