Optimal flat funtions and local right inverses for the asymptotic Borel map in ultraholomorphic classes

Javier Sanz (University of Valladolid, Spain)

Joint work with J. Jiménez-Garrido (Univ. Cantabria), I. Miguel (Univ. Valladolid) and G. Schindl (Univ. Vienna)

August 31st 2023, CDDE II, Bedlewo (Poland)

Sectors and weight sequences

$\mathcal{R}$ will denote the Riemann surface of the logarithm.

Given $\gamma > 0$, we consider unbounded sectors

$$S_\gamma := \{z \in \mathcal{R};\ |\arg(z)| < \pi\gamma/2\}.$$

Sectors and weight sequences

$\mathcal{R}$ will denote the Riemann surface of the logarithm.

Given $\gamma > 0$, we consider unbounded sectors

$$S_\gamma := \{z \in \mathcal{R};\ |\arg(z)| < \pi\gamma/2\}.$$

$\mathbb{N}_0 = \{0, 1, 2, ...\}$.

Let $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers, with $M_0 = 1$.

$\mathbb{M}$ is said to be logarithmically convex or (lc) if $M_n^2 \le M_{n-1}M_{n+1},\ n \ge 1$; equivalently, the sequence of quotients of $\mathbb{M}$, $\boldsymbol{m} = (m_n := \frac{M_{n+1}}{M_n})_{n\in\mathbb{N}_0}$, is nondecreasing.

We always assume that $\mathbb{M}$ is (lc) and $\lim_{n\to\infty} m_n = \infty$: we say $\mathbb{M}$ is a weight sequence.

Examples of weight sequences

Examples:

- $\mathbb{M} = (\prod_{k=0}^{n} \log^{\beta}(e+k))_{n\in\mathbb{N}_0}$, $\beta > 0$.
- $\mathbb{M}_{\alpha} = (n!^{\alpha})_{n\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$.
- $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.
- For $q > 1$ and $\sigma > 1$, $\mathbb{M}_{q,\sigma} := (q^{n^{\sigma}})_{n\in\mathbb{N}_0}$
  (for $\sigma = 2$, $q-$Gevrey sequence).
- For $\tau > 0$ and $\sigma > 1$, $\mathbb{M}^{\tau,\sigma} = (n^{\tau n^{\sigma}})_{n\in\mathbb{N}_0}$ $(M_0^{\tau,\sigma} = 1)$.

Uniform asymptotics

$f : S \to \mathbb{C}$ (holomorphic in a sector S) admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-uniform asymptotic expansion at 0 if there exist $C, A > 0$ such that for every $z \in S$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \leq C A^n M_n |z|^n. \qquad [f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)]$$

Uniform asymptotics

$f : S \to \mathbb{C}$ (holomorphic in a sector S) admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-uniform asymptotic expansion at 0 if there exist $C, A > 0$ such that for every $z \in S$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \le C A^n M_n |z|^n. \qquad [f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)]$$

The norm

$$\|f\|_{\mathbb{M},A,\widetilde{u}} := \sup_{z\in S, n\in\mathbb{N}_0} \frac{|f(z) - \sum_{k=0}^{n-1} a_k z^k|}{A^n M_n |z|^n}$$

makes it a Banach space ($\frac{1}{A}$ may be called the type).

$\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S) := \bigcup_{A>0} \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$ is an (LB) space.

The Borel map

$\mathbb{C}\,[[z]]$ formal complex power series.

$$\mathbb{C}[[z]]_{\{\mathbb{M}\},A} = \Big\{ \widehat{f} = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]] : \Big|\widehat{f}\Big|_{\mathbb{M},A} := \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{A^p M_p} < \infty \Big\}.$$

$(\mathbb{C}[[z]]_{\{\mathbb{M}\},A}, |\cdot|_{\mathbb{M},A})$ is a Banach space.

$\mathbb{C}[[z]]_{\{\mathbb{M}\}} := \bigcup_{A>0} \mathbb{C}[[z]]_{\{\mathbb{M}\},A}$ is an (LB) space.

The Borel map

$\mathbb{C}[[z]]$ formal complex power series.

$$\mathbb{C}[[z]]_{\{\mathbb{M}\},A}=\Big\{\widehat{f}=\sum_{n=0}^{\infty}a_nz^n\in\mathbb{C}[[z]]:\ \Big|\widehat{f}\Big|_{\mathbb{M},A}:=\sup_{p\in\mathbb{N}_0}\frac{|a_p|}{A^pM_p}<\infty\Big\}.$$

$(\mathbb{C}[[z]]_{\{\mathbb{M}\},A},|\cdot|_{\mathbb{M},A})$ is a Banach space.

$\mathbb{C}[[z]]_{\{\mathbb{M}\}}:=\bigcup_{A>0}\mathbb{C}[[z]]_{\{\mathbb{M}\},A}$ is an (LB) space.

We consider the asymptotic Borel map (continuous homomorphism of algebras)

$$\begin{aligned}\widetilde{\mathcal{B}}:\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S) &\longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}}\\ f &\mapsto \hat{f}=\sum_{n=0}^{\infty}a_nz^n.\end{aligned}$$

It may also be considered from $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$ into $\mathbb{C}[[z]]_{\{\mathbb{M}\},A}$.

In this talk we are interested in surjectivity and existence of (local or global) linear continuous extension operators, right inverses for $\widetilde{\mathcal{B}}$.

Surjectivity intervals and its non-triviality

$$\widetilde{S}^u_{\{\mathbb{M}\}} := \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}} \text{ is surjective}\}.$$

$\widetilde{S}^u_{\{\mathbb{M}\}}$ is either empty, or an interval having 0 as left-endpoint.

Surjectivity intervals and its non-triviality

$$\widetilde{S}^u_{\{\mathbb{M}\}} := \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}} \text{ is surjective}\}.$$

$\widetilde{S}^u_{\{\mathbb{M}\}}$ is either empty, or an interval having 0 as left-endpoint.

$\mathbb{M}$ is strongly non-quasianalytic (snq) if there exists $B > 0$ such that

$$\sum_{k\geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B\frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0. \qquad [\widehat{\mathbb{M}} := (n!M_n)_{n\in\mathbb{N}_0} \text{ has } (M3)]$$

H.-J. Petzsche, Math. Ann. 282 (1988), no. 2, 299–313.
V. Thilliez, Results Math. 44 (2003), 169–188.

V. Thilliez (2003)

If $\mathbb{M}$ does not satisfy (snq), $\widetilde{S}^u_{\{\mathbb{M}\}} = \emptyset$.

