Stability properties for ultraholomorphic classes defined in unbounded sectors

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Sectors and sequences

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$\mathbb{N}_0 = \{0, 1, 2, ...\}$.

Let $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers, with $M_0 = 1$.

We denote by $\check{\mathbb{M}} = (\widecheck{M}_n)_{n\in\mathbb{N}_0}$ the sequence defined by $\widecheck{M}_n := \frac{M_n}{n!}$.

Example:

- For $a \in \mathbb{R}$ we set

$$\mathbb{G}^a := (j!^a)_{j\in\mathbb{N}_0}, \qquad \overline{\mathbb{G}}^a := (j^{ja})_{j\in\mathbb{N}_0},$$

i.e. for $a > 0$ the sequence $\mathbb{G}^a$ is the Gevrey-sequence of index a.

Log-convexity

$\mathbb{M}$ is said to be logarithmically convex or (lc) if $M_n^2 \leq M_{n-1}M_{n+1},\ n \geq 1$; equivalently, the sequence of quotients of $\mathbb{M}$, $\boldsymbol{m} = (m_n := \frac{M_{n+1}}{M_n})_{n\in\mathbb{N}_0}$, is nondecreasing.

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If $\mathbb{M}$ satisfies $\lim_{j\to+\infty}(M_j)^{1/j} = +\infty$, we denote by $\mathbb{M}^{\text{lc}}$ the log-convex minorant of $\mathbb{M}$, i.e. each log-convex sequence $\mathbb{L}$ with $\mathbb{L} \leq \mathbb{M}$ satisfies $\mathbb{L} \leq \mathbb{M}^{\text{lc}}$ (and $\mathbb{M}^{\text{lc}} \equiv \mathbb{M}$ if and only if $\mathbb{M}$ is log-convex).

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We say $\mathbb{M}$ is a weight sequence, if $\mathbb{M}$ is (lc) and $\lim_{n\to\infty} m_n = \infty$.

Properties of the sequences

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$\mathbb{M}$ has the condition *root almost increasing*, denoted by (rai), if

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$\mathbb{M}$ has the *Faà-di-Bruno property*, denoted by (FdB), if

$$\exists\, C \geq 1\, \exists\, h \geq 1\, \forall\, j \in \mathbb{N}_0 : \quad \breve{M}_j^{\circ} \leq C h^j \breve{M}_j,$$

with $\breve{\mathbb{M}}^{\circ} := (\breve{M}_j^{\circ})_{j\in\mathbb{N}_0}$ the sequence defined by

$$\breve{M}_k^{\circ} := \max\left\{ \breve{M}_\ell \cdot \breve{M}_{j_1} \cdots \breve{M}_{j_\ell} : j_i \in \mathbb{N}, \sum_{i=1}^{\ell} j_i = k \right\}, \qquad \breve{M}_0^{\circ} := 1.$$

Equivalent sequences

Let $\mathbb{M}, \mathbb{L}$ two sequences, we write $\mathbb{M} \preceq \mathbb{L}$ if $\sup_{j\in\mathbb{N}} (M_j/L_j)^{1/j} < +\infty$ and call $\mathbb{M}$ and $\mathbb{L}$ *equivalent*, denoted by $\mathbb{M} \approx \mathbb{L}$, if $\mathbb{M} \preceq \mathbb{L}$ and $\mathbb{L} \preceq \mathbb{M}$; equivalently, there exist some constant $A, B > 0$ such that $A^n M_n \leq L_n \leq B^n M_n$ for all $n \in \mathbb{N}$.

Ultraholomorphic (Carleman-Roumieu) classes

Given $\mathbb{M}$, $A > 0$ and a sector S, we consider

$$\mathcal{A}_{\{\mathbb{M}\},A}(S) = \left\{ f \in \mathcal{H}(S) \colon \|f\|_{\mathbb{M},A} := \sup_{z \in S, n \in \mathbb{N}_0} \frac{|f^{(n)}(z)|}{A^n M_n} < \infty \right\}.$$

$(\mathcal{A}_{\{\mathbb{M}\},A}(S), \| \ \|_{\mathbb{M},A})$ is a Banach space.

$\mathcal{A}_{\{\mathbb{M}\}}(S) := \bigcup_{A>0} \mathcal{A}_{\{\mathbb{M}\},A}(S)$ is an (LB) space.

