Integral transformations of hypergeometric functions with several variables

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Plan of my talk

1. Integral transformations

- Examples

2. More integral transformations

- Examples

3. Transformations of differential equations

4. KZ equations

- An example

§ Dirichlet's integral formula

$$\int_0^1 t^{\alpha-1}(1-t)^{\mu-1}dt = \frac{\Gamma(\alpha)\Gamma(\mu)}{\Gamma(\alpha+\mu)}$$

$$\int_{\substack{t_1>0,\ldots,t_n>0\\ t_1+\cdots+t_n<1}} t_1^{\alpha_1-1}\cdots t_n^{\alpha_n-1}(1-t_1-\cdots-t_n)^{\mu-1}dt$$

$$= \int_0^1 t_1^{\alpha_1}dt_1 \int_0^{1-t_1} t_2^{\alpha_2}dt_2 \cdots \int_0^{1-t_1-\cdots-t_{n-1}} t_n^{\alpha_n-1}(1-t_1-\cdots-t_{n-1}-t_n)^{\mu-1}dt_n$$

$$(t_n = (1-t_1-\cdots-t_{n-1})s)$$

$$= \int_0^1 t_1^{\alpha_1-1}dt_1 \int_0^{1-t_1} t_2^{\alpha_2-1}dt_2 \cdots \int_0^1 (1-t_1-\cdots-t_{n-1})^{\alpha_n+\mu}s^{\alpha_n-1}(1-s)^{\mu-1}ds$$

$$= \frac{\Gamma(\mu)\Gamma(\alpha_n)}{\Gamma(\alpha_n+\mu)}\int_0^1 t_1^{\alpha_1-1}dt_1\cdots\int_0^{1-t_1-\cdots-t_{n-2}} t_{n-1}^{\alpha_{n-1}-1}(1-t_1-\cdots-t_{n-1})^{\alpha_n+\mu-1}dt_{n-1}$$

$$= \frac{\Gamma(\mu)\Gamma(\alpha_n)}{\Gamma(\alpha_n+\mu)} \times \frac{\Gamma(\alpha_n+\mu)\Gamma(\alpha_{n-1})}{\Gamma(\alpha_{n-1}+\alpha_n+\mu)} \times \cdots \times \frac{\Gamma(\alpha_1)\Gamma(\alpha_2+\cdots+\alpha_n+\mu)}{\Gamma(\alpha_1+\cdots+\alpha_n+\mu)}$$

$$= \frac{\Gamma(\mu)\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\cdots+\alpha_n+\mu)}$$

§ Integral transformation

$$\mathbf{m} = (m_1, \ldots, m_n) \geq 0 \;\Leftrightarrow\; m_1 \geq 0, \ldots, m_n \geq 0$$

$$|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n,\ \boldsymbol{\alpha} + c = (\alpha_1 + c, \ldots, \alpha_n + c),\ \mathbf{m}! = m_1! \cdots m_n!$$

$$x^{\boldsymbol{\alpha}} = \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ (c - \mathbf{x})^{\boldsymbol{\alpha}} = (c - x_1)^{\alpha_1} \cdots (c - x_n)^{\alpha_n}$$

$$\Gamma(\boldsymbol{\alpha}) = \Gamma(\alpha_1) \cdots \Gamma(\alpha_n),\ (\boldsymbol{\alpha})_{\mathbf{m}} = \frac{\Gamma(\boldsymbol{\alpha} + \mathbf{m})}{\Gamma(\boldsymbol{\alpha})} = (\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n}$$

$$\int_{t \geq 0,\ |t| < 1} t^{\boldsymbol{\alpha} - 1} (1 - |t|)^{\mu - 1} dt = \frac{\Gamma(\mu)\Gamma(\boldsymbol{\alpha})}{\Gamma(|\boldsymbol{\alpha}| + \mu)}$$

$$(1 - |\mathbf{x}|)^{-\lambda} = \sum_{\mathbf{m} \geq 0} \frac{(\lambda)_{|\boldsymbol{m}|}}{\mathbf{m}!} x^{\mathbf{m}} \quad \text{and} \quad e^{|\mathbf{x}|} = \sum_{\mathbf{m} \geq 0} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!}$$

$$K_x^{\mu, \boldsymbol{\lambda}} u(x) := \frac{1}{\Gamma(\mu)} \int_{\substack{t_1 > 0, \ldots, t_n > 0 \\ t_1 + \cdots + t_n < 1}} t^{\boldsymbol{\lambda} - 1} (1 - |t|)^{\mu - 1} u(t_1 x_1, \ldots, t_n x_n) dt_1 \ldots dt_n$$

$$K_x^{\mu, \boldsymbol{\lambda}} x^{\boldsymbol{\alpha}} = \frac{\Gamma(\boldsymbol{\alpha} + \boldsymbol{\lambda})}{\Gamma(|\boldsymbol{\alpha} + \boldsymbol{\lambda}| + \mu)} x^{\boldsymbol{\alpha}} \quad (\mu \in \mathbb{C},\ \boldsymbol{\alpha},\ \boldsymbol{\lambda} \in \mathbb{C}^n)$$

$$K_x^{\mu, \boldsymbol{\lambda}} : \mathcal{O}_0 \to \mathcal{O}_0 \quad \text{(convergent power series)}$$

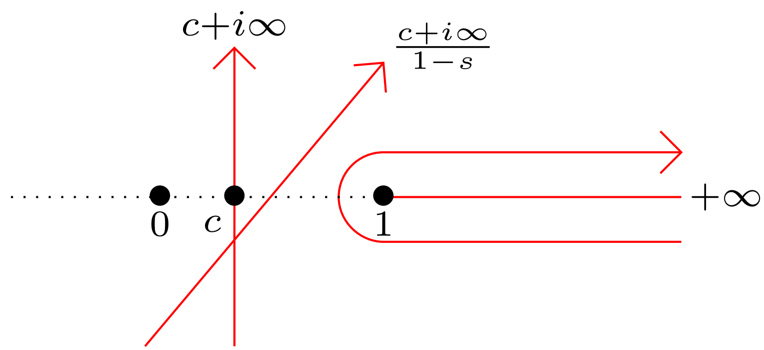
$$K_x^{\mu, \boldsymbol{\lambda}} : u(x) = \sum_{\mathbf{m} \geq 0} c_{\mathbf{m}} x^{\mathbf{m}} \mapsto \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(|\boldsymbol{\lambda}| + \mu)} \sum_{\mathbf{m} \geq 0} \frac{(\boldsymbol{\lambda})_{\mathbf{m}}}{(|\boldsymbol{\lambda}| + \mu)_{|\mathbf{m}|}} c_{\mathbf{m}} x^{\boldsymbol{m}}$$