Example: no surjectivity result for $\mathbb{M} = (\prod_{k=0}^n \log^\beta(e+k))_{n\in\mathbb{N}_0}$, $\beta > 0$.

Thilliez's index

V. Thilliez (2003) introduces a growth index $\gamma(\mathbb{M})$. Now we know:
A sequence $(c_p)_{p\in\mathbb{N}_0}$ of positive real numbers, is almost increasing if there exists $a>0$ such that for every $p\in\mathbb{N}_0$ we have that $c_p\le ac_q$ for every $q\ge p$.
We have proved that

$$\gamma(\mathbb{M}) = \sup\{\gamma>0 : (m_p/(p+1)^{\gamma})_{p\in\mathbb{N}_0} \text{ is almost increasing}\}$$
$$=: \text{lower Matuszewska index of } \boldsymbol{m}.$$

Thilliez's index

V. Thilliez (2003) introduces a growth index $\gamma(\mathbb{M})$. Now we know:
A sequence $(c_p)_{p\in\mathbb{N}_0}$ of positive real numbers, is almost increasing if there exists $a>0$ such that for every $p\in\mathbb{N}_0$ we have that $c_p\leq ac_q$ for every $q\geq p$.
We have proved that

$$\gamma(\mathbb{M})=\sup\{\gamma>0:(m_p/(p+1)^{\gamma})_{p\in\mathbb{N}_0}\text{ is almost increasing}\}$$
$$=:\text{lower Matuszewska index of }\boldsymbol{m}.$$

Moreover, $\gamma(\mathbb{M})>0$ if and only if $\mathbb{M}$ is (snq).

The associated function $h_{\mathbb{M}}$ and optimal flat functions

$f \in \mathcal{H}(S)$ is flat (at 0) if f has a null asymptotic expansion.

$f \in \widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\},A}(S)$: $\left|f(z)\right| = \left|f(z) - \sum_{k=0}^{n-1} a_k z^k\right| \le CA^n M_n |z|^n$ for every n.

The associated function $h_{\mathbb{M}}$ and optimal flat functions

$f \in \mathcal{H}(S)$ is flat (at 0) if f has a null asymptotic expansion.
$f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$: $\left|f(z)\right| = \left|f(z) - \sum_{k=0}^{n-1} a_k z^k\right| \leq CA^n M_n |z|^n$ for every n.

$$h_{\mathbb{M}}(t) := \inf_{k \in \mathbb{N}_0} M_k t^k, \quad t > 0; \quad h_{\mathbb{M}}(0) = 0.$$

Let $f \in \mathcal{H}(S)$, the following are equivalent:

1. $f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S)$ and it is flat.
2. $|f(z)| \leq C h_{\mathbb{M}}(A|z|)$, for some $C, A \in \mathbb{R}$, and for all $z \in S$.

The associated function $h_{\mathbb{M}}$ and optimal flat functions

$f \in \mathcal{H}(S)$ is flat (at 0) if f has a null asymptotic expansion.
$f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},A}(S)$: $\left|f(z)\right| = \left|f(z) - \sum_{k=0}^{n-1} a_k z^k\right| \le CA^n M_n |z|^n$ for every n.

$$h_{\mathbb{M}}(t) := \inf_{k\in\mathbb{N}_0} M_k t^k, \quad t>0; \quad h_{\mathbb{M}}(0)=0.$$

Let $f \in \mathcal{H}(S)$, the following are equivalent:

1. $f \in \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S)$ and it is flat.
2. $|f(z)| \le C h_{\mathbb{M}}(A|z|)$, for some $C, A \in \mathbb{R}$, and for all $z \in S$.

Definition

Let S be an unbounded sector bisected by the positive real line $(0,+\infty)$. A function $G \in \mathcal{H}(S)$ is an optimal $\mathbb{M}$-flat function, if

(i) $\exists K_1, K_2 > 0: \quad K_1 h_{\mathbb{M}}(K_2 x) \le G(x)$ for all $x > 0$, (symmetry!)

(ii) $\exists K_3, K_4 > 0: \quad |G(z)| \le K_3 h_{\mathbb{M}}(K_4|z|)$ for all $z \in S$.

Surjectivity intervals for strongly regular sequences

$\mathbb{M}$ is strongly regular if it is (lc), (snq) and has moderate growth (mg) or (M2): there exists $A > 0$ such that $M_{n+p} \leq A^{n+p} M_n M_p$, $n, p \in \mathbb{N}_0$.

Example: $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

Surjectivity intervals for strongly regular sequences

$\mathbb{M}$ is strongly regular if it is (lc), (snq) and has moderate growth (mg) or (M2): there exists $A > 0$ such that $M_{n+p} \leq A^{n+p} M_n M_p$, $n, p \in \mathbb{N}_0$.

Example: $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

Theorem (V. Thilliez, 2003)

Let $\mathbb{M}$ be a strongly regular sequence. Then, $\gamma(\mathbb{M}) \in (0, \infty)$. Moreover, each of the following statements implies the next one:

(i) $0 < \gamma < \gamma(\mathbb{M})$,

(ii) *the space $\widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\}}(S_\gamma)$ contains optimal $\mathbb{M}$-flat functions,*

(iii) *there exists $c \geq 1$, depending on $\mathbb{M}$ and γ, such that for every $A > 0$ there exists a right inverse for $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\{\mathbb{M}\},A} \to \widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\},cA}(S_\gamma)$,*

(iv) $\widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^{u}_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective,*

Surjectivity intervals for strongly regular sequences

$\mathbb{M}$ is strongly regular if it is (lc), (snq) and has moderate growth (mg) or (M2): there exists $A > 0$ such that $M_{n+p} \le A^{n+p} M_n M_p$, $n, p \in \mathbb{N}_0$.

Example: $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.

Theorem (V. Thilliez, 2003; J. Jiménez-Garrido, J. S., G. Schindl, 2019)

Let $\mathbb{M}$ be a strongly regular sequence. Then, $\gamma(\mathbb{M}) \in (0,\infty)$. Moreover, each of the following statements implies the next one:

(i) $0 < \gamma < \gamma(\mathbb{M})$,

(ii) *the space $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma)$ contains optimal $\mathbb{M}$-flat functions,*

(iii) *there exists $c \ge 1$, depending on $\mathbb{M}$ and γ, such that for every $A > 0$ there exists a right inverse for $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\{\mathbb{M}\},A} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},cA}(S_\gamma)$,*

(iv) $\widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective,*

(v) $\gamma \le \gamma(\mathbb{M})$.

