Differential transcendence of solutions for second order linear q-difference equations

Hiroshi Ogawara

Josai University

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Introduction

In this talk, we discuss differential transcendence of solutions to second order linear q-difference equations.

A function $f(x)$ is differentially algebraic over $\mathbb{C}(x)$ if $f(x)$ satisfies a non-trivial algebraic differential equation with coefficients in $\mathbb{C}(x)$, i.e., ${}^{\exists}G(X, Y_0, Y_1, \ldots, Y_n) \in \mathbb{C}[X, Y_0, Y_1, \ldots, Y_n] \backslash \{0\}$ s.t.

$$G\left(x, f, \frac{df}{dx}, \ldots, \frac{d^n f}{dx^n}\right) \equiv 0. \tag{1}$$

A function $f(x)$ is differentially transcendental over $\mathbb{C}(x)$ if $f(x)$ is not differentially algebraic over $\mathbb{C}(x)$.

Previous works

There are results on differential transcendence w.r.t eq. of q-special functions:

- q-Airy: $y(q^2x) + xy(qx) - y(x) = 0$ ([Nishioka, 2018])
- Ramanujan: $qxy(q^2x) - y(qx) + y(x) = 0$ ([Ogawara, 2023])
- Hahn-Exton q-Bessel when q : tr. / $\mathbb{Q}$ (essentially proved in [Nishioka, 2016])

- $q \in \mathbb{C}\setminus\{0\}$: not a root of unity.
- $\tau : \varphi(x) \mapsto \varphi(qx)$ the q-shift operator.

Consider a second order linear q-difference equation with coefficients $a_i \in \mathbb{C}[x]^\times$,

$$a_2\tau^2(y) + a_1\tau(y) + a_0 y = 0. \tag{2}$$

Define a linear fractional transformation as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} * y := \frac{\alpha y + \beta}{\gamma y + \delta}.$$

We transform the q-difference equation into a certain q-difference Riccati equation:

$$z := \begin{pmatrix} a_1 & a_0 \\ a_1 & 0 \end{pmatrix} * \frac{\tau(y)}{y}, \quad a := -\frac{a_2\tau(a_0)}{a_1\tau(a_1)}, \tag{3}$$

$$(2) \quad \rightsquigarrow \quad \tau(z) = \begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix} * z. \tag{4}$$

The equation (4) is called Tietze's normal form.

Theorem 1 ([Nishioka, 2018, Theorem 9 (q-difference ver.)])

Put

$$A = \begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix}, \qquad a \in \mathbb{C}(x) \backslash \mathbb{C}$$

Consider the following q-difference Riccati equation of Tietze's normal form:

$$\tau(y) = A * y = 1 + \frac{a}{y}, \tag{5}$$

Suppose that

1. $^{\exists} p \in \mathbb{P} = \mathbb{C} \cup \{\infty\}$ s.t. $\mathrm{ord}_p(\tau^k a) > 0$ for $^{\forall} k \in \mathbb{Z}_{\geq 0}$. $(\mathrm{ord}_p(x-p)^n := n)$
2. Put $A_k = (\tau^{k-1}A)(\tau^{k-2}A)\cdots(\tau A)A$.
   For $^{\forall} k \in \mathbb{Z}_{\geq 1}$, $\tau^k(y) = A_k * y$ has no algebraic solution.
3. The following third order linear q-difference equation has no rational solution,

$$q^3\tau^2(a)\tau^3(y) + q^2(\tau(a)+1)\tau^2(y) - q(\tau(a)+1)\tau(y) - ay + q\tau\left(\frac{d}{dx}\log a\right) = 0. \tag{6}$$

Then (5) has no differentially algebraic solution over $\mathbb{C}(x)$.

More precisely, the criterion is described by means of difference fields.