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For $f \in \mathcal{A}_{\{\mathbb{M}\}}(S)$ and for every $n \in \mathbb{N}_0$, there exists

$$f^{(n)}(0) := \lim_{z \to 0, z \in S} f^{(n)}(z).$$

Ider-Sidiqqi's result

Theorem

Let $\mathbb{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ *be a sequence such that* $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j)^{1/j} = \infty$ *and for* $0 < \alpha \leq 1$, *let* $\mathbb{M}^{(\alpha)} = \overline{\mathbb{G}}^{\alpha-1}\left(\overline{\mathbb{G}}^{1-\alpha}\mathbb{M}\right)^{lc}$. *Then the following assertions are equivalent:*

(a) *The sequence* $\mathbb{M}^{(\alpha)}$ *has the* (*rai*) *property.*

(b) *The class* $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ *is holomorphically closed, i.e, if for all* $f \in \mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ *and* $g \in \mathcal{H}(U)$ *where* $U \subseteq \mathbb{C}$ *is an open set containing the closure of the range of* f, *we have that* $g \circ f \in \mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$.

(c) *The class* $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ *is inverse-closed, i.e, if for all* $f \in \mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ *such that* $\inf_{z\in S_\alpha}|f(z)| > 0$ *we have that* $1/f \in \mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$.

Definition

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Let $\mathbb{L}$ and S be given. A function $f \in \mathcal{A}_{\{\mathbb{L}\}}(S)$ is said to be **characteristic** in the class $\mathcal{A}_{\{\mathbb{L}\}}(S)$, if the following holds true: If for some sequence $\mathbb{M}$ we have $f \in \mathcal{A}_{\{\mathbb{M}\}}(S) \subseteq \mathcal{A}_{\{\mathbb{L}\}}(S)$, then already $\mathcal{A}_{\{\mathbb{M}\}}(S) = \mathcal{A}_{\{\mathbb{L}\}}(S)$.

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Theorem

Let $\mathbb{L}$ and S be given. Let $f \in \mathcal{A}_{\{\mathbb{L}\}}(S)$ with $C_n(f) := \sup_{z\in S} |f^{(n)}(z)|$ for all $n \in \mathbb{N}_0$, then each condition implies the next one:

1. *The sequence $(|f^{(j)}(0)|)_{j\in\mathbb{N}_0}$ is equivalent to $\mathbb{L}$.*
2. *The sequence $(C_j(f))_{j\in\mathbb{N}_0}$ is equivalent to $\mathbb{L}$.*
3. *f is characteristic in the class $\mathcal{A}_{\{\mathbb{L}\}}(S)$.*

Basic functions I

Definition

Let us consider $\alpha \in (0,1]$, and denote by $\widetilde{E}_\alpha(z)$ the function defined by

$$\widetilde{E}_\alpha(z) := E_{2-\alpha,4-\alpha}(-z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{\Gamma((2-\alpha)j+4-\alpha)}, \qquad z \in \mathbb{C}.$$

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Theorem (Salinas (1962))

Let $\widetilde{E}_\alpha(z)$ be the above function and $\alpha \in (0,1]$, then

$$\forall\ z \in S_\alpha\ \forall\ n \in \mathbb{N}_0 : \quad \left|\widetilde{E}_\alpha^{(n)}(z)\right| \leq 2\frac{n!e^n}{n^{(2-\alpha)n}}.$$

Consequently, $\widetilde{E}_\alpha \in \mathcal{A}_{\{\overline{\mathbb{G}}^{\alpha-1}\}}(S_\alpha)$. Moreover, $\widetilde{E}_\alpha$ is a characteristic function in the class $\mathcal{A}_{\{\overline{\mathbb{G}}^{\alpha-1}\}}(S_\alpha)$.

Basic functions II

Definition

Let $\alpha > 1$ and take $\alpha' > \alpha$. For all $z \in S_\alpha$ we define

$$g_{\alpha,\alpha'}(z) := \int_0^{\infty(-\phi)} e^{-zv^{\alpha'-1}} e^{-v} dv,$$

where we choose $\phi \in (-\frac{(\alpha-1)}{(\alpha'-1)}\frac{\pi}{2}, \frac{(\alpha-1)}{(\alpha'-1)}\frac{\pi}{2})$ with $|\arg(z) - (\alpha' - 1)\phi| < \pi/2$.

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Theorem (Salinas (1962))

Consider $\alpha > 1$ and take $\alpha' > \alpha$. Let $g_{\alpha,\alpha'}$ be the above function, then

$$\exists\, C, A \geq 1\ \forall\, z \in S_\alpha\ \forall\, n \in \mathbb{N}_0 : \quad \left|g^{(n)}_{\alpha,\alpha'}(z)\right| \leq CA^n\Gamma((\alpha'-1)n+1).$$

Consequently, $g_{\alpha,\alpha'} \in \mathcal{A}_{\{\overline{\mathbb{G}}^{\alpha'-1}\}}(S_\alpha)$ and, moreover, $g_{\alpha,\alpha'}$ is a characteristic function of the class $\mathcal{A}_{\{\overline{\mathbb{G}}^{\alpha'-1}\}}(S_\alpha)$.