J. Jiménez-Garrido, J. S., G. Schindl, J. Math. Anal. Appl. 469 (2019), 136–168.

Condition (mg) and regular sequences in the sense of E. M. Dyn'kin

Conjecture: (i)-(iv) are equivalent (true if $\gamma(\mathbb{M}) \in \mathbb{Q}$ or in the Gevrey case).

Condition (mg) and regular sequences in the sense of E. M. Dyn'kin

Conjecture: (i)-(iv) are equivalent (true if $\gamma(\mathbb{M}) \in \mathbb{Q}$ or in the Gevrey case).

Several crucial steps in (i)⇒(ii) (estimates for harmonic extensions) and (ii)⇒(iii) (Whitney extension results; ramification argument; estimates for M_{kp}, $k \in \mathbb{N}$) work because of condition (mg), but it is considered to be a technical issue, unlike (snq).

Main aim: To weaken (or even suppress) the (mg) condition.

Condition (mg) and regular sequences in the sense of E. M. Dyn'kin

Conjecture: (i)-(iv) are equivalent (true if $\gamma(\mathbb{M}) \in \mathbb{Q}$ or in the Gevrey case).

Several crucial steps in (i)⇒(ii) (estimates for harmonic extensions) and (ii)⇒(iii) (Whitney extension results; ramification argument; estimates for M_{kp}, $k \in \mathbb{N}$) work because of condition (mg), but it is considered to be a technical issue, unlike (snq).

Main aim: To weaken (or even suppress) the (mg) condition.

$\mathbb{M}$ is derivation closed (dc) if there exists a constant $A > 0$ such that

$$M_{n+1} \leq A^{n+1} M_n, \quad n \in \mathbb{N}_0.$$

$\widehat{\mathbb{M}} := (n! M_n)_{n \in \mathbb{N}_0}$ is regular (following E. M. Dyn'kin) if $\mathbb{M}$ is a weight sequence and satisfies (dc).
If $\mathbb{M}$ is strongly regular, the corresponding $\widehat{\mathbb{M}}$ is regular.

Condition (mg) and regular sequences in the sense of E. M. Dyn'kin

Conjecture: (i)-(iv) are equivalent (true if $\gamma(\mathbb{M}) \in \mathbb{Q}$ or in the Gevrey case).

Several crucial steps in (i)⇒(ii) (estimates for harmonic extensions) and (ii)⇒(iii) (Whitney extension results; ramification argument; estimates for M_{kp}, $k \in \mathbb{N}$) work because of condition (mg), but it is considered to be a technical issue, unlike (snq).

Main aim: To weaken (or even suppress) the (mg) condition.

$\mathbb{M}$ is derivation closed (dc) if there exists a constant $A > 0$ such that

$$M_{n+1} \leq A^{n+1} M_n, \quad n \in \mathbb{N}_0.$$

$\widehat{\mathbb{M}} := (n! M_n)_{n\in\mathbb{N}_0}$ is regular (following E. M. Dyn'kin) if $\mathbb{M}$ is a weight sequence and satisfies (dc).
If $\mathbb{M}$ is strongly regular, the corresponding $\widehat{\mathbb{M}}$ is regular.

No proof of surjectivity had been given for regular $\widehat{\mathbb{M}}$, except for the q-Gevrey sequences $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$, $q > 1$, see C. Zhang, Ann. Inst. Fourier 49 (1999), 227–261.

Connection with the Stieltjes moment problem

Stieltjes moments for a function f: $\displaystyle\int_0^\infty f(t)t^n\,dt$.

A. Debrouwere, J. Jiménez-Garrido, J. S., RACSAM 113 (2019), 3341–3358.

A. Debrouwere, Studia Math. 254 (2020), 295-323.

He has got a characterization of the surjectivity of the Stieltjes moment mapping for regular sequences by using (non constructive) functional-analytic methods.

Theorem (A. Debrouwere, 2020)

Let $\widehat{\mathbb{M}}$ be regular. The following are equivalent:

(i) $\widetilde{\mathcal{B}}\colon \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_1) \to \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective.*

(ii) $\gamma(\mathbb{M}) > 1$.

Surjectivity intervals for regular sequences

J. Jiménez-Garrido, J. S., G. Schindl, RACSAM (2021), 115:181.

By using Balser's moment summability methods, with associated Laplace and Borel transforms, we prove

Theorem (J. Jiménez-Garrido, J. S., G. Schindl, 2021)

Let $\widehat{\mathbb{M}}$ be a regular sequence. Then,

$$(0, \gamma(\mathbb{M})) \subseteq \widetilde{S}^u_{\{\mathbb{M}\}} \subseteq (0, \gamma(\mathbb{M})].$$

Surjectivity intervals for regular sequences

J. Jiménez-Garrido, J. S., G. Schindl, RACSAM (2021), 115:181.

By using Balser's moment summability methods, with associated Laplace and Borel transforms, we prove

Theorem (J. Jiménez-Garrido, J. S., G. Schindl, 2021)

Let $\widehat{\mathbb{M}}$ be a regular sequence. Then,

$$(0, \gamma(\mathbb{M})) \subseteq \widetilde{S}^u_{\{\mathbb{M}\}} \subseteq (0, \gamma(\mathbb{M})].$$

Conjecture: $\widetilde{S}^u_{\{\mathbb{M}\}} = (0, \gamma(\mathbb{M}))$ in general (true if $\gamma(\mathbb{M}) \in \mathbb{N}$).

Construction of extension operators

New aim: Obtain a constructive proof for the surjectivity of the Borel map for regular sequences via extension operators.

Construction of extension operators

New aim: Obtain a constructive proof for the surjectivity of the Borel map for regular sequences via extension operators.

A. Lastra, S. Malek, J. S., J. Math. Anal. Appl. 396 (2012), 724–740.

Alternative proof for Thilliez's result: Suppose G is an optimal $\mathbb{M}$-flat function in S_γ, put $e(z) := G(1/z)$ and $\mathfrak{m}(p) := \int_0^\infty t^p e(t)\, dt$.

Construction of extension operators

New aim: Obtain a constructive proof for the surjectivity of the Borel map for regular sequences via extension operators.

A. Lastra, S. Malek, J. S., J. Math. Anal. Appl. 396 (2012), 724–740.

Alternative proof for Thilliez's result: Suppose G is an optimal $\mathbb{M}$-flat function in S_γ, put $e(z) := G(1/z)$ and $\mathfrak{m}(p) := \int_0^\infty t^p e(t)\,dt$.