Theorem 2 ([Nishioka, 2018, Theorem 9])

Let

- $\mathcal{F} = (F, D_0, \tau_0)$: a DTC field with $D_0\tau_0 = s\tau_0 D_0$ for a certain $s \in F^\times$.
- F/K : an algebraic function field of one variable.
- $A = \begin{pmatrix} 1 & r \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(F), \quad (D_0 r \neq 0).$

Suppose that

1. ${}^\exists$ a place P of F/K s.t. $v_P(\tau_0^i r) > 0$ for ${}^\forall i \in \mathbb{Z}_{\geq 0}$.
2. Put $A_k = (\tau^{k-1}A)(\tau^{k-2}A)\cdots(\tau A)A$.
   For ${}^\forall i \in \mathbb{Z}_{\geq 1}$, $\tau_0^i(y) = A_i * y$ has no algebraic solution over F.

Let $\mathcal{U} = (U, D, \tau)$ be a DTC overfield of $\mathcal{F}$ with $D\tau = s\tau D$.
If ${}^\exists$ a differentially algebraic solution $f \in U$ over F satisfying $\tau_0(y) = A * y$,
then ${}^\exists g \in F$ s.t.

$$\tau^2(sr)\tau(s)s\tau^3(g) + (\tau(r)+1)\tau(s)s\tau^2(g) - (\tau(r)+1)s\tau(g) - rg + s\tau\left(\frac{Dr}{r}\right) = 0. \quad (7)$$

Theorem 1 ([Nishioka, 2018, Theorem 9 (q-difference ver.)]) has redundant assumptions than Tietze's theorem.

Theorem 3 ([Tietze, 1905])

Consider the following difference Riccati equation

$$y(x+1) = 1 + \frac{a(x)}{y(x)}, \quad a \in \mathbb{C}(x) \setminus \mathbb{C}. \tag{8}$$

Suppose that

1. $a \to 0 \quad (x \to \infty)$, i.e., $\operatorname{ord}_\infty a > 0$.
2. (8) has no rational solution.

Then (8) has no differentially algebraic solution over $\mathbb{C}(x)$.

$\rightsquigarrow$ We shall simplify the assumptions of Nishioka's theorem to the same extent as those of Tietze's theorem.

Criterion for second order linear q-difference equation

Consider a second order linear q-difference equation

$$a_2\tau^2(y) + a_1\tau(y) + a_0 y = 0, \quad \left(a_i \in \mathbb{C}[x]^\times\right). \tag{9}$$

Tietze's normal form of the q-difference Riccati equation w.r.t. (9) is given by

$$\tau(y) = A * y = 1 + \frac{a}{y}, \qquad a = -\frac{a_2\tau(a_0)}{a_1\tau(a_1)}. \tag{10}$$

Proposition 1

Suppose that the coefficients a_0, a_1, a_2 satisfy at least one of the following inequilities:

$$(I_0) \quad \operatorname{ord}_0 a_2 + \operatorname{ord}_0 a_0 - 2\operatorname{ord}_0 a_1 > 0 \qquad (\Leftrightarrow \operatorname{ord}_0 a > 0)$$

$$(I_\infty) \quad 2\deg a_1 - \deg a_0 - \deg a_2 > 0 \qquad (\Leftrightarrow \operatorname{ord}_\infty a > 0)$$

Then the following 3rd order linear q-difference equation has no rational solution,

$$q^3\tau^2(a)\tau^3(y) + q^2(\tau(a)+1)\tau^2(y) - q(\tau(a)+1)\tau(y) - ay + q\tau\left(\frac{d}{dx}\log a\right) = 0 \tag{11}$$

Put $A = \begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix}$, $a \in \mathbb{C}(x) \backslash \mathbb{C}$. Let

$$A_1 := A, \qquad\qquad A_k := (\tau^{k-1} A)(\tau^{k-2} A) \cdots (\tau A) A.$$

Define

- $V(A) := \{ f \in \mathbb{C}(x) \mid \tau(f) = A * f \}$,
  the set of rational solutions to Tietze's normal form.
- $\overline{V}_k(A) := \left\{ f \in \overline{\mathbb{C}(x)} \mid \tau^k(f) = A_k * f \right\}$,
  the set of alg. sol. to k-iterated Tietze's normal form.