Construction of Characteristic functions

Definition

Let $\mathbb{M}$ be a (lc) sequence, $\mathbb{L}$ a general sequence and S be a sector, take $f \in \mathcal{A}_{\{\mathbb{L}\}}(S)$. Then we define the $\mathcal{T}_{\mathbb{M}}-$transform of f by

$$\mathcal{T}_{\mathbb{M}}(f)(z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{M_j}{m_j^j} f(m_j z), \qquad z \in S.$$

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Theorem

Let $\mathbb{M}$ be a general sequence and $\alpha > 0$. We assume that $\overline{\mathbb{G}}^{1-\alpha'}\mathbb{M} := (j^{(1-\alpha')j} M_j)_{j \in \mathbb{N}_0}$ is equivalent to a (lc) sequence $\mathbb{L}$ depending on α', where $\alpha' = \alpha$, if $\alpha \leq 1$, or $\alpha' > \alpha$, if $\alpha > 1$. Then, the following assertions hold true:

1. *If $\alpha \leq 1$, then $\mathcal{T}_{\mathbb{L}}(\widetilde{E}_\alpha)$ is characteristic in the class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$.*
2. *If $\alpha > 1$, then $\mathcal{T}_{\mathbb{L}}(g_{\alpha,\alpha'})$ is characteristic in the class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$.*

Weight matrices

Definition

A *weight matrix* $\mathcal{M}$ is a (one parameter) family of sequences $\mathcal{M} := \{\mathbb{M}^{(\alpha)} : \alpha > 0\}$, such that

$$\mathbb{M}^{(\alpha)} \leq \mathbb{M}^{(\beta)} \text{ for } \alpha \leq \beta, \qquad M_0^{(\alpha)} = 1, \quad \forall\, \alpha > 0.$$

Moreover, we put $\widetilde{M}_j^{(\alpha)} := \frac{M_j^{(\alpha)}}{j!}$ for $j \in \mathbb{N}_0$, and $m_j^{(\alpha)} := \frac{M_{j+1}^{(\alpha)}}{M_j^{(\alpha)}}$ for $j \in \mathbb{N}_0$.

Properties I

Let $\mathcal{M} := \{\mathbb{M}^{(\alpha)} : \alpha > 0\}$ be a weight matrix, we said that $\mathcal{M}$ has:

($\mathcal{M}_{\mathsf{lc}}$) if $\mathbb{M}^{(\alpha)}$ is a log-convex sequence for all $\alpha > 0$,
($\mathcal{M}_{\mathsf{sc}}$) if $\mathbb{M}^{(\alpha)}$ is a normalized weight sequence for all $\alpha > 0$,

($\mathcal{M}_{\{\mathsf{C}^\omega\}}$) $\exists\, \alpha > 0 : \quad \liminf_{j\to\infty}(\breve{M}_j^{(\alpha)})^{1/j} > 0,$

($\mathcal{M}_{\mathcal{H}}$) $\forall\, \alpha > 0 : \quad \liminf_{j\to\infty}(\breve{M}_j^{(\alpha)})^{1/j} > 0,$

($\mathcal{M}_{\{\mathsf{rai}\}}$)

$\forall\, \alpha > 0\, \exists\, C > 0\, \exists\, \beta > 0\, \forall\, 1 \le j \le k : \quad (\breve{M}_j^{(\alpha)})^{1/j} \le C(\breve{M}_k^{(\beta)})^{1/k},$

($\mathcal{M}_{\{\mathsf{FdB}\}}$) $\forall\, \alpha > 0\, \exists\, \beta > 0 : \quad (\breve{\mathbb{M}}^{(\alpha)})^\circ \preceq \breve{\mathbb{M}}^{(\beta)},$

with $(\breve{\mathbb{M}}^{(\alpha)})^\circ := ((\breve{M}_j^{(\alpha)})^\circ)_j$ the sequence defined by

$$(\breve{M}_k^{(\alpha)})^\circ := \max\left\{\breve{M}_\ell^{(\alpha)} \cdot \breve{M}_{j_1}^{(\alpha)} \cdots \breve{M}_{j_\ell}^{(\alpha)} : j_i \in \mathbb{N}, \sum_{i=1}^{\ell} j_i = k\right\}, \qquad (\breve{M}_0^{(\alpha)})^\circ := 1,$$