Another integral formula

$$\int_{c-i\infty}^{c+i\infty} t^{-\alpha}(1-s-t)^{-\tau}\frac{dt}{t} = (1-s)^{-\alpha-\tau}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{1-s}\right)^{-\alpha}\left(1-\frac{t}{1-s}\right)^{-\tau}\frac{dt}{t}$$

$$(0 \le \mathrm{Re}\, s < 1,\ 0 < c < 1-\mathrm{Re}\, s)$$

$$= (1-s)^{-\alpha-\tau}\int_{\frac{c-i\infty}{1-s}}^{\frac{c+i\infty}{1-s}} t^{-\alpha}(1-t)^{-\tau}\frac{dt}{t}$$



$$= (1-s)^{-\alpha-\tau}\int_{c-i\infty}^{c+i\infty} t^{-\alpha}(1-t)^{-\tau}\frac{dt}{t}$$

$$= (1-s)^{-\alpha-\tau}(-e^{-\tau\pi i}+e^{\tau\pi i})\int_{1}^{\infty} t^{-\alpha}(t-1)^{-\tau}\frac{dt}{t}$$

$$= (1-s)^{-\alpha-\tau}\cdot 2i\sin\tau\pi\int_0^1 \left(\frac{1}{u}\right)^{-\alpha}\left(\frac{1}{u}-1\right)^{-\tau}\frac{du}{u} \quad \left(u=\frac{1}{t}\right)$$

$$= \frac{2\pi i(1-s)^{-\alpha-\tau}}{\Gamma(\tau)\Gamma(1-\tau)}\int_0^1 u^{\alpha+\tau-1}(1-u)^{-\tau}du$$

$$= 2\pi i\frac{\Gamma(\alpha+\tau)}{\Gamma(\tau)\Gamma(\alpha+1)}(1-s)^{-(\alpha+\tau)}$$

$$\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty}\cdots\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty} t^{1-\boldsymbol{\alpha}}(1-t_1-\cdots-t_n)^{-\tau}\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}$$

$$= (2\pi i)^n \frac{\Gamma(\alpha_1-1+\tau)}{\Gamma(\tau)\Gamma(\alpha_1)}\frac{\Gamma(\alpha_1+\alpha_2-2+\tau)}{\Gamma(\alpha_1-1+\tau)\Gamma(\alpha_2)}\cdots\frac{\Gamma(|\boldsymbol{\alpha}|-n+\tau)}{\Gamma(\alpha_2+\cdots+\alpha_{n-1}-n+1+\tau)\Gamma(\alpha_n)}$$

$$= (2\pi i)^n \frac{\Gamma(|\boldsymbol{\alpha}|+\tau-n)}{\Gamma(\tau)\Gamma(\boldsymbol{\alpha})} \qquad (\tau=\mu+n)$$

$$(L_x^{\mu,\boldsymbol{\lambda}}\phi)(x) := \frac{\Gamma(\mu+n)}{(2\pi i)^n}\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty}\cdots\int_{\frac{1}{n+1}-i\infty}^{\frac{1}{n+1}+i\infty} t^{1-\boldsymbol{\lambda}}(1-|\mathbf{t}|)^{-\mu-n}\phi\left(\tfrac{x_1}{t_1},\ldots,\tfrac{x_n}{t_n}\right)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}$$

$$L_x^{\mu,\boldsymbol{\lambda}}x^{\boldsymbol{\alpha}} = \frac{\Gamma(|\boldsymbol{\alpha}+\boldsymbol{\lambda}|+\mu)}{\Gamma(\boldsymbol{\alpha}+\boldsymbol{\lambda})}x^{\boldsymbol{\alpha}}, \quad K_x^{\mu,\boldsymbol{\lambda}}x^{\boldsymbol{\alpha}} = \frac{\Gamma(\boldsymbol{\alpha}+\boldsymbol{\lambda})}{\Gamma(|\boldsymbol{\alpha}+\boldsymbol{\lambda}|+\mu)}x^{\boldsymbol{\alpha}}$$

$$x^{-\boldsymbol{\lambda}'}\circ L_x^{\mu,\boldsymbol{\lambda}}\circ x^{\boldsymbol{\lambda}'} = L_x^{\mu,\boldsymbol{\lambda}+\boldsymbol{\lambda}'},\; x^{-\boldsymbol{\lambda}'}\circ K_x^{\mu,\boldsymbol{\lambda}}\circ x^{\boldsymbol{\lambda}'} = K_x^{\mu,\boldsymbol{\lambda}+\boldsymbol{\lambda}'}$$

$$L_x^{\mu} := L_x^{\mu,1},\; K_x^{\mu} := K_x^{\mu,1}$$

$$K_x^{\mu,\boldsymbol{\lambda}}\sum_{\mathbf{m}\geq 0} c_{\mathbf{m}}x^{\mathbf{m}} = \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(|\boldsymbol{\lambda}|+\mu)}\sum_{\mathbf{m}\geq 0}\frac{(\boldsymbol{\lambda})_{\mathbf{m}}}{(|\boldsymbol{\lambda}|+\mu)_{|\mathbf{m}|}}c_{\mathbf{m}}x^{\mathbf{m}}$$

$$L_x^{\mu,\boldsymbol{\lambda}}\sum_{\mathbf{m}\geq 0} c_{\mathbf{m}}x^{\mathbf{m}} = \frac{\Gamma(|\boldsymbol{\lambda}|+\mu)}{\Gamma(\boldsymbol{\lambda})}\sum_{\mathbf{m}\geq 0}\frac{(|\boldsymbol{\lambda}|+\mu)_{|\mathbf{m}|}}{(\boldsymbol{\lambda})_{\mathbf{m}}}c_{\mathbf{m}}x^{\mathbf{m}}$$

§ Examples

$$n=1\ :\ (K_x^{\mu}u)(x) = \frac{1}{\Gamma(\mu)}\int_0^1 (1-t)^{\mu-1}u(tx)dt$$

$$= \frac{1}{\Gamma(\mu)}\int_0^x (1-\tfrac{s}{x})^{\mu-1}u(s)\tfrac{ds}{x} \quad (s=tx)$$

$$= x^{-\mu}\frac{1}{\Gamma(\mu)}\int_0^x (x-s)^{\mu-1}u(s)ds \quad (I_0^{\mu}u : \text{Riemann-Liouville integral})$$

$$K_x^{\mu,\lambda_1}(1-x)^{-\lambda_0} = \frac{\Gamma(\lambda_1)}{\Gamma(\lambda_1+\mu)}\sum_{m=0}^{\infty}\frac{(\lambda_1)_m}{(\lambda_1+\mu)_m}\frac{(\lambda_0)_m}{m!}x^m$$

$$= \frac{\Gamma(\lambda_1)}{\Gamma(\lambda_1+\mu)}F(\lambda_0,\lambda_1,\lambda_1+\mu;x) \quad (\text{Gauss HG})$$

$$K_x^{\mu,\lambda}e^x = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{(\lambda+\mu)_m m!}x^m$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)}{}_1F_1(\lambda,\lambda+\mu;x) \quad (\text{Kummer Conf. HG})$$

$$K_{x,y}^{\mu,\lambda_1,\lambda_2}(1-x-y)^{-\lambda_0} = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2+\mu)}\sum_{i,j\ge 0}\frac{(\lambda_1)_i(\lambda_2)_j}{(\lambda_1+\lambda_2+\mu)_{i+j}}\frac{(\lambda_0)_{i+j}}{i!j!}x^iy^j$$

$$= \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2+\mu)}F_1(\lambda_0;\lambda_1,\lambda_2;\lambda_1+\lambda_2+\mu;x,y)$$