O. Blasco, Funct. Approx. Comment. Math. 59 (2018), 175–189.

There exist $B_1, B_2 > 0$ such that $\mathfrak{m}(0)B_1^p M_p \le \mathfrak{m}(p) \le \mathfrak{m}(0)B_2^p M_{p+2}$.
So, $\mathbb{M}$ and $\{\mathfrak{m}(p)\}_p$ are equivalent if $\mathbb{M}$ satisfies (dc).

Given $\widehat{f} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{\mathbb{M},A}$, put $g := \sum_{p\ge 0} \frac{a_p}{\mathfrak{m}(p)} z^p$ and

$$T_{\mathbb{M},A}(\widehat{f})(z) := \frac{1}{z}\int_0^{R_0} e\left(\frac{u}{z}\right) g(u)\,du, \qquad z \in S_\gamma.$$

Construction of extension operators

New aim: Obtain a constructive proof for the surjectivity of the Borel map for regular sequences via extension operators.

A. Lastra, S. Malek, J. S., J. Math. Anal. Appl. 396 (2012), 724–740.

Alternative proof for Thilliez's result: Suppose G is an optimal $\mathbb{M}$-flat function in S_γ, put $e(z) := G(1/z)$ and $\mathfrak{m}(p) := \int_0^\infty t^p e(t)\,dt$.

O. Blasco, Funct. Approx. Comment. Math. 59 (2018), 175–189.

There exist $B_1, B_2 > 0$ such that $\mathfrak{m}(0)B_1^p M_p \le \mathfrak{m}(p) \le \mathfrak{m}(0)B_2^p M_{p+2}$.
So, $\mathbb{M}$ and $\{\mathfrak{m}(p)\}_p$ are equivalent if $\mathbb{M}$ satisfies (dc).

Given $\widehat{f} = \sum_{p=0}^{\infty} a_p z^p \in \mathbb{C}[[z]]_{\mathbb{M},A}$, put $g := \sum_{p\ge 0} \frac{a_p}{\mathfrak{m}(p)} z^p$ and

$$T_{\mathbb{M},A}(\widehat{f})(z) := \frac{1}{z}\int_0^{R_0} e\left(\frac{u}{z}\right) g(u)\,du, \qquad z \in S_\gamma.$$

Modified new aim: Construct optimal $\mathbb{M}$-flat functions in narrow sectors.

Construction of optimal flat functions: Preliminaries, I

S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Paris, 1952.
H. Komatsu, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 25–105.

Given a weight sequence $\mathbb{M}$, we consider two classical associated functions:

- $\omega_{\mathbb{M}}(t) := \sup_{p\in\mathbb{N}_0} \log\left(\dfrac{t^p}{M_p}\right)$ for all $t \geq 0$.
  Note that $h_{\mathbb{M}}(t) = \exp(-\omega_{\mathbb{M}}(1/t))$ for all $t > 0$.

Construction of optimal flat functions: Preliminaries, I

S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Paris, 1952.
H. Komatsu, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 25–105.

Given a weight sequence $\mathbb{M}$, we consider two classical associated functions:

- $\omega_{\mathbb{M}}(t) := \sup_{p\in\mathbb{N}_0} \log\left(\frac{t^p}{M_p}\right)$ for all $t \geq 0$.
  Note that $h_{\mathbb{M}}(t) = \exp(-\omega_{\mathbb{M}}(1/t))$ for all $t > 0$.
- The counting function associated with $\boldsymbol{m}$, that is
  $\nu_{\boldsymbol{m}}(t) := \#\{n \in \mathbb{N}_0 : m_n \leq t\}$ for all $t \geq 0$.

$$\omega_{\mathbb{M}}(x) = \int_0^x \frac{\nu_{\boldsymbol{m}}(t)}{t} dt.$$

Construction of optimal flat functions: Preliminaries, I

S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Paris, 1952.
H. Komatsu, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 20 (1973), 25–105.

Given a weight sequence $\mathbb{M}$, we consider two classical associated functions:

- $\omega_{\mathbb{M}}(t) := \sup_{p\in\mathbb{N}_0} \log\left(\frac{t^p}{M_p}\right)$ for all $t \geq 0$.
  Note that $h_{\mathbb{M}}(t) = \exp(-\omega_{\mathbb{M}}(1/t))$ for all $t > 0$.
- The counting function associated with $\boldsymbol{m}$, that is
  $\nu_{\boldsymbol{m}}(t) := \#\{n \in \mathbb{N}_0 : m_n \leq t\}$ for all $t \geq 0$.

$$\omega_{\mathbb{M}}(x) = \int_0^x \frac{\nu_{\boldsymbol{m}}(t)}{t} dt.$$

For all $r \geq 0$ and $B \geq 0$,

$$\omega_{\mathbb{M}}(e^B r) = \int_0^{e^B r} \frac{\nu_{\boldsymbol{m}}(u)}{u} du = \omega_{\mathbb{M}}(r) + \int_r^{e^B r} \frac{\nu_{\boldsymbol{m}}(u)}{u} du \geq \omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r).$$

Construction of optimal flat functions: Preliminaries, II

The harmonic extension of $\omega_{\mathbb{M}}$, described next, will play a crucial role.

A nondecreasing function $\sigma : [0,\infty) \to [0,\infty)$ is nonquasianalytic if

$$\int_1^\infty \frac{\sigma(t)}{t^2}\, dt < \infty.$$

Construction of optimal flat functions: Preliminaries, II

The harmonic extension of $\omega_{\mathbb{M}}$, described next, will play a crucial role.

A nondecreasing function $\sigma : [0, \infty) \to [0, \infty)$ is nonquasianalytic if

$$\int_1^\infty \frac{\sigma(t)}{t^2}\, dt < \infty.$$

Let $\sigma : [0, \infty) \to [0, \infty)$ be a nondecreasing nonquasianalytic function. The harmonic extension P_σ of σ to the open upper and lower halfplanes of $\mathbb{C}$ is defined by

$$P_\sigma(x + iy) = \begin{cases} \sigma(|x|) & \text{if } x \in \mathbb{R},\ y = 0, \\ \dfrac{|y|}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sigma(|t|)}{(t-x)^2 + y^2} dt & \text{if } x \in \mathbb{R},\ y \neq 0. \end{cases}$$

For every $z \in \mathbb{C}$ one has $\sigma(|z|) \leq P_\sigma(z)$.