($\mathcal{M}_{\{\mathsf{dc}\}}$) $\forall\, \alpha > 0\, \exists\, C > 0\, \exists\, \beta > 0\, \forall\, j \in \mathbb{N}_0 : M_{j+1}^{(\alpha)} \le C^{j+1} M_j^{(\beta)}.$

Properties II

Lemma

Let $\mathcal{M} = \{\mathbb{M}^{(\alpha)} : \alpha > 0\}$ *be a weight matrix. Then we have the following:*

(i) $(\mathcal{M}_{\{\mathsf{rai}\}})$ *implies* $(\mathcal{M}_{\mathcal{H}})$.

(ii) $(\mathcal{M}_{\{\mathsf{dc}\}})$ *and* $(\mathcal{M}_{\{\mathsf{rai}\}})$ *imply* $(\mathcal{M}_{\{\mathsf{FdB}\}})$.

(iii) *If*

$$\forall\, \alpha > 0\, \exists\, H \geq 1\, \forall\, 1 \leq j \leq k : \quad (M_j^{(\alpha)})^{1/j} \leq H(M_k^{(\alpha)})^{1/k},$$

i.e. each sequence $((M_j^{(\alpha)})^{1/j})_j$ *is almost increasing, then* $(\mathcal{M}_{\mathcal{H}})$ *and* $(\mathcal{M}_{\{\mathsf{FdB}\}})$ *imply* $(\mathcal{M}_{\{\mathsf{rai}\}})$.
In particular, the inequality holds true (with $H = 1$ *for any* α*) provided that* $\mathcal{M}$ *is log-convex.*

R-equivalence

Let $\mathcal{M} = \{\mathbb{M}^{(\alpha)} : \alpha > 0\}$ and $\mathcal{L} = \{\mathbb{L}^{(\alpha)} : \alpha > 0\}$ be given. We write $\mathcal{M}\{\preceq\}\mathcal{L}$ if

$$\forall\, \alpha > 0 \,\exists\, \beta > 0 : \quad \mathbb{M}^{(\alpha)} \preceq \mathbb{L}^{(\beta)},$$

and call $\mathcal{M}$ and $\mathcal{L}$ R-equivalent, if $\mathcal{M}\{\preceq\}\mathcal{L}$ and $\mathcal{L}\{\preceq\}\mathcal{M}$.

Some properties like $(\mathcal{M}_{\{\mathsf{rai}\}})$ and $(\mathcal{M}_{\{\mathsf{FdB}\}})$ are stable under R-equivalence.

Ultraholomorphic (Carleman-Roumieu) classes

Given a weight matrix $\mathcal{M} = \{\mathbb{M}^{(\alpha)} : \alpha > 0\}$ and a sector S we may introduce the class $\mathcal{A}_{\{\mathcal{M}\}}(S)$ of *Roumieu type* as

$$\mathcal{A}_{\{\mathcal{M}\}}(S) := \bigcup_{\alpha>0} \mathcal{A}_{\{\mathbb{M}^{(\alpha)}\}}(S).$$

R-equivalent weight matrices yield the same function class on each sector S.

The matrix $\mathcal{M}^\alpha$

Definition

Let $\mathcal{M} = \{\mathbb{M}^{(p)} : p > 0\}$ be a weight matrix. Given $\alpha > 0$ we assume that $\lim_{j\to+\infty}(j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ for all $p > 0$. Let us introduce the matrix

$$\mathcal{M}^\alpha := \{\mathbb{M}^{(p,\alpha)} : p > 0\}$$

given by the sequences satisfying the relation

$$\left(\overline{\mathbb{G}}^{1-\alpha}\mathbb{M}^{(p)}\right)^{\mathsf{lc}} = \overline{\mathbb{G}}^{1-\alpha}\mathbb{M}^{(p,\alpha)}.$$

We have that $\mathcal{M}^\alpha$ is a weight matrix. However, in general $\mathcal{M}^\alpha$ is not log-convex anymore.