Lauricella hypergeometric series

$$F_D(\lambda_0,\boldsymbol{\lambda},\mu;\mathbf{x}) := \sum_{\mathbf{m}\geq 0} \frac{(\lambda_0)_{|\mathbf{m}|}(\boldsymbol{\lambda})_{\mathbf{m}}}{(\mu)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} = \frac{\Gamma(\mu)}{\Gamma(\boldsymbol{\lambda})} K_x^{\mu-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-|\mathbf{x}|)^{-\lambda_0}$$

$$F_A(\lambda_0,\boldsymbol{\mu},\boldsymbol{\lambda};\mathbf{x}) := \sum_{\mathbf{m}\geq 0} \frac{(\lambda_0)_{|\mathbf{m}|}(\boldsymbol{\mu})_{\mathbf{m}}}{(\boldsymbol{\lambda})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}$$

$$= \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(\boldsymbol{\mu})} K_{x_1}^{\lambda_1-\mu_1,\mu_1}\cdots K_{x_n}^{\lambda_n-\mu_n,\mu_n}(1-|\mathbf{x}|)^{-\lambda_0}$$

$$= \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(\lambda_0)} L_x^{\lambda_0-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-\mathbf{x})^{-\boldsymbol{\mu}}$$

$$F_B(\boldsymbol{\lambda},\boldsymbol{\lambda}',\mu;\mathbf{x}) := \sum_{\mathbf{m}\geq 0} \frac{(\boldsymbol{\lambda})_{\mathbf{m}}(\boldsymbol{\lambda}')_{\mathbf{m}}}{(\mu)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{m} = \frac{\Gamma(\mu)}{\Gamma(\boldsymbol{\lambda})} K_x^{\mu-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-\mathbf{x})^{-\boldsymbol{\lambda}'}$$

$$F_C(\mu,\lambda_0,\boldsymbol{\lambda};\mathbf{x}) := \sum_{\mathbf{m}\geq 0} \frac{(\mu)_{|\mathbf{m}|}(\lambda_0)_{|\mathbf{m}|}}{(\boldsymbol{\lambda})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} = \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(\mu)} L_x^{\mu-|\boldsymbol{\lambda}|,\boldsymbol{\lambda}}(1-|\mathbf{x}|)^{-\lambda_0}$$

$$n=2 \;\Rightarrow (F_D,F_A,F_B,F_C) = (F_1,F_2,F_3,F_4)$$

Horn's series (confluent hypergeometric functions)

$$\Phi_2(\beta,\beta';\gamma;x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m(\beta')_n}{(\gamma)_{m+n}m!n!}x^m y^n$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')}K_{x,y}^{\gamma-\beta-\beta',\beta,\beta'}e^{x+y}$$

$$\Psi_1(\alpha;\beta;\gamma,\gamma';x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n m!n!}x^m y^n$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)}L_{x,y}^{\alpha-\gamma-\gamma',\gamma,\gamma'}(1-x)^{-\beta}e^{y}$$

$$\Psi_2(\alpha;\gamma',\gamma';x,y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n m!n!}x^m y^n$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)}L_{x,y}^{\alpha-\gamma-\gamma',\gamma,\gamma'}e^{x+y}$$

§ More transformations

$x \mapsto R(x)$: a coordinare transformation of $\mathbb{C}^n$

$(T_{x \to R(x)}\phi)(x) := \phi(R(x))$

$\mathbf{y} = (x_{i_1}, \ldots, x_{i_k})$ for a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$

$\mu \in \mathbb{C}$ and $\boldsymbol{\lambda} \in \mathbb{C}^k$ we define

$$K^{\mu,\boldsymbol{\lambda}}_{\mathbf{y},x\to R(x)} := T^{-1}_{x\to R(x)} \circ K^{\mu,\boldsymbol{\lambda}}_{\mathbf{y}} \circ T_{x\to R(x)}$$

$$L^{\mu,\boldsymbol{\lambda}}_{\mathbf{y},x\to R(x)} := T^{-1}_{x\to R(x)} \circ L^{\mu,\boldsymbol{\lambda}}_{\mathbf{y}} \circ T_{x\to R(x)}$$

$$\mathbf{p} = \left(p_{i,j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \in GL(n, \mathbb{Z})$$

$$p_{i_\nu,j} \ge 0 \qquad (1 \le \nu \le k,\ 1 \le j \le n)$$

$$x \mapsto x^{\mathbf{p}} = \mathbf{x}^{\mathbf{p}} = (x^{p_{*,1}}, \ldots, x^{p_{*,n}}) = \left(\prod_{\nu=1}^{n} x_\nu^{p_{\nu,1}}, \ldots, \prod_{\nu=1}^{n} x_\nu^{p_{\nu,n}}\right),$$

$$\mathbf{pm} = (p_{1,*}\mathbf{m}, \ldots, p_{n,*}\mathbf{m}) = \left(\sum_{\nu=1}^{n} p_{1,\nu}m_\nu, \ldots, \sum_{\nu=1}^{n} p_{n,\nu}m_\nu\right)$$

$$\left(T^{-1}_{x\to x^{\mathbf{p}}}\circ T_{x\to(t_1x_1,\dots,t_nx_n)}\circ T_{x\to x^{\mathbf{p}}}\phi\right)(x)=\phi\Big(x_1\prod_{\nu=1}^{p}t_\nu^{p_{\nu,1}},\dots,x_n\prod_{\nu=1}^{n}t_\nu^{p_{\nu,n}}\Big),$$

$$\left(K^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}\phi\right)(x)$$

$$=\frac{1}{\Gamma(\mu)}\int_{\substack{t_1>0,\dots t_k>0\\ t_1+\dots+t_k<1}}\mathbf{t}^{\boldsymbol{\lambda}-1}(1-|\boldsymbol{t}|)^{\mu-1}\phi\Big(x_1\prod_{\nu=1}^{k}t_\nu^{p_{i_\nu,1}},\dots,x_n\prod_{\nu=1}^{k}t_\nu^{p_{i_\nu,n}}\Big)dt$$

$$\left(L^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}\phi\right)(x)=\frac{\Gamma(\mu+k)}{(2\pi i)^k}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}\mathbf{t}^{1-\boldsymbol{\lambda}}(1-|\boldsymbol{t}|)^{-\mu-k}$$

$$\phi\Big(\frac{x_1}{\prod_{\nu=1}^{k}t_\nu^{p_{i_\nu,1}}},\dots,\frac{x_n}{\prod_{\nu=1}^{k}t_\nu^{p_{i_\nu,n}}}\Big)\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}\quad\text{with}\quad c=\tfrac{1}{k+1}$$

$$(\mathbf{pm})_{i_1,\dots,i_k}:=\Big(\sum_{\nu=1}^{n}p_{i_1,\nu}m_\nu,\dots,\sum_{\nu=1}^{n}p_{i_k,\nu}m_\nu\Big),$$

$$K^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}x^{\mathbf{m}}=\frac{\Gamma(\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k})}{\Gamma(|\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k}|+\mu)}x^{\mathbf{m}},$$

$$L^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}}x^{\mathbf{m}}=\frac{\Gamma(|\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k}|+\mu)}{\Gamma(\boldsymbol{\lambda}+(\mathbf{pm})_{i_1,\dots,i_k})}x^{\mathbf{m}}$$

$$K^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}} \sum_{\mathbf{m}\geq 0} c_{\mathbf{m}} x^{\mathbf{m}} = \frac{\Gamma(\boldsymbol{\lambda})}{\Gamma(|\boldsymbol{\lambda}|+\mu)} \sum_{\mathbf{m}\geq 0} \frac{(\boldsymbol{\lambda})_{(\mathbf{pm})_{i_1,\dots,i_k}}}{(|\boldsymbol{\lambda}|+\mu)_{|(\mathbf{pm})_{i_1,\dots,i_k}|}} c_{\mathbf{m}} x^{\mathbf{m}}$$

$$L^{\mu,\boldsymbol{\lambda}}_{(x_{i_1},\dots,x_{i_k}),x\to x^{\mathbf{p}}} \sum_{\mathbf{m}\geq 0} c_{\mathbf{m}} x^{\mathbf{m}} = \frac{\Gamma(|\boldsymbol{\lambda}|+\mu)}{\Gamma(\boldsymbol{\lambda})} \sum_{\mathbf{m}\geq 0} \frac{(|\boldsymbol{\lambda}|+\mu)_{|(\mathbf{pm})_{i_1,\dots,i_k}|}}{(\boldsymbol{\lambda})_{(\mathbf{pm})_{i_1,\dots,i_k}}} c_{\mathbf{m}} x^{\mathbf{m}}$$