Langenbruch's condition

For a weight sequence $\mathbb{M}$, the condition

$$\sum_{p=0}^{\infty} \frac{1}{m_p} < \infty \qquad (M3)'$$

amounts to ν_m and/or $\omega_{\mathbb{M}}$ being nonquasianalytic (H. Komatsu, 1973).

Langenbruch's condition

For a weight sequence $\mathbb{M}$, the condition

$$\sum_{p=0}^{\infty} \frac{1}{m_p} < \infty \qquad (M3)'$$

amounts to ν_m and/or $\omega_{\mathbb{M}}$ being nonquasianalytic (H. Komatsu, 1973).

M. Langenbruch, Manuscripta Math. 83 (1994), no. 2, 123–143.

M. Langenbruch (1994)

Let $\mathbb{M}$ be a weight sequence with (M3)'. We say that $\mathbb{M}$ satisfies the Langenbruch's condition if there exists $C > 0$ such that

$$P_{\omega_{\mathbb{M}}}(iy) \leq \omega_{\mathbb{M}}(Cy) + C, \quad y \geq 0.$$

Optimal flat functions in a halfplane

J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl, Optimal flat functions in Carleman-Roumieu ultraholomorphic classes in sectors, Results Math. (2023), 78:98.

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\mathbb{M}$ be a weight sequence with (M3)' which satisfies Langenbruch's condition, then the function

$$G(z) := \exp(-P_{\omega_{\mathbb{M}}}(i/z) - iQ_{\omega_{\mathbb{M}}}(i/z))$$

is an optimal $\mathbb{M}$-flat function in S_1, where $Q_{\omega_{\mathbb{M}}}$ is the harmonic conjugate of $P_{\omega_{\mathbb{M}}}$ in the upper half plane.

Optimal flat functions in a halfplane

J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl, Optimal flat functions in Carleman-Roumieu ultraholomorphic classes in sectors, Results Math. (2023), 78:98.

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\mathbb{M}$ be a weight sequence with (M3)' which satisfies Langenbruch's condition, then the function

$$G(z) := \exp(-P_{\omega_{\mathbb{M}}}(i/z) - iQ_{\omega_{\mathbb{M}}}(i/z))$$

is an optimal $\mathbb{M}$-flat function in S_1, where $Q_{\omega_{\mathbb{M}}}$ is the harmonic conjugate of $P_{\omega_{\mathbb{M}}}$ in the upper half plane.

- For $z \in S_1$, $|G(z)| = \exp(-P_{\omega_{\mathbb{M}}}(i/z)) \leq \exp(-\omega_{\mathbb{M}}(1/|z|)) = h_{\mathbb{M}}(|z|)$.
- Let consider $x > 0$, then by Langenbruch's condition,

$$G(x) = \exp(-P_{\omega_{\mathbb{M}}}(i/x)) \geq \exp(-\omega_{\mathbb{M}}(C/x) - C) = \exp(-C)h_{\mathbb{M}}(x/C).$$

Langenbruch's condition and the index $\gamma(\mathbb{M})$

D. N. Nenning, A. Rainer, G. Schindl, RACSAM (2023), 117:40.

Proposition (D. N. Nenning, A. Rainer, G. Schindl (2023))

Let $\widehat{\mathbb{M}}$ be a regular sequence. The following are equivalent:

- $\gamma(\mathbb{M}) > 1$.
- $\mathbb{M}$ *satisfies Langenbruch's condition.*

Langenbruch's condition and the index $\gamma(\mathbb{M})$

D. N. Nenning, A. Rainer, G. Schindl, RACSAM (2023), 117:40.

Proposition (D. N. Nenning, A. Rainer, G. Schindl (2023))

Let $\widehat{\mathbb{M}}$ be a regular sequence. The following are equivalent:

- $\gamma(\mathbb{M}) > 1$.
- $\mathbb{M}$ *satisfies Langenbruch's condition.*

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

(1) $\gamma(\mathbb{M}) > 1$.

(2) $\gamma(\mathbb{M}) > 0$, *and $\mathbb{M}$ satisfies both (M3)' and Langenbruch's condition.*

Auxiliary results for the proof, I

Let $\sigma : [0,\infty) \to [0,\infty)$ be a nondecreasing nonquasianalytic function. The κ_σ function is defined by

$$\kappa_\sigma(y) = \int_1^\infty \frac{\sigma(ys)}{s^2}\, ds, \qquad y \geq 0.$$

R. Meise, B. A. Taylor, Ark. Mat. 26 (1988), no. 2, 265–287.

J. Bonet, R. Meise, B. A. Taylor, North-Holland Mathematics Studies - Progress in Functional Analysis 170 (1992), 97–111.

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\sigma : [0,\infty) \to [0,\infty)$ be a nondecreasing nonquasianalytic function. Then, we have

$$\frac{1}{\pi}\kappa_\sigma(y) \leq P_\sigma(iy) \leq \kappa_\sigma(y), \qquad y \geq 0.$$

Auxiliary results for the proof, II

J. Jiménez-Garrido, J. S., G. Schindl, RACSAM 113 (4) (2019), 3659–3697.

$\gamma(\mathbb{M}) > 0$ if and only if $\nu_{\boldsymbol{m}}$ satisfies the condition $\nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$ as t tends to ∞.

Auxiliary results for the proof, II

J. Jiménez-Garrido, J. S., G. Schindl, RACSAM 113 (4) (2019), 3659–3697.

$\gamma(\mathbb{M}) > 0$ if and only if $\nu_{\boldsymbol{m}}$ satisfies the condition $\nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$ as t tends to ∞.

$\gamma(\mathbb{M}) > 1$ if and only if $\nu_{\boldsymbol{m}}$ satisfies the condition (ω_{snq}), i.e., there exists $D > 0$ such that

$$\kappa_{\nu_{\boldsymbol{m}}}(y) = \int_1^\infty \frac{\nu_{\boldsymbol{m}}(ys)}{s^2}\, ds \le D\nu_{\boldsymbol{m}}(y) + D, \qquad y \ge 0.$$

Auxiliary results for the proof, II

J. Jiménez-Garrido, J. S., G. Schindl, RACSAM 113 (4) (2019), 3659–3697.

$\gamma(\mathbb{M}) > 0$ if and only if $\nu_{\boldsymbol{m}}$ satisfies the condition $\nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$ as t tends to ∞.