Stability Properties

Definition

Let $\mathbb{M}$ be a sequence and $U \subseteq \mathbb{C}$ be an open set. Given $K \subset U$ a compact set, we define

$$\mathcal{H}_{\mathbb{M},h}(K) := \{f \in \mathcal{H}(U) : \|f\|_{\mathbb{M},K,h} := \sup_{z\in K, j\in\mathbb{N}_0} \frac{|f^{(j)}(z)|}{h^j M_j} < +\infty\}.$$

We put

$$\mathcal{H}_{\{\mathbb{M}\}}(K) := \bigcup_{h>0} \mathcal{H}_{\mathbb{M},h}(K).$$

Moreover, given a weight matrix $\mathcal{M} = \{\mathbb{M}^{(p)} : p > 0\}$, we may introduce the class $\mathcal{H}_{\{\mathcal{M}\}}(U)$ as

$$\mathcal{H}_{\{\mathcal{M}\}}(U) := \bigcap_{K\subset U} \bigcup_{p>0} \mathcal{H}_{\{\mathbb{M}^{(p)}\}}(K).$$

Stability Properties

Definition

Let $\mathcal{M} = \{\mathbb{M}^{(p)} : p > 0\}$ be a weight matrix and $\alpha > 0$. The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is said to be:

- (i) holomorphically closed, if for all $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ and $g \in \mathcal{H}(U)$ where $U \subseteq \mathbb{C}$ is an open set containing the closure of the range of f, we have that $g \circ f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.
- (ii) inverse-closed, if for all $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ such that $\inf_{z \in S_\alpha} |f(z)| > 0$ we have that $1/f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.
- (iii) closed under composition, if for all $f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ and for all $g \in \mathcal{H}_{\{\mathcal{M}\}}(U)$ where $U \subseteq \mathbb{C}$ is an open set containing the closure of the range of f, we have that $g \circ f \in \mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$.

Preparatory results

Theorem (Salinas (1962))

Let $0 < \alpha \leq 1$ *and* $f \in \mathcal{H}(S_\alpha)$. *If* $C_n(f) = \sup_{z \in S_\alpha} |f^{(n)}(z)|$, *then the sequence* $B_n = n^{(1-\alpha)n} C_n(f)$ *verifies*

$$B_n \leq A q^{(1-\alpha)n} B_{n_1}^{\frac{n_2-n}{n_2-n_1}} B_{n_2}^{\frac{n-n_1}{n_2-n_1}}, \qquad n_1 < n < n_2,$$

where $A = 4$ *and* $q = 1$ *if* $\alpha = 1$, *or* $A = 8\pi$ *and* $q = 2e(2-\alpha)/(1-\alpha)$ *for the remainder cases.*

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Theorem (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\mathcal{M} = \{\mathbb{M}^{(p)} : p > 0\}$ *be a weight matrix and* $0 < \alpha \leq 1$ *be given such that* $\lim_{j \to +\infty} (j^{(1-\alpha)j} M_j^{(p)})^{1/j} = \infty$ *for all* $p > 0$. *Let us consider the matrix* $\mathcal{M}^\alpha = \{\mathbb{M}^{(p,\alpha)} : p > 0\}$. *Then, we have that*

$$\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha) = \mathcal{A}_{\{\mathcal{M}^\alpha\}}(S_\alpha).$$

Main result: Case $0 < \alpha \le 1$

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- (a) *The matrix* $\mathcal{M}^\alpha$ *satisfies the property* $(\mathcal{M}_{\{\mathsf{rai}\}})$.
- (b) *The class* $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ *is holomorphically closed.*
- (c) *The class* $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ *is inverse-closed.*

If $\mathcal{M}$ *has in addition* $(\mathcal{M}_{\{C^\omega\}})$ *and* $\mathcal{M}^\alpha$ *has* $(\mathcal{M}_{\{\mathsf{dc}\}})$, *then the list of equivalences can be extended by*

- (d) *The class* $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ *is closed under composition.*
- (e) *The matrix* $\mathcal{M}^\alpha$ *satisfies the property* $(\mathcal{M}_{\{\mathsf{FdB}\}})$.

Main result: Case $0 < \alpha \leq 1$

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- (a) *The matrix* $\mathcal{M}^\alpha$ *satisfies the property* $(\mathcal{M}_{\{\mathsf{rai}\}})$*.*
- (b) *The class* $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ *is holomorphically closed.*
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- (d) *The class* $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ *is closed under composition.*
- (e) *The matrix* $\mathcal{M}^\alpha$ *satisfies the property* $(\mathcal{M}_{\{\mathsf{FdB}\}})$*.*

If $\mathcal{M}$ has $(\mathcal{M}_{\{\mathsf{dc}\}})$ then $\mathcal{M}^\alpha$ has it too (the converse is not clear in general).
If $\alpha = 0$, we obtain the same result by replacing the sector by the positive real line in the previous theorems.