$\mathbf{p} = \left(\begin{smallmatrix} p_1 & p_2 \\ q_1 & q_2 \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$ with $p_1,\ p_2,\ q_1,\ q_2 \geq 0$. Put $\tilde{\mathbf{p}} = \mathbf{p} \oplus I_{n-2} \in GL(n,\mathbb{Z})$.

$$K^{\mu,(\lambda_1,\lambda_2)}_{(x_1,x_2),x\to x^{\tilde{\mathbf{p}}}} x^{\mathbf{m}} = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2+\mu)} \frac{(\lambda_1)_{p_1m_1+p_2m_2}(\lambda_2)_{q_1m_1+q_2m_2}}{(\lambda_1+\lambda_2+\mu)_{(p_1+q_1)m_1+(p_2+q_2)m_2}} x^{\mathbf{m}}$$

$$K^{\mu,\lambda}_{x_1,x\to x^{\tilde{\mathbf{p}}}} x^{\mathbf{m}} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)} \frac{(\lambda)_{p_1m_1+p_2m_2}}{(\lambda+\mu)_{p_1m_1+p_2m_2}} x^{\mathbf{m}}$$

$$\begin{aligned} K^{\mu,\lambda}_{x,(x,y)\mapsto(x,\frac{x}{y})} (1-x)^{-\alpha}(1-y)^{-\beta} &= \frac{1}{\Gamma(\mu)} \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1-tx)^{-\alpha}(1-ty)^{-\beta}dt \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)} \sum_{\mathbf{m}\geq 0} \frac{(\lambda)_{m_1+m_2}}{(\lambda+\mu)_{m_1+m_2}} \frac{(\alpha)_{m_1}(\beta)_{m_2}}{m_1!m_2!} x^{m_1}y^{m_2} \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+\mu)} F_1(\lambda,\alpha,\beta,\lambda+\mu;x,y) \end{aligned}$$

§ Differential equations

Notation : $\partial := \frac{d}{dx}$, $\vartheta := x\partial$, $\partial_i := \frac{\partial}{\partial x_i}$, $\vartheta_i := x_i\partial_i$, $W[x] := \mathbb{C}[x]\otimes\mathbb{C}[\boldsymbol{\partial}]$

$n=1$: $K_x^\mu = x^{-u} I_0^\mu$, $L_x^\mu = I_0^{-\mu} x^\mu$ $\quad (I_0^\tau \circ I_0^\mu = I_0^{\tau+\mu},\ I_0^0 = \mathrm{id})$

$P(x,\partial)u = 0 \ \Rightarrow\ \partial^\gamma P = \sum c_{i,j}\partial^i\vartheta^j \in W[x] \quad (\exists\,\gamma \in \mathbb{Z}_{\ge 0})$

$\Rightarrow\ \mathrm{mc}_\mu(P) := \partial^{-\delta}\sum c_{i,j}\partial^i(\vartheta-\mu)^j \in W[x]$ (maximal $\delta \in \mathbb{Z}_{\ge 0}$)

$Pu = 0 \ \Rightarrow\ \mathrm{mc}_\mu(P) I_0^\mu u = 0$ (middle convolution)

General case : $\quad x_i\partial_i\big(u(tx)\big) = \big(x_i\partial_i u(x)\big)|_{x\mapsto tx}$

$$(\vartheta_i K_x^\mu u)(x) = \frac{1}{\Gamma(\mu)}\int_0^1 (1-|\mathbf{t}|)^{\mu-1} x_i t_i (\partial_i u)(tx)dt = (K_x^\mu \vartheta_i u)(x)$$

$$\partial_i\big((1-|\mathbf{t}|)^{\mu-1}u(tx)\big) = -(\mu-1)(1-|\mathbf{t}|)^{\mu-2}u(tx) + (1-t)^{\mu-1}x_i(\partial_i u)(tx)$$

$$x_i K_x^\mu \partial_i = (\mu-1)K_x^{\mu-1} = x_\nu K_x^\mu \partial_\nu$$

$$\mu\int_0^1 (1-|\mathbf{t}|)^{\mu-1}u(tx)dt = x_i\int_0^1 (1-|\mathbf{t}|)^{\mu}(\partial_i u)(tx)dt \quad (\uparrow\ \mu\mapsto\mu+1)$$

$$= x_i\int_0^1 (1-|\mathbf{t}|)^{\mu-1}(1-t_1-\cdots-t_n)(\partial_i u)(tx)dt$$

$$x_i K_x^\mu \partial_i u = \mu K_x^\mu u + \sum_{\nu=1}^n \frac{x_i}{\Gamma(\mu)} \frac{1}{x_\nu} \int_0^1 (1-|\mathbf{t}|)^{\mu-1} \big((x_\nu \partial_i u)|_{x\to tx}\big) dt$$

$$= \mu K_x^\mu u + \sum_{\nu=1}^n \frac{x_i}{x_\nu} K_x^\mu \partial_i x_\nu u - K_x^\mu u$$

$$= (\mu-1) K_x^\mu u + \sum_{\nu=1}^n K_x^\mu \partial_\nu x_\nu u = (\mu+n-1) K_x^\mu u + \sum_{\nu=1}^n \vartheta_\nu K_x^\mu u$$

$$\boxed{\begin{aligned} &K_x^\mu \circ \vartheta_j = \vartheta_j \circ K_x^\mu \\ &K_x^\mu \circ \partial_j = \tfrac{1}{x_j}(\vartheta_1 + \cdots + \vartheta_n + \mu + n - 1) \circ K_x^\mu \end{aligned}}$$

Def. Suppose $u(x) \in \mathcal{O}_0$ and $P \in \mathbb{C}(x) \otimes W[x]$ satisfies $Pu = 0$. Define

$$\mathrm{R}P \in W[x] \cap (\mathbb{C}[x] \setminus \{0\}) P \quad (\deg_x \mathrm{R}P \text{ is minimal})$$

Choose minimal $\boldsymbol{\gamma} \in \mathbb{Z}_{\geq 0}^n$ so that

$$\partial^{\boldsymbol{\gamma}} \mathrm{R}P = \sum_{\alpha,\, \beta \in \mathbb{Z}_{\geq 0}^n} c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} \vartheta^{\boldsymbol{\beta}} \quad (c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{C}).$$