Theorem 4

Suppose (I_0) or (I_∞). If f is an algebraic solution to $\tau^k(y) = A_k * y$, then f is a rational solution to $\tau(y) = A * y$, i.e.,

$$V(A) = \overline{V}_1(A) = \overline{V}_2(A) = \cdots = \overline{V}_k(A) = \cdots . \tag{12}$$

Key ideas of proof: By [Hendriks, 1997, Lemma 10] or [Nishioka, 2010, Lemma 8], algebraic solutions are rational solutions in the variable $x^{1/l}$ for some $l \in \mathbb{Z}_{\geq 1}$. Computing the ramification order $\operatorname{ram} \left(\sum_i \varphi_i x^{i/l} \right) := \min \{ i \in \mathbb{Z} \mid \varphi_i \neq 0,\ l \not| \, i \}$, we find $\operatorname{ram} f = \infty$, i.e., $l = 1$.

Main result

Theorem 5

Consider a second order linear q-difference equation with $a_i \in \mathbb{C}[x]^\times$,

$$a_2\tau^2(y) + a_1\tau(y) + a_0 y = 0. \tag{13}$$

Tietze's normal form of the q-difference Riccati equation w.r.t. (13) is given by

$$\tau(y) = A * y = 1 + \frac{a}{y}, \qquad \left(a = -\frac{a_2\tau(a_0)}{a_1\tau(a_1)}\right). \tag{14}$$

Suppose that

1. The coefficients a_0, a_1, a_2 satisfy at least one of the following inequilities::

$$\begin{array}{ll} (I_0) & \operatorname{ord}_0 a_2 + \operatorname{ord}_0 a_0 - 2\operatorname{ord}_0 a_1 > 0 \quad (\Leftrightarrow \operatorname{ord}_0 a > 0) \\ (I_\infty) & 2\deg a_1 - \deg a_0 - \deg a_2 > 0 \quad (\Leftrightarrow \operatorname{ord}_\infty a > 0) \end{array}$$

2. (14) has no rational solution.

Then (13) has no non-trivial differentially algebraic solution over $\mathbb{C}(x)$.

$$
\begin{tikzcd}
 & & \text{Weber} \arrow[dr] & \\
\text{Gauss} \arrow[r] & \text{Kummer} \arrow[ur] \arrow[dr] & & \text{Airy} \\
 & & \text{Bessel} \arrow[ur] &
\end{tikzcd}
$$

There are three unified approaches to construct the degeneration diagram:

- Confluence of singularities
- Separation of variables of the Laplacian by orthogonal coordinates
- Classifying differential equations of the Laplace type

Degeneration diagram of Heine q-hypergeometric equation

Ohyama constructed a degeneration diagram of Heine q-hypergeometric equation by using a q-analogue of classifying differential equations of the Laplace type [Ohyama, 2011].

$$\begin{tikzcd}
 & & \text{Hahn-Exton } q\text{-Bessel} \arrow[r] & q\text{-Airy} \\
\text{Heine} \arrow[r] & q\text{-confluent} \arrow[ur] \arrow[r] \arrow[dr] & \text{Jackson } q\text{-Bessel} \arrow[dr] & \\
 & & q\text{-Hermite-Weber} \arrow[r] & \text{Ramanujan}
\end{tikzcd}$$

We check the conditions $(I_0), (I_\infty)$ for second order linear q-difference equations of the hypergeometric type (cf. [Ohyama, 2011]).

Equation	Irregular singular pt.	Conditions
Heine q-hypergeometric	$\emptyset$	none
q-confluent	$x = \infty$	(I_∞)
Jackson q-Bessel	$\emptyset$	none
Hahn-Exton q-Bessel	$x = \infty$	(I_∞)
q-Hermite-Weber	$x = 0, \infty$	$(I_0), (I_\infty)$
q-Airy	$x = \infty$	(I_∞)
Ramanujan	$x = 0, \infty^2$ (apparent)	(I_0)

In order to prove differential transcendence of solutions to five of these q-difference equations, we only have to show that there is no rat. sol. to $\tau(y) = A * y$ by virtue of Theorem 5.