Ider-Sidiqqi's general result I

Corollary (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\mathbb{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ be a sequence, and $0 < \alpha \leq 1$ be given such that $\lim_{j\to+\infty}(j^{(1-\alpha)j}M_j)^{1/j} = \infty$. Let $\mathbb{M}^{(\alpha)} = \overline{\mathbb{G}}^{\alpha-1}\left(\overline{\mathbb{G}}^{1-\alpha}\mathbb{M}\right)^{lc}$. Then the following assertions are equivalent:

- (a) *The sequence $\mathbb{M}^{(\alpha)}$ has the (rai) property.*
- (b) *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ is holomorphically closed.*
- (c) *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ is inverse-closed.*

If $\liminf_{j\to\infty}(\widecheck{M}_j)^{1/j} > 0$ and the sequence $\mathbb{M}^{(\alpha)}$ is (dc), then the list of equivalences can be extended by

- (d) *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ is closed under composition.*
- (e) *The sequence $\mathbb{M}^{(\alpha)}$ has the (FdB) property.*

Main result: Case $\alpha > 1$

Theorem (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\mathcal{M} = \{\mathbb{M}^{(p)} : p > 0\}$ be a weight matrix and consider $\alpha > 1$. For each $p > 0$, we suppose that there exist some $\alpha_p > \alpha$ such that $\overline{\mathbb{G}}^{1-\alpha_p}\mathbb{M}^{(p)}$ is equivalent to a (lc) sequence $\mathbb{L}^{(p)}$ depending on α_p. Then the following assertions are equivalent:

- (a) *The matrix $\mathcal{M}$ satisfies the property $(\mathcal{M}_{\{rai\}})$.*
- (b) *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is holomorphically closed.*
- (c) *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is inverse-closed.*

If $\mathcal{M}$ has in addition $(\mathcal{M}_{\{C^\omega\}})$ and $(\mathcal{M}_{\{dc\}})$, then the list of equivalences can be extended by

- (d) *The class $\mathcal{A}_{\{\mathcal{M}\}}(S_\alpha)$ is closed under composition.*
- (e) *The matrix $\mathcal{M}$ satisfies the property $(\mathcal{M}_{\{FdB\}})$.*

Differences between the two cases

Note that there exist some differences between the statements of the previous theorems.

Proposition (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\mathcal{M} = \{\mathbb{M}^{(p)} : p > 0\}$ *be a given weight matrix. Suppose that for every* $p > 0$ *there exists* $\alpha_p > 0$ *such that* $\overline{\mathbb{G}}^{1-\alpha_p}\mathbb{M}^{(p)}$ *is equivalent to a (lc) sequence* $\mathbb{L}^{(p)}$*, and that there exists* $\beta \in \mathbb{R}$ *such that* $\beta < \alpha_p$ *for all* $p > 0$*. Then, for every* $p > 0$ *one has* $\lim_{j\to+\infty}(j^{(1-\beta)j}M_j^{(p)})^{1/j} = \infty$, $\mathcal{M}$ *and* $\mathcal{M}^\beta$ *are* R*-equivalent, and therefore* $\mathcal{M}$ *satisfies the property* $(\mathcal{M}_{\{rai\}})$ *(resp.*$(\mathcal{M}_{\{FdB\}})$*) if and only if the matrix* $\mathcal{M}^\beta$ *satisfies this condition too. Moreover,* $\mathcal{A}_{\{\mathcal{M}^\beta\}}(S_\gamma) = \mathcal{A}_{\{\mathcal{M}\}}(S_\gamma)$*, for all* $\gamma > 0$*.*

Ider-Sidiqqi's general result II

Corollary (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\mathbb{M} \in \mathbb{R}_{>0}^{\mathbb{N}_0}$ and $\alpha > 1$. Suppose there exists $\alpha' > \alpha$ such that $\overline{\mathbb{G}}^{1-\alpha'}\mathbb{M}$ is equivalent to an (lc) sequence $\mathbb{L}$ (depending on α'). Then the following assertions are equivalent:

- (a) *The sequence $\mathbb{M}$ has the (rai) property.*
- (b) *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ is holomorphically closed.*
- (c) *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ is inverse-closed.*

If $\liminf_{j\to\infty}(\widetilde{M}_j)^{1/j} > 0$ and $\mathbb{M}$ is (dc), then the list of equivalences can be extended by

- (d) *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\alpha)$ is closed under composition.*
- (e) *The sequence $\mathbb{M}$ has the (FdB) property.*

Weight functions

Definition

A function $\omega : [0, +\infty) \to [0, +\infty)$ is called a *weight function*, if it is continuous, nondecreasing, $\omega(0) = 0$ and $\lim_{t\to+\infty} \omega(t) = +\infty$.
If ω satisfies in addition $\omega(t) = 0$ for all $t \in [0, 1]$, then we call ω a *normalized weight function*.