Then $K_x^\mu(\partial^\gamma \mathrm{R} P) K_x^\mu u(x) = 0$ with

$$K_x^\mu\Big(\sum c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} \vartheta^{\boldsymbol{\beta}}\Big) := \mathrm{R} \sum c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \Big(\prod_{k=1}^{n} \big(\tfrac{1}{x_k}(\vartheta_1 + \cdots + \vartheta_n + \mu + n - 1)\big)^{\alpha_k}\Big) \vartheta^{\boldsymbol{\beta}}$$

$L_x^\mu(\partial^\gamma \mathrm{R} P) L_x^\mu u(x) = 0$ with replacing P by $(x_j, \partial_j) \mapsto (x_j^{-1}, -x_j(\vartheta_j + 1))$ and

$$L_x^\mu\Big(\sum c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \partial^{\boldsymbol{\alpha}} \vartheta^{\boldsymbol{\beta}}\Big) := \mathrm{R} \sum c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \Big(\prod_{k=1}^{n} \big(x_k(\mu - \vartheta_1 - \cdots - \vartheta_n)\big)^{\alpha_k}\Big) (-\boldsymbol{\vartheta} - 1)^{\boldsymbol{\beta}}$$

Remark. $P_1,\ P_2 \in W[x]$

$$\{u \in \mathcal{O}_0 \mid P_1 u = 0\} = \{0\} \ \Rightarrow\ \{u \in \mathcal{O}_0 \mid P_1 P_2 u = 0\} = \{u \in \mathcal{O}_0 \mid P_2 u = 0\}$$

$n = 1 :\ P_1 = \vartheta + \mu \Rightarrow$ middle convolution $\Rightarrow$ keeps irreducibility $\quad (\mu \notin \mathbb{Z})$

General case : $K_x^\mu \Rightarrow P_1 = \vartheta_1 + \cdots + \vartheta_n + \mu + m \quad (m \in \mathbb{Z}) \Rightarrow$?

$L_x^\mu \Rightarrow P_1 = \vartheta_1 + \cdots + \vartheta_n - \mu + m \quad (m \in \mathbb{Z}) \Rightarrow$?

§ KZ (Knizhnik-Zamolodchikov) equation

$$\mathcal{M} : \frac{\partial u}{\partial x_i} = \sum_{\substack{0 \le \nu \le q \\ \nu \ne i}} \frac{A_{i,\nu}}{x_i - x_\nu} u \quad (i = 0, \dots, q)$$

$$A_{i,j} = A_{j,i} \in M(N, \mathbb{C}) \quad (i,\, j \in \{0, 1, \dots, q+1\}),$$

$$A_{i,i} = A_\emptyset = A_i = 0, \quad A_{i,q+1} := -\sum_{\nu=0}^{q} A_{i,\nu},$$

$$A_{i_1, i_2, \dots, i_k} := \sum_{1 \le \nu < \nu' \le k} A_{i_\nu, i_{\nu'}} \quad (\{i_1, \dots, i_k\} \subset \{0, \dots, q+1\}),$$

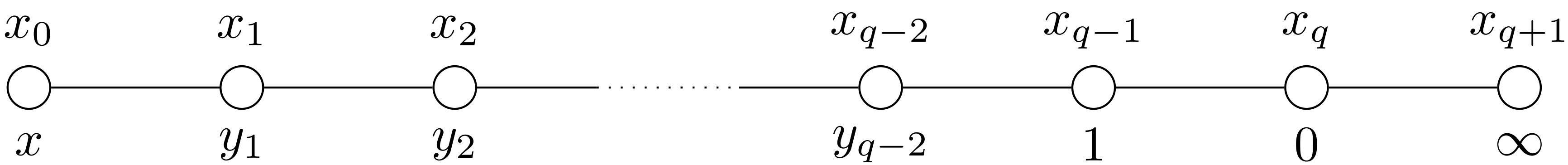
Compatibility condition (cf. [Ok]):

$$[A_I, A_J] = 0 \quad \text{if } I \cap J = \emptyset \text{ or } I \subset J \text{ with } I,\, J \subset \{0, \dots, q+1\}$$

We may assume $\mathcal{M}$ is homogeneous :

$$A_I = 0 \quad (\# I = q + 1)$$

$\mathfrak{S}_{q+2}$ acts on the space of KZ systems as the permutaions of the indices



Rigid irreducible Fuchsian system

$$\frac{du}{dx} = \sum_{i=1}^{q} \frac{A_i}{x - x_i} u$$

can be extended to a KZ equation $\mathcal{M}$ with $x = x_0$ and $A_i = A_{0,i}$ (cf. [Ha])

$n = q - 1 = 2$: $\quad A_{01} + A_{01} + A_{03} + A_{12} + A_{13} + A_{23} = 0$

$(A_{01}, A_{02}, A_{03}, A_{12}, A_{13})$ determines $\mathcal{M}$

$$\begin{array}{rcl}
(x_0, x_1, x_2, x_3, x_4) & \to & (x, y, 1, 0, \infty) \\
x_0 \leftrightarrow x_1 & \to & (x, y) \leftrightarrow (y, x) \\
x_1 \leftrightarrow x_2 & \to & (x, y) \leftrightarrow (\frac{x}{y}, \frac{1}{y}) \\
x_2 \leftrightarrow x_3 & \to & (x, y) \leftrightarrow (1 - x, 1 - y) \\
x_3 \leftrightarrow x_4 & \to & (x, y) \leftrightarrow (\frac{1}{x}, \frac{1}{y})
\end{array}$$

$$\hat{K}_x^{\mu}: u(x,y) \mapsto \hat{u}(x,y) = \begin{pmatrix} xK_x^{\mu+1}\frac{u(x,y)}{x-y} \\ xK_x^{\mu+1}\frac{u(x,y)}{x-1} \\ xK_x^{\mu+1}\frac{u(x,y)}{x} \end{pmatrix}, \quad \frac{\partial \hat{u}}{\partial x_i} = \sum_{\substack{0\le\nu\le q \\ \nu\neq i}} \frac{\hat{A}_{i,\nu}}{x_i - x_\nu}\hat{u}$$

$$\hat{A}_{01} = \begin{pmatrix} A_{01}+\mu & A_{02} & A_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{A}_{02} = \begin{pmatrix} 0 & 0 & 0 \\ A_{01} & A_{02}+\mu & A_{03} \\ 0 & 0 & 0 \end{pmatrix}, \ \hat{A}_{03} = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ A_{01} & A_{02} & A_{03} \end{pmatrix}$$

$$\hat{A}_{12} = \begin{pmatrix} A_{02}+A_{12} & -A_{02} & 0 \\ -A_{01} & A_{01}+A_{12} & 0 \\ 0 & 0 & A_{12} \end{pmatrix}, \ \hat{A}_{13} = \begin{pmatrix} A_{03}+A_{13} & 0 & -A_{03} \\ 0 & A_{13} & 0 \\ -A_{01} & 0 & A_{01}+A_{13} \end{pmatrix}$$

$$\hat{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ 0 & A_{03}+A_{23} & -A_{03} \\ 0 & -A_{02} & A_{02}+A_{23} \end{pmatrix}, \ \hat{A}_{04} = -\begin{pmatrix} A_{01} & A_{02} & A_{03} \\ A_{01} & A_{02} & A_{03} \\ A_{01} & A_{01} & A_{03} \end{pmatrix}$$

$$\hat{A}_{14} = \begin{pmatrix} A_{23}-\mu & 0 & 0 \\ A_{01} & A_{02}+A_{03}+A_{23} & 0 \\ A_{01} & 0 & A_{02}+A_{03}+A_{23} \end{pmatrix}$$