Theorem 6 ([Arreche et al., 2021, Theorem 3.5 (q-difference ver.)])

Let

- $K = \bigcup_{l \in \mathbb{Z}_{\geq 1}} \mathbb{C}(x^{1/l})$: the field of ramified rational functions.
- $\tau : \varphi(x) \mapsto \varphi(qx)$

Consider a second-order linear q-difference equation

$$\tau^2(y) + a\tau(y) + by = 0 \quad (a \in K, b \in K^\times) \tag{15}$$

Suppose that

- $\tau(u) = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} * u$ has no solution u in K.
- For $^\forall \mathcal{L} \in \mathbb{C}[\delta]^\times, \delta = x\frac{d}{dx}$, the q-difference equation

$$\mathcal{L}\left(\frac{\delta b}{b}\right) = \tau(g) - g$$

  has no solution g in K.

Then (15) has no non-trivial differentially algebraic solution over K.

Let $a \in \mathbb{C}\backslash\{0\} = \mathbb{C}^{\times}$. q-Hermite-Weber equation is

$$ax\tau^2(y) + (1-x)\tau(y) - y = 0.$$

Tietze's normal form of q-Hermite-Weber equation is

$$\tau(y) = \begin{pmatrix} 1 & \alpha \\ 1 & 0 \end{pmatrix} * y, \qquad \alpha = \frac{ax}{(1-x)(1-qx)}. \tag{16}$$

We rewrite (16) as

$$Fy\tau(y) + Gy + H = 0, \tag{17}$$

where $(F, G, H) = ((1-x)(1-qx),\ -(1-x)(1-qx),\ -ax)$.

$$
\begin{gathered}
Fy\tau(y) + Gy + H = 0, \qquad (17)\\
(F, G, H) = ((1-x)(1-qx),\ -(1-x)(1-qx),\ -ax).
\end{gathered}
$$

By Hendriks' algorithm [Hendriks, 1997, Section 4.1], we shall find a rational solution u to (17). Write $u \in \mathbb{C}(x)$ as

$$
u = cx^m \frac{P}{T}, \qquad (c \in \mathbb{C}^\times, m \in \mathbb{Z})
$$

where $P, T \in \mathbb{C}[x]^{\mathrm{monic}}$ s.t. $\gcd(P, T) = \gcd(P, x) = \gcd(T, x) = 1$.
Let $R \in \mathbb{C}[x]^{\mathrm{monic}}$ be the greatest monic divisor of T satisfying $\tau R \mid P$.
Then we have ${}^{\exists}t, p \in \mathbb{C}[x]^{\mathrm{monic}}$ s.t. $T = tR, P = p\tau R$ and

$$
u = cx^m \frac{p}{t} \frac{\tau R}{R}. \qquad (18)
$$

Substituting (18) to (17), we find $p \mid H$ and $t \mid \tau^{-1}F$. Hence we define

$$
\begin{aligned}
S_p &:= \left\{\tilde{p} \in \mathbb{C}[x]^{\mathrm{monic}}\ ;\ \tilde{p} \mid H,\ \gcd(\tilde{p}, x) = 1\right\} = \{1\},\\
S_t &:= \left\{\tilde{t} \in \mathbb{C}[x]^{\mathrm{monic}}\ ;\ \tilde{t} \mid \tau^{-1}F,\ \gcd(\tilde{t}, x) = 1\right\}\\
&= \{1, x-1, x-q, (x-1)(x-q)\}.
\end{aligned}
$$

Application to q-Hermite-Weber equation

$$F y \tau(y) + G y + H = 0, \tag{17}$$
$$(F, G, H) = ((1-x)(1-qx),\ -(1-x)(1-qx),\ -ax).$$