$\gamma(\mathbb{M}) > 1$ if and only if $\nu_{\boldsymbol{m}}$ satisfies the condition (ω_{snq}), i.e., there exists $D > 0$ such that

$$\kappa_{\nu_{\boldsymbol{m}}}(y) = \int_1^\infty \frac{\nu_{\boldsymbol{m}}(ys)}{s^2}\,ds \le D\nu_{\boldsymbol{m}}(y) + D, \qquad y \ge 0.$$

$\gamma(\mathbb{M}) > 1$ implies $\mathbb{M}$ satisfies $(M3)'$.

Sketch of the proof: $(2) \Rightarrow (1)$

Facts: (1) $\sigma(|z|) \leq P_\sigma(z)$
(2) $\kappa_\sigma(y) \leq \pi P_\sigma(iy)$
(3) $\omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r) \leq \omega_{\mathbb{M}}(e^B r)$
(4) $\sigma_1 \leq \sigma_2 \implies P_{\sigma_1} \leq P_{\sigma_2}$
(5) $\gamma(\mathbb{M}) > 0 \iff \nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$
(6) $\nu_{\boldsymbol{m}}$ satisfies (ω_{snq}) if and only if $\gamma(\mathbb{M}) > 1$

Sketch of the proof: $(2) \Rightarrow (1)$

Facts: (1) $\sigma(|z|) \leq P_\sigma(z)$
(2) $\kappa_\sigma(y) \leq \pi P_\sigma(iy)$
(3) $\omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r) \leq \omega_{\mathbb{M}}(e^B r)$
(4) $\sigma_1 \leq \sigma_2 \implies P_{\sigma_1} \leq P_{\sigma_2}$
(5) $\gamma(\mathbb{M}) > 0 \iff \nu_{\boldsymbol{m}}(2t) = O(\nu_{\boldsymbol{m}}(t))$
(6) $\nu_{\boldsymbol{m}}$ satisfies (ω_{snq}) if and only if $\gamma(\mathbb{M}) > 1$

For all $y \geq 0$ we have

$$\omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y) \overset{(1)+(2)}{\leq} P_{\omega_{\mathbb{M}}+\pi\nu_{\boldsymbol{m}}}(iy) \overset{(3)+(4)}{\leq} P_{\omega_{\mathbb{M}}(e^\pi\cdot)}(iy)$$
$$= P_{\omega_{\mathbb{M}}}(ie^\pi y) \overset{\text{Langenbr.}}{\leq} \omega_{\mathbb{M}}(Ce^\pi y) + C.$$

$$\kappa_{\nu_{\boldsymbol{m}}}(y) \leq \omega_{\mathbb{M}}(Ce^\pi y) - \omega_{\mathbb{M}}(y) + C = \int_y^{Ce^\pi y} \frac{\nu_{\boldsymbol{m}}(u)}{u} du + C$$
$$\leq \nu_{\boldsymbol{m}}(Ce^\pi y) + \ln(Ce^\pi) + C \overset{(5)}{\leq} D\nu_{\boldsymbol{m}}(y) + D, \qquad y \geq 0,\ D > 0.$$

Sketch of the proof: $(1) \Rightarrow (2)$

Facts: (1) $P_{\omega_{\mathbb{M}}}(iy) \leq \kappa_{\omega_{\mathbb{M}}}(y)$
(2) $\kappa_{\omega_{\mathbb{M}}}(y) = \omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y)$ (H. Komatsu, 1973)
(3) $\gamma(\mathbb{M}) > 1$ if and only if $\nu_{\boldsymbol{m}}$ satisfies (ω_{snq})
(4) $\omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r) \leq \omega_{\mathbb{M}}(e^B r)$

Sketch of the proof: $(1) \Rightarrow (2)$

Facts: (1) $P_{\omega_{\mathbb{M}}}(iy) \leq \kappa_{\omega_{\mathbb{M}}}(y)$
(2) $\kappa_{\omega_{\mathbb{M}}}(y) = \omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y)$ (H. Komatsu, 1973)
(3) $\gamma(\mathbb{M}) > 1$ if and only if $\nu_{\boldsymbol{m}}$ satisfies (ω_{snq})
(4) $\omega_{\mathbb{M}}(r) + B\nu_{\boldsymbol{m}}(r) \leq \omega_{\mathbb{M}}(e^B r)$

$$P_{\omega_{\mathbb{M}}}(iy) \overset{(1)}{\leq} \kappa_{\omega_{\mathbb{M}}}(y) \overset{(2)}{=} \omega_{\mathbb{M}}(y) + \kappa_{\nu_{\boldsymbol{m}}}(y) \overset{(3)}{\leq} \omega_{\mathbb{M}}(y) + C\nu_{\boldsymbol{m}}(y) + C$$
$$\overset{(4)}{\leq} \omega_{\mathbb{M}}(e^C y) + C, \qquad y \geq 0.$$

Optimal flat functions in general sectors: ramification

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\mathbb{M}$ *be a weight sequence with* $\gamma(\mathbb{M}) > 0$. *Then, for any* $0 < \gamma < \gamma(\mathbb{M})$ *there exists an optimal* $\mathbb{M}$*-flat function in* S_γ.

Optimal flat functions in general sectors: ramification

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\mathbb{M}$ be a weight sequence with $\gamma(\mathbb{M}) > 0$. Then, for any $0 < \gamma < \gamma(\mathbb{M})$ there exists an optimal $\mathbb{M}$-flat function in S_γ.

- Fix $s > 0$ such that $\gamma < 1/s < \gamma(\mathbb{M})$. Then, $\gamma(\mathbb{M}^s) = s\gamma(\mathbb{M}) > 1$.
- By the last result, $\mathbb{M}^s$ satisfies Langenbruch's condition, and so there exists an optimal $\mathbb{M}^s$-flat function G in S_1.
- Now, we consider the function $F(z) = (G(z^s))^{1/s}$ for all $z \in S_\gamma$. Since

$$\omega_{\mathbb{M}}(t^{1/s}) = \frac{1}{s}\omega_{\mathbb{M}^s}(t), \qquad t \geq 0,$$

we prove that the function F is an optimal $\mathbb{M}$-flat function in S_γ.

Optimal flat functions in general sectors: ramification

Proposition (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2023))

Let $\mathbb{M}$ be a weight sequence with $\gamma(\mathbb{M}) > 0$. Then, for any $0 < \gamma < \gamma(\mathbb{M})$ there exists an optimal $\mathbb{M}$-flat function in S_γ.