(cf. [DR, Ha])

$$\hat{A}_{24} = \begin{pmatrix} A_{01}+A_{13}+A_{03} & A_{02} & 0 \\ 0 & A_{13}-\mu & 0 \\ 0 & A_{02} & A_{01}+A_{13}+A_{03} \end{pmatrix}$$

$$\hat{A}_{34} = \begin{pmatrix} A_{01}+A_{02}+A_{12}+\mu & 0 & A_{03} \\ 0 & A_{01}+A_{02}+A_{12}+\mu & A_{03} \\ 0 & 0 & A_{12} \end{pmatrix}$$

$$\hat{K}_x^{\mu,\lambda} := x^{-\lambda}\circ\hat{K}_x^{\mu}\circ x^{\lambda}, \quad \hat{K}_x^{\mu,\lambda}u = \begin{pmatrix} K_x^{\mu+1,\lambda}\frac{xu(x,y)}{x-y} \\ K_x^{\mu+1,\lambda}\frac{xu(x,y)}{y} \\ K_x^{\mu+1,\lambda}u(x,y) \end{pmatrix}, \quad \mathcal{L} := \begin{pmatrix} \ker A_{01} \\ \ker A_{02} \\ 0 \end{pmatrix}$$

$$\hat{K}_y^{\mu,\lambda} : u(x,y) \mapsto \hat{K}_x^{\mu,\lambda} u(y,x)\Big|_{(x,y)\mapsto(y,x)}, \quad \hat{K}_y^{\mu,\lambda} u = \begin{pmatrix} K_y^{\mu+1,\lambda} \frac{y u(x,y)}{y-x} \\ K_y^{\mu+1,\lambda} \frac{y u(x,y)}{x} \\ K_y^{\mu+1,\lambda} u(x,y) \end{pmatrix}$$

$$\mathcal{L} := \begin{pmatrix} \ker A_{01} \\ \ker A_{12} \\ 0 \end{pmatrix} \quad : \text{ invariant subspace for generic } \lambda \text{ and } \mu \qquad (x_0 \leftrightarrow x_1)$$

$$\hat{K}_{x,y}^{\mu,\lambda} : u(x,y) \mapsto \left(\hat{K}_x^{\mu,\lambda} u(x, \tfrac{x}{y})\right)\Big|_{y\mapsto \frac{x}{y}} : \; x_0 \leftrightarrow x_2, \; x_3 \leftrightarrow x_4$$

$$\hat{A}_{01} = \begin{pmatrix} A_{01}+A_{02} & -A_{02} & 0 \\ -A_{12} & A_{01}+A_{12} & 0 \\ 0 & 0 & A_{01} \end{pmatrix}, \quad \hat{A}_{02} = \begin{pmatrix} 0 & 0 & 0 \\ A_{12} & A_{02}+\mu & A_{24}+\lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{A}_{03} = \begin{pmatrix} A_{03} & A_{02} & 0 \\ 0 & A_{14}-\mu-\lambda & 0 \\ 0 & A_{02} & A_{03} \end{pmatrix}, \quad \hat{A}_{04} = \begin{pmatrix} A_{04}+A_{24} & 0 & 0 \\ 0 & A_{04}+A_{24}+\lambda & -A_{24}-\lambda \\ 0 & -A_{02} & A_{02}+A_{12} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} A_{12}+\mu & A_{02} & A_{24}+\lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} A_{04}-\mu-\lambda & 0 & 0 \\ A_{12} & A_{13} & 0 \\ A_{12} & 0 & A_{13} \end{pmatrix}$$

$$\hat{A}_{14} = \begin{pmatrix} A_{14}+A_{24}+\lambda & 0 & -A_{24}-\lambda \\ 0 & A_{14}+A_{24} & 0 \\ -A_{12} & 0 & A_{12}+A_{14} \end{pmatrix}, \quad \hat{A}_{23} = \begin{pmatrix} -A_{12}+\lambda & -A_{02} & -A_{24}-\lambda \\ -A_{12} & -A_{02}+\lambda & -A_{24}-\lambda \\ -A_{12} & -A_{02} & -A_{24} \end{pmatrix}$$

$$\hat{A}_{24} = \begin{pmatrix} -\mu-\lambda & 0 & 0 \\ 0 & -\mu-\lambda & 0 \\ A_{12} & A_{02} & A_{24} \end{pmatrix}, \quad \hat{A}_{34} = \begin{pmatrix} A_{01}+A_{02}+A_{12}+\mu & 0 & A_{24}+\lambda \\ 0 & A_{01}+A_{02}+A_{12}+\mu & A_{24}+\lambda \\ 0 & 0 & A_{01} \end{pmatrix}$$

$$\mathcal{L} := \begin{pmatrix} \ker A_{12} \\ \ker A_{02} \\ 0 \end{pmatrix} \quad : \text{ invariant subspace for generic } \lambda \text{ and } \mu$$

$$\frac{du}{dx} = \frac{A_y}{x-y}u + \frac{A_1}{x-1}u + \frac{A_0}{x}u \quad \xrightarrow{\text{rigid}} \quad \frac{\partial u}{\partial y} = \frac{A_y}{x-y}u + \frac{B_1}{y-1}u + \frac{B_0}{y}u$$

$$\xrightarrow{\hat{K}^{\mu}_{x},\ \hat{K}^{\mu}_{y},\ \hat{K}^{\mu}_{x,y}}$$

$$\frac{d\hat{u}}{dx} = \frac{\begin{pmatrix} A_y+\mu & A_1 & A_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{x-y}\hat{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ A_y & A_1+\mu & A_0 \\ 0 & 0 & 0 \end{pmatrix}}{x-1}\hat{u} + \frac{\begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ A_y & A_1 & A_0 \end{pmatrix}}{x}\hat{u}$$

$$\frac{d\hat{u}}{dx} = \frac{\begin{pmatrix} A_y+\mu & B_1 & B_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}{x-y}\hat{u} + \frac{\begin{pmatrix} A_1+B_1 & -B_1 & 0 \\ -A_y & A_1+A_y & 0 \\ 0 & 0 & A_1 \end{pmatrix}}{x-1}\hat{u} + \frac{\begin{pmatrix} A_0+B_0 & 0 & -B_0 \\ 0 & A_0 & 0 \\ -A_y & 0 & A_0+A_y \end{pmatrix}}{x}\hat{u}$$

$$\frac{d\hat{u}}{dx} = \frac{\begin{pmatrix} A_y+B_1 & -B_1 & 0 \\ -A_1 & A_y+A_1 & 0 \\ 0 & 0 & A_y \end{pmatrix}}{x-y}\hat{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ B_0 & A_0+\mu & A_{24} \\ 0 & 0 & 0 \end{pmatrix}}{x-1}\hat{u} + \frac{\begin{pmatrix} A_0 & A_1 & 0 \\ 0 & A_{14}-\mu & 0 \\ A_y & A_1 & A_0 \end{pmatrix}}{x}\hat{u}$$

$$A_{14} = -A_y - B_0 - B_1$$

$$A_{24} = A_y + A_0 + B_0$$

$$\mathrm{idx}_x\mathcal{M} := 2N^2 - \sum_{i=1}^{q+1}(N^2 - \dim Z_{M(N,\mathbb{C})}A_{0,i}) = 2 \Leftrightarrow \text{ rigid}$$