For any $s > 0$ we put ω^s to be the function given by $\omega^s(t) := \omega(t^s)$.
(If $s = 0$, then we put $\omega^0(t) := \omega(1)$.)

Associated weight function

Definition

Let $\mathbb{M}$ be a sequence such that $\lim_{j\to+\infty}(M_j)^{1/j} = +\infty$, then the *associated weight function* $\omega_{\mathbb{M}} : [0,+\infty) \to [0,+\infty)$ is defined by

$$\omega_{\mathbb{M}}(t) := \sup_{j\in\mathbb{N}_0} \log\left(\frac{t^j}{M_j}\right) \quad \text{for } t>0, \qquad \omega_{\mathbb{M}}(0) := 0.$$

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If $\mathbb{M}$ is a sequence which satisfies $\lim_{j\to+\infty}(M_j)^{1/j} = +\infty$, we can construct the log-convex minorant $\mathbb{M}^{\mathrm{lc}}$ of $\mathbb{M}$, more precisely

$$M_j^{\mathrm{lc}} = \sup_{t\geq 0} \frac{t^j}{\exp(\omega_{\mathbb{M}}(t))}, \quad j\in\mathbb{N}_0.$$

Properties

Let ω be a weight function, we say that ω has:

- (ω_0) if ω is a normalized weight.
- (ω_1) $\omega(2t) = O(\omega(t))$ as $t \to +\infty$, i.e.
  $\exists\, L \geq 1\, \forall\, t \geq 0: \quad \omega(2t) \leq L(\omega(t) + 1)$.
- (ω_2) $\omega(t) = O(t)$ as $t \to +\infty$.
- (ω_3) $\log(t) = o(\omega(t))$ as $t \to +\infty$.
- (ω_4) $\varphi_\omega : t \mapsto \omega(e^t)$ is a convex function on $\mathbb{R}$.
- (ω_5) $\omega(t) = o(t)$ as $t \to +\infty$.
- (α_0) $\exists\, C \geq 1\, \exists\, t_0 \geq 0\, \forall\, \lambda \geq 1\, \forall\, t \geq t_0 : \quad \omega(\lambda t) \leq C\lambda\omega(t)$.

For convenience we define the sets

$$\mathcal{W}_0 := \{\omega : [0,\infty) \to [0,\infty) : \omega \text{ has } (\omega_0), (\omega_3), (\omega_4)\},$$
$$\mathcal{W} := \{\omega \in \mathcal{W}_0 : \omega \text{ has } (\omega_1)\}.$$

Weight matrices associated with weight functions

For any $\omega \in \mathcal{W}_0$ we define the *Legendre-Fenchel-Young-conjugate* of φ_ω by

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y \geq 0\}, \quad x \geq 0.$$

Definition

Given $\omega \in \mathcal{W}_0$ we can associate a weight matrix $\mathcal{M}_\omega := \{\mathbb{W}^{(\ell)} : \ell > 0\}$ by

$$W_j^{(\ell)} := \exp\left(\frac{1}{\ell}\varphi_\omega^*(\ell j)\right), \quad \forall j \in \mathbb{N}_0.$$

This matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{\mathsf{sc}})$ and $(\mathcal{M}_{\{\mathsf{dc}\}})$.

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$$W^{(\ell)}_j := \exp\left(\frac{1}{\ell}\varphi^*_\omega(\ell j)\right), \quad \forall j \in \mathbb{N}_0.$$

This matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{\mathsf{sc}})$ and $(\mathcal{M}_{\{\mathsf{dc}\}})$.

$\mathcal{M}_\omega$ satisfies $(\mathcal{M}_{\mathcal{H}})$ if and only if ω has in addition (ω_2).
In particular, if $\omega \in \mathcal{W}_0$ has (ω_2) then properties $(\mathcal{M}_{\{\mathsf{rai}\}})$ and $(\mathcal{M}_{\{\mathsf{FdB}\}})$ for $\mathcal{M}_\omega$ are equivalently satisfied.