- Fix $s > 0$ such that $\gamma < 1/s < \gamma(\mathbb{M})$. Then, $\gamma(\mathbb{M}^s) = s\gamma(\mathbb{M}) > 1$.
- By the last result, $\mathbb{M}^s$ satisfies Langenbruch's condition, and so there exists an optimal $\mathbb{M}^s$-flat function G in S_1.
- Now, we consider the function $F(z) = (G(z^s))^{1/s}$ for all $z \in S_\gamma$. Since

$$\omega_{\mathbb{M}}(t^{1/s}) = \frac{1}{s}\omega_{\mathbb{M}^s}(t), \qquad t \geq 0,$$

  we prove that the function F is an optimal $\mathbb{M}$-flat function in S_γ.

Remark: The existence of optimal $\mathbb{M}$-flat functions may have significant applications, as in the recent work
S. Fürdös, G. Schindl, Ellipticity and the problem of iterates in Denjoy-Carleman classes, available at https://arxiv.org/abs/2212.12260.

Surjectivity intervals for regular sequences

Theorem (J. Jiménez-Garrido, I. Miguel, J. S., G. Schindl (2022))

Let $\widehat{\mathbb{M}}$ *be a regular sequence with* $\gamma(\mathbb{M}) \in (0, \infty]$. *Each of the following statements implies the next one:*

(i) $0 < \gamma < \gamma(\mathbb{M})$,

(ii) *The space* $\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma)$ *contains optimal* $\mathbb{M}$*-flat functions,*

(iii) *There exists* $c \geq 1$, *depending on* $\mathbb{M}$ *and* γ, *such that for every* $A > 0$ *there exists a right inverse for* $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\{\mathbb{M}\},A} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\},cA}(S_\gamma)$,

(iv) $\widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\{\mathbb{M}\}}$ *is surjective,*

(v) $0 < \gamma \leq \gamma(\mathbb{M})$.

We need (dc) condition only for (ii)⇒(iii) and (iv)⇒(v).

Examples: $\mathbb{M}_{q,\sigma} := (q^{n^\sigma})_{n\in\mathbb{N}_0}$

For $q > 1$ and $1 < \sigma \le 2$, $\mathbb{M}_{q,\sigma} := (q^{n^\sigma})_{n\in\mathbb{N}_0}$.

$\widehat{\mathbb{M}}_{q,\sigma}$ is regular, but $\mathbb{M}_{q,\sigma}$ is not strongly regular.
There exists a unique $s \ge 2$ such that $\sigma = s/(s-1)$, put

$$b_{q,s} := \frac{1}{s}\left(\frac{s-1}{s\ln(q)}\right)^{s-1}.$$

For $0 < t \le q^{-2s/(s-1)}$ we have

$$\exp\left(-b_{q,s}\ln^s\left(\frac{1}{t}\right)\right) \le h_{\mathbb{M}_{q,\sigma}}(t) \le q^{s/(s-1)}\exp\left(-b_{q,s}\ln^s\left(\frac{1}{q^{s/(s-1)}t}\right)\right).$$

So,

$$G_2^{q,s}(z) := \exp\left(-b_{q,s}\log^s\left(1+\frac{1}{z}\right)\right), \quad z \in S_2,$$

is an optimal $\mathbb{M}_{q,\sigma}$-flat function in S_2.
In wider sectors, ramification provides suitable optimal flat functions.

Example: $\mathbb{M}^{\tau,\sigma} = (n^{\tau n^\sigma})_{n\in\mathbb{N}_0}$

S. Pilipović, N. Teofanov, F. Tomić, J. Pseudo-Differ. Oper. Appl. 11, 593–612 (2020).

J. Jiménez-Garrido, A. Lastra, J. S., Constr. Approx. (2023), https://doi.org/10.1007/s00365-023-09663-z.

For $1 < \sigma < 2$ and $\tau > 0$, let $\mathbb{M}^{\tau,\sigma} = (n^{\tau n^\sigma})_{n\in\mathbb{N}_0}$; $\widehat{\mathbb{M}}^{\tau,\sigma}$ is regular.

The Lambert function W is the complex function satisfying $W(z)e^{W(z)} = z$. Put

$$a_{\tau,\sigma} := \left(\frac{\sigma-1}{\tau\sigma}\right)^{\frac{1}{\sigma-1}}, \qquad b_{\tau,\sigma} := e^{\frac{\sigma-1}{\sigma}}\frac{\sigma-1}{\tau\sigma},$$

and for every $z \in S_2$,

$$e_{\tau,\sigma}(z) := z\exp\left(-a_{\tau,\sigma}W^{-\frac{1}{\sigma-1}}(b_{\tau,\sigma}\operatorname{Log}(z+1))\operatorname{Log}^{\frac{\sigma}{\sigma-1}}(z+1)\right).$$

The function $e_{\tau,\sigma}(1/z)$ is an optimal $\mathbb{M}^{\tau,\sigma}$-flat function in the corresponding ultraholomorphic class.

Example: convolved sequences (Komatsu, 1973)

Let $\mathbb{M}^1 = (M_n^1)_{n\in\mathbb{N}_0}$, $\mathbb{M}^2 = (M_n^2)_{n\in\mathbb{N}_0}$ be two weight sequences.
The convolved sequence $\mathbb{L} := \mathbb{M}^1 \star \mathbb{M}^2$ is $(L_n)_{n\in\mathbb{N}_0}$ is given by

$$L_n := \min_{0\le q\le n} M_q^1 M_{n-q}^2, \quad n \in \mathbb{N}_0.$$

$\mathbb{M}^1 \star \mathbb{M}^2$ is again a weight sequence, and For all $n \in \mathbb{N}_0$ we have $L_n \le \min\{M_n^1, M_n^2\}$.

If either $\mathbb{M}^1$ or $\mathbb{M}^2$ has (dc), then $\mathbb{M}^1 \star \mathbb{M}^2$ as well.

Moreover,

$$\nu_{\boldsymbol{m}^1\star\boldsymbol{m}^2}(t) = \nu_{\boldsymbol{m}^1}(t) + \nu_{\boldsymbol{m}^2}(t), \quad \omega_{\mathbb{M}^1\star\mathbb{M}^2}(t) = \omega_{\mathbb{M}^1}(t) + \omega_{\mathbb{M}^2}(t), \quad t \ge 0.$$

If optimal flat functions $G_{\mathbb{M}^1}$ and $G_{\mathbb{M}^2}$ exist in the corresponding classes, then $G_{\mathbb{M}^1\star\mathbb{M}^2} := G_{\mathbb{M}^1} \cdot G_{\mathbb{M}^2}$ is an optimal flat function (on the same sector S) in the class associated with the sequence $\mathbb{M}^1 \star \mathbb{M}^2$.