An example $(F_1 : p = q = r = 1)$

$$F_{p,q,r}\left(\begin{smallmatrix}\boldsymbol\alpha & \boldsymbol\beta & \boldsymbol\gamma\\ \boldsymbol\alpha' & \boldsymbol\beta' & \boldsymbol\gamma'\end{smallmatrix}; x, y\right) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_m\prod_{j=1}^{q}(\beta_j)_n\prod_{k=1}^{r}(\gamma_k)_{m+n}x^m y^n}{\prod_{i=1}^{p}(1-\alpha'_i)_m\prod_{j=1}^{q}(1-\beta'_j)_n\prod_{k=1}^{r}(1-\gamma'_k)_{m+n}}$$

$$= C\prod_{i=2}^{p}K_x^{1-\alpha'_i-\alpha_i,\alpha_i}\prod_{j=2}^{q}K_y^{1-\beta'_j-\beta_j,\beta_j}\prod_{k=1}^{r}K_{x,y}^{1-\gamma'_k-\gamma_k,\gamma_k}(1-x)^{\alpha_1}(1-y)^{-\beta_1}$$

$$\alpha''_i := \alpha_i + \alpha'_i,\ \beta''_j := \beta_j + \beta'_j,\ \gamma''_k := \gamma_k + \gamma'_k,\ \alpha'_1 = \beta'_1 = 0$$

$$\alpha'' = \sum_{i=1}^{p}\alpha''_i,\ \beta'' = \sum_{j=1}^{q}\beta''_j,\ \gamma'' = \sum_{k=1}^{r}\gamma''_k$$

Riemann scheme of KZ equation : $(\mathrm{rank}\,\mathcal{M} = pq + qr + rp)$

$$\left\{\begin{matrix} A_{01} : {\scriptstyle x=y} & A_{02} : {\scriptstyle x=1} & A_{03} : {\scriptstyle x=0} & A_{04} : {\scriptstyle x=\infty} & A_{12} : {\scriptstyle y=1} \\ [0]_{pq+(p+q-1)r} & [0]_{pr+(p+r-1)q} & [\alpha'_i]_{q+r} & [\alpha_i]_{q+r} & [0]_{qr+(q+r-1)p} \\ [-\alpha''-\beta'']_r & [-\alpha''-\gamma'']_q & \beta_j+\gamma'_k & \beta'_j+\gamma_k & [-\beta''-\gamma'']_p \end{matrix}\right.$$

$$\left.\begin{matrix} A_{13} : {\scriptstyle y=0} & A_{23} & A_{14} : {\scriptstyle y=\infty} & A_{24} & A_{34} \\ [\beta'_j]_{p+r} & [\gamma_k]_{p+q} & [\beta_j]_{p+r} & [\gamma'_k]_{p+q} & [0]_{pq+qr+rp-(p+q+r)+1} \\ \alpha_i+\gamma'_k & \alpha_i+\beta_j & \alpha'_i+\gamma_k & \alpha'_i+\beta'_j & [-\alpha''-\beta''-\gamma'']_2 \\ & & & & [-\alpha''-\beta'']_{r-1} \\ & & & & [-\beta''-\gamma'']_{p-1} \\ & & & & [-\alpha''-\gamma'']_{q-1} \end{matrix}\right\}$$

$\mathrm{idx}_x\mathcal{M} = 2 - 2(q-1)(r-1)(q+r+1)$ (rigid $\Leftrightarrow q = r = 1$)

\# solutions up to constant multiple at (0,0) with *simple monodromy*

$=$ # eigenvalues of A_{24} with *free multiplicity* $(= pq)$

Involutive coordinate transformations

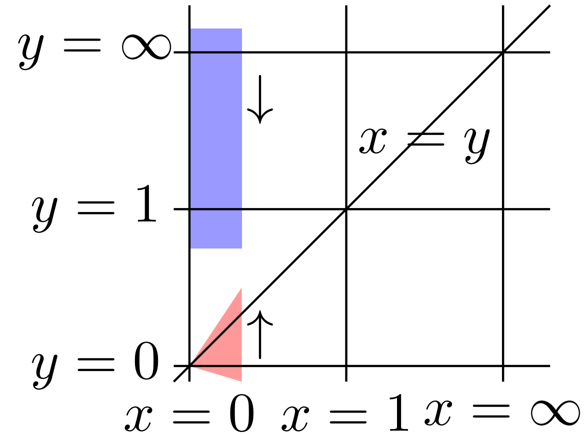
$$(x_0, x_1, x_2, x_3, x_4) \quad \to \quad (x, y, 1, 0, \infty)$$

$$(x_0, x_1, x_2, x_3, x_4) \leftrightarrow (x_2, x_1, x_0, x_4, x_3) \quad \to \quad (x, y) \leftrightarrow (x, \tfrac{x}{y})$$

$\Rightarrow$ blowing up of the singularities of $\mathcal{M}$ at the origin:

$\mathfrak{S}_5 \ni x_0 \leftrightarrow x_1 \quad x_1 \leftrightarrow x_2 \quad x_2 \leftrightarrow x_3 \quad x_3 \leftrightarrow x_4$

$(x, y) \mapsto (y, x) \quad (\frac{x}{y}, \frac{1}{y}) \quad (1-x, 1-y) \quad (\frac{1}{x}, \frac{1}{y})$



$$(x, y) \leftrightarrow (x, \tfrac{x}{y})$$

$$\{|x| < \epsilon,\ |y| < C|x|\} \leftrightarrow \{|x| < \epsilon,\ |y| > C^{-1}\}$$

$$x = y = 0 \leftrightarrow x = 0$$

Theorem ([Oi]). In general $\{i, j, k, s, t\} = \{0, 1, 2, 3, 4\} \Rightarrow$

a *simple solution* at $x_i = x_j = x_k \leftrightarrow$ a *simple solution* along $x_s = x_t$

A simple solution $\overset{\text{def}}{\Leftrightarrow}$ It spans 1-dimenional space under local analytic continuation

Theorem([O-Ma]). irreducible $\Leftrightarrow \alpha_i + \alpha'_{i'},\ \beta_j + \beta'_{j'},\ \alpha_i + \beta_j + \gamma'_{k'},\ \alpha'_l + \beta'_j + \gamma_k \notin \mathbb{Z}$

spectral type (multiplicities of eigenvalues)

$p=q=r=1$: Appell's F_1

	x_0	x_1	x_2	x_3	x_4	idx
x_0		21	21	21	21	2
x_1	21		21	21	21	2
x_2	21	21		21	21	2
x_3	21	21	21		21	2
x_4	21	21	21	21		2

$p=q=r=2,$ rank $=12$

	$x_0=x$	$x_1=y$	$x_2=1$	$x_3=0$	$x_4=\infty$	idx
x_0		(10)2	(10)2	441111	441111	-8
x_1	(10)2		(10)2	441111	441111	-8
x_2	(10)2	(10)2		441111	441111	-8
x_3	441111	441111	441111		72111	-124
x_4	441111	441111	441111	72111		-124