In addition to S_p, S_t, we define

$$\begin{aligned} S_0 &:= \{\text{all possibilities for the first term of } u \text{ expressed in } \mathbb{C}((x))\} \\ &= \{1, -ax\}, \\ S_\infty &:= \big\{\text{all possibilities for the first term of } u \text{ expressed in } \mathbb{C}((x^{-1}))\big\} \\ &= \left\{1, -\frac{a}{q}x^{-1}\right\} \end{aligned}$$

Put $e := \deg R \in \mathbb{Z}_{\geq 0}$. From the form u, we obtain

$$\frac{ut}{cx^m p} = \frac{\tau R}{R} \equiv \begin{cases} 1 & \mod x, \\ q^e & \mod x^{-1}. \end{cases} \tag{19}$$

$\rightsquigarrow$ For each possibility $(p, t, u_0, u_\infty) \in S_p \times S_t \times S_0 \times S_\infty$ and the parameter $a \in \mathbb{C}^\times$, we determine c, m, R by using (17) and (19).

Application to q-Hermite-Weber equation

Tietze's normal form of q-Hermite-Weber equation

$$\tau y = \begin{pmatrix} 1 & \alpha \\ 1 & 0 \end{pmatrix} * y, \qquad \alpha = \frac{ax}{(1-x)(1-qx)}. \tag{16}$$

Proposition 2

Let $a \in \mathbb{C}^\times$.

- When $a = q^{e+1}, e \in \mathbb{Z}_{\geq 0}$, (16) has a rational solution

$$u = -\frac{1}{x-1}\frac{\tau R}{R}, \qquad R = \sum_{k=0}^{e} \binom{e}{k}_q x^k.$$

- When $a = q^{-e}, e \in \mathbb{Z}_{\geq 0}$, (16) has a rational solution

$$u = q^{-e}\frac{x}{x-1}\frac{\tau R}{R}, \qquad R = \sum_{k=0}^{e} \binom{e}{k}_q q^{k(k-e)} x^k.$$

- When $a \in \mathbb{C}^\times \backslash q^{\mathbb{Z}}$, (16) has no rational solution.

From Proposition 2 and our main result, we obtain the following theorem:

Theorem 7

Let $a \in \mathbb{C}^{\times}$. Consider q-Hermite-Weber equation

$$ax\tau^2(y) + (1-x)\tau(y) - y = 0.$$

Every non-trivial solution to q-Hermite-Weber equation is differentially transcendental over $\mathbb{C}(x)$ if $a \notin q^{\mathbb{Z}}$.

Application to Hahn-Exton q-Bessel equation

Let $\nu \in \mathbb{C}$. Hahn-Exton q-Bessel equation is the following linear q-difference equation

$$\tau(y) + \left(\frac{x^2}{4} - q^\nu - q^{-\nu}\right)y + \tau^{-1}y = 0. \tag{20}$$

This equation is transformed into the q-difference Riccati equation

$$u\tau u + \left(\frac{x^2}{4} - \alpha\right)u + 1 = 0, \tag{21}$$

where $\alpha = q^\nu + q^{-\nu}$.
Nishioka showed (21) has no algebraic solution when q is transcendental over $\mathbb{Q}$ in [Nishioka, 2016, Proposition 16].
In the same way as before, it follows from Hendriks' algorithm that (21) has no rational solution for any parameter α.

Theorem 8

For any $\nu \in \mathbb{C}$, every non-trivial solution to Hahn-Exton q-Bessel equation is differentially transcendental over $\mathbb{C}(x)$.

Conclusion and future work

Conclusion

We have examined the following subjects:

- Simplifying the conditions of Nishioka's criterion w.r.t. the q-shift operator to the same extent as those of Tietze's theorem.
- Applying our result to q-Hermite-Weber equation and Hahn-Exton q-Bessel equation.
- Determining when every non-trivial solution for these equations is differentially transcendental over $\mathbb{C}(x)$ by using Hendriks' algorithm.

Future work

Investigating to determine differential transcendence of q-confluent equation.

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Thank you for your attention!