Global extension operators in a half-plane

In the ultradifferentiable setting, H.-J. Petzsche (1988) introduced the condition

$$\forall\varepsilon>0,\ \exists k\in\mathbb{N},\ k>1: \limsup_{p\to\infty}\left(\frac{M_{kp}}{M_p}\right)^{\frac{1}{(k-1)p}}\frac{1}{m_{kp-1}}\le\varepsilon, \qquad (\beta_2)$$

which again appeared in the results about the existence of global extension operators in the ultraholomorphic framework of

J. Schmets, M. Valdivia, Studia Math. 143 (3) (2000), 221–250.

Global extension operators in a half-plane

In the ultradifferentiable setting, H.-J. Petzsche (1988) introduced the condition

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N},\ k > 1 : \limsup_{p\to\infty} \left(\frac{M_{kp}}{M_p}\right)^{\frac{1}{(k-1)p}} \frac{1}{m_{kp-1}} \leq \varepsilon, \qquad (\beta_2)$$

which again appeared in the results about the existence of global extension operators in the ultraholomorphic framework of

J. Schmets, M. Valdivia, Studia Math. 143 (3) (2000), 221–250.

Theorem (A. Debrouwere, 2020)

Suppose $\widehat{\mathbb{M}}$ *is a regular sequence. The following are equivalent:*

(i) *There exists a global extension operator* $U_{\mathbb{M}} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \mathcal{A}_{\{\widehat{\mathbb{M}}\}}(S_1)$.

(ii) $\gamma(\mathbb{M}) > 1$, *and* (β_2) *is satisfied.*

Global extension operators in a fixed sector

The use of Laplace and Borel transforms of arbitrary positive order allows us to generalize this statement.

Theorem (J. Jiménez-Garrido, J. S., G. Schindl (2021))

Suppose $\widehat{\mathbb{M}}$ is a regular sequence, and let $r>0$. Each of the following statements implies the next one:

(i) $r<\gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

(ii) *There exists a global extension operator* $V_{\mathbb{M},r}:\mathbb{C}[[z]]_{\{\mathbb{M}\}}\to\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iii) $r\le\gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

Global extension operators in a fixed sector

The use of Laplace and Borel transforms of arbitrary positive order allows us to generalize this statement.

Theorem (J. Jiménez-Garrido, J. S., G. Schindl (2021))

Suppose $\widehat{\mathbb{M}}$ *is a regular sequence, and let* $r > 0$. *Each of the following statements implies the next one:*

(i) $r < \gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

(ii) *There exists a global extension operator* $V_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iii) $r \leq \gamma(\mathbb{M})$, *and* (β_2) *is satisfied.*

Conjecture: (i) and (ii) are equivalent.

A result of J. Schmets and V. Valdivia

Aim: Determine the weight sequences for which (local or global) extension operators exist for sectors of arbitrary opening.

From the previous results, we should have $\gamma(\mathbb{M}) = \infty$.

A result of J. Schmets and V. Valdivia

Aim: Determine the weight sequences for which (local or global) extension operators exist for sectors of arbitrary opening.

From the previous results, we should have $\gamma(\mathbb{M})=\infty$.

Theorem (J. Schmets, M. Valdivia, 2000)

Let $\mathbb{M}$ be a weight sequence such that

$$\text{for every } r\in\mathbb{N},\ (m_n/n^r)_{n\in\mathbb{N}} \text{ is eventually increasing.}$$

The following are equivalent:

(i) *For every $r\in\mathbb{N}$, there exists a global extension operator $U_{\mathbb{M},r}:\mathbb{C}[[z]]_{\{\mathbb{M}\}}\to\widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.*

(ii) *$\mathbb{M}$ satisfies (β_2).*

Rapidly varying sequences

Proposition (J. Jiménez-Garrido, J. S., G. Schindl (2021))

Let $\mathbb{M}$ *be a weight sequence. Each of the following statements implies the next one, and only the implication (ii)* $\Longrightarrow$ *(iii) may be reversed:*

(i) *For every* $r \in \mathbb{N}$, $(m_n/n^r)_{n\in\mathbb{N}}$ *is eventually increasing.*

(ii) $\gamma(\mathbb{M}) = \infty$.

(iii) *There exists* $k_0 \in \mathbb{N}$, $k_0 \geq 2$, *such that* $\lim_{n\to\infty} \dfrac{m_{k_0 n}}{m_n} = \infty$.

(iv) $\mathbb{M}$ *satisfies* (β_2).

(v) $\lim_{n\to\infty} \dfrac{m_n}{M_n^{1/n}} = \infty$.

(vi) $\omega(\mathbb{M}) := \liminf_{n\to\infty} \dfrac{\log(m_n)}{\log(n)} = \infty$ *(known as the lower order of* $\boldsymbol{m}$*).*

(vii) $\alpha(\boldsymbol{m}) = \infty$, *where* $\alpha(\boldsymbol{m})$ *is the upper Matuszewska index of* $\boldsymbol{m}$*. Equivalently,* $\mathbb{M}$ *does not satisfy (mg).*

Surjectivity and global extension operators for rapidly varying sequences

First consequence: For strongly regular sequences surjectivity does hold and local extension operators exist with an scaling in the type for small openings, but no global extension operator is possible.

Surjectivity and global extension operators for rapidly varying sequences

First consequence: For strongly regular sequences surjectivity does hold and local extension operators exist with an scaling in the type for small openings, but no global extension operator is possible.

Theorem (J. Jiménez-Garrido, J. S., G. Schindl (2021))

Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

(i) $\gamma(\mathbb{M}) = \infty$.

(ii) *For every $r > 0$, there exists a global extension operator* $V_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iii) $\widetilde{S}^u_{\{\mathbb{M}\}} = (0, \infty)$.

Examples: For $q > 1$ and $\sigma > 1$, $\mathbb{M}_{q,\sigma} := (q^{n^\sigma})_{n\in\mathbb{N}_0}$.
For $\tau > 0$ and $\sigma > 1$, $\mathbb{M}^{\tau,\sigma} = (n^{\tau n^\sigma})_{n\in\mathbb{N}_0}$.

Surjectivity and global extension operators for rapidly varying sequences

First consequence: For strongly regular sequences surjectivity does hold and local extension operators exist with an scaling in the