Ultraholomorphic (Braun-Meise-Taylor) classes

Let $\omega \in \mathcal{W}_0$, S be an unbounded sector, and for every $\ell > 0$, we first define

$$\mathcal{A}_{\omega,\ell}(S) := \{f \in \mathcal{H}(S) : \|f\|_{\omega,\ell} := \sup_{z\in S, j\in\mathbb{N}_0} \frac{|f^{(j)}(z)|}{\exp(\frac{1}{\ell}\varphi^*_\omega(\ell j))} < +\infty\}.$$

$(\mathcal{A}_{\omega,\ell}(S), \|\cdot\|_{\omega,\ell})$ is a Banach space and we put

$$\mathcal{A}_{\{\omega\}}(S) := \bigcup_{\ell>0} \mathcal{A}_{\omega,\ell}(S).$$

$\mathcal{A}_{\{\omega\}}(S)$ is called the ultraholomorphic class (of Braun-Meise-Taylor type) associated with ω in the sector S (it is a (LB) space).

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$\mathcal{A}_{\{\omega\}}(S)$ is called the ultraholomorphic class (of Braun-Meise-Taylor type) associated with ω in the sector S (it is a (LB) space).

Let $\omega \in \mathcal{W}$ be given and let $\mathcal{M}_\omega$ be the associated weight matrix, then

$$\mathcal{A}_{\{\omega\}}(S) = \mathcal{A}_{\{\mathcal{M}_\omega\}}(S)$$

holds as locally convex vector spaces.

Auxiliary Lemma

Lemma

Let $\omega \in \mathcal{W}_0$ be given with associated weight matrix $\mathcal{M}_\omega := \{\mathbb{W}^{(\ell)} : \ell > 0\}$. Then the following are equivalent:

(a) *The matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{\{\mathsf{rai}\}})$, i.e. (recall $\widetilde{W}_j^{(\ell)} = W_j^{(\ell)}/j!$)*

$$\forall\, \ell > 0\, \exists\, \ell' > 0\, \exists\, H \geq 1\, \forall\, 1 \leq j \leq k : \quad (\widetilde{W}_j^{(\ell)})^{1/j} \leq H(\widetilde{W}_k^{(\ell')})^{1/k}.$$

(b) *ω satisfies the condition (α_0), so*

$$\exists\, C \geq 1\, \exists\, t_0 \geq 0\, \forall\, \lambda \geq 1\, \forall\, t \geq t_0 : \quad \omega(\lambda t) \leq C\lambda\omega(t).$$

Main result: Case $0 < \alpha \leq 1$

Theorem (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\omega \in \mathcal{W}$ be given with associated weight matrix $\mathcal{M}_\omega := \{\mathbb{W}^{(\ell)} : \ell > 0\}$ and let $0 < \alpha \leq 1$. Then the following are equivalent:

- (a) *The matrix $\mathcal{M}_\omega$ has $(\mathcal{M}_{\{rai\}})$.*
- (b) *ω satisfies the condition (α_0).*
- (c) *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is holomorphically closed.*
- (d) *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is inverse-closed.*

If ω has in addition (ω_2), then the list of equivalences can be extended by:

- (e) *The class $\mathcal{A}_{\{\omega\}}(S_\alpha)$ is closed under composition.*
- (f) *The matrix $\mathcal{M}_\omega$ satisfies the condition $(\mathcal{M}_{\{FdB\}})$.*

Main result: Case $\alpha > 1$

Theorem (J. Jiménez-Garrido, I. M-C, J. Sanz, G. Schindl (2023))

Let $\omega \in \mathcal{W}_0$ *be given with associated weight matrix* $\mathcal{M}_\omega := \{\mathbb{W}^{(\ell)} : \ell > 0\}$ *and let* $\alpha > 1$. *Suppose there exists* $s > \alpha - 1$ *such that, for* $\omega^s(t) := \omega(t^s)$, *one has:*

(i) ω^s *has* (ω_5).

(ii) ω^s *satisfies the condition* (α_0).

Then the following are equivalent:

(a) *The matrix* $\mathcal{M}_\omega$ *has* $(\mathcal{M}_{\{rai\}})$.

(b) ω *satisfies the condition* (α_0).

(c) *The class* $\mathcal{A}_{\{\omega\}}(S_\alpha)$ *is holomorphically closed.*

(d) *The class* $\mathcal{A}_{\{\omega\}}(S_\alpha)$ *is inverse-closed.*

If ω *has in addition* (ω_2), *then the list of equivalences can be extended by:*

(e) *The class* $\mathcal{A}_{\{\omega\}}(S_\alpha)$ *is closed under composition.*

(f) *The matrix* $\mathcal{M}_\omega$ *satisfies the condition* $(\mathcal{M}_{\{FdB\}})$.

THANK YOU VERY MUCH FOR YOUR ATTENTION!