$p=q=r=3,$ rank $=27$

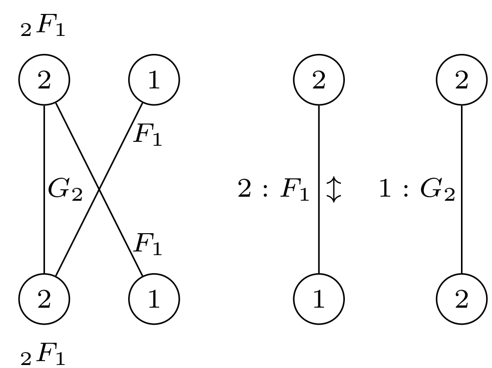
	x_0	x_1	x_2	x_3	x_4	idx
x_0		(24)3	(24)3	$6^3 1^9$	$6^3 1^9$	-54
x_1	(24)3		(24)3	$6^3 1^9$	$6^3 1^9$	-54
x_2	(24)3	(24)3		$6^3 1^9$	$6^3 1^9$	-54
x_3	$6^3 1^9$	$6^3 1^9$	$6^3 1^9$		$(19)22^3$	-730
x_4	$6^3 1^9$	$6^3 1^9$	$6^3 1^9$	$(19)22^3$		-730

local solutions at a normally crossing point

$x_i = x_j$: 21

$p=q=r=1$:

$x_k = x_\ell$: 21

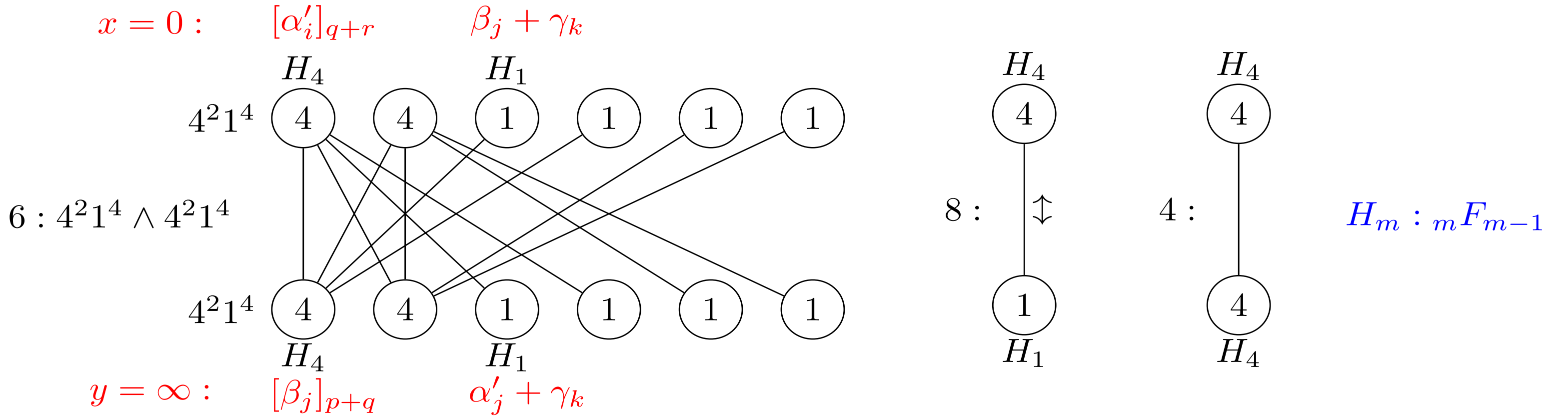


$\exists$ 15 normally crossing singular points : $\{x_i = x_j\} \wedge \{x_k = x_\ell\}$

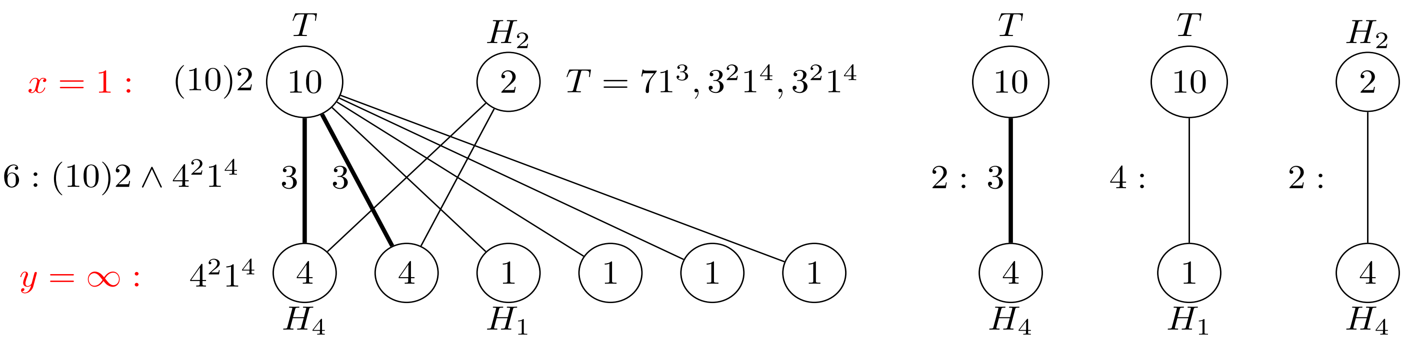
$\supset$ 6 points are multiplicity free $\Rightarrow$ 6 sets of natural bases of local solutions

$p=q=r=2$: $1^{12}, 1^{12}, 1^{12}, 1^{12}, 1^{12}, 1^{12}, 3^2 1^6, 3^2 1^6, 3^2 1^6, 3^2 1^6, 3^2 1^6, 3^2 1^6, 71^5, 71^5, 71^5$

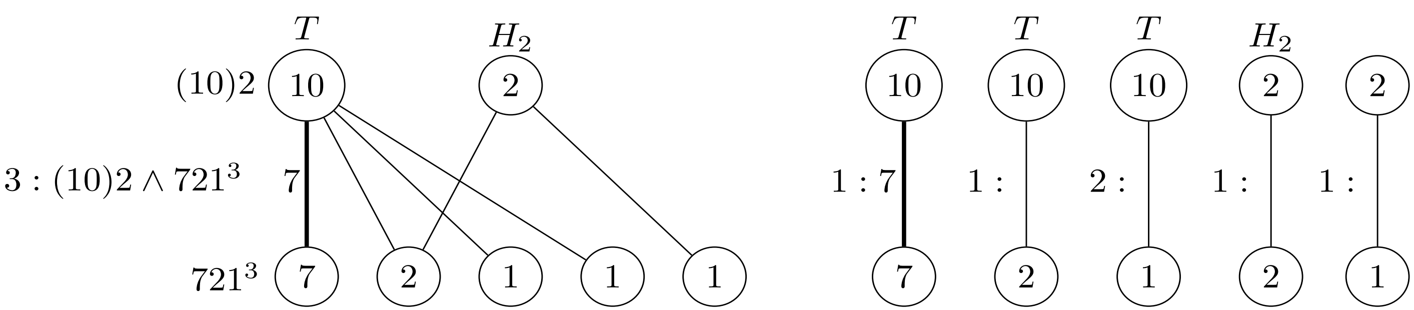
$$[1^4 + 1^4 + 1 + 1 + 1 + 1] \wedge [1^4 + 1^4 + 1 + 1 + 1 + 1] : 1^{12}$$



$$([3^2 1^4] + [1^2]) \wedge [31 + 31 + 1 + 1 + 1 + 1] : 3^2 1^6$$



$$[7 \cdot 1^3 + 1^2] \wedge [7 + 2 + 1 + 1 + 1] : 71^5$$



Thank you for your attention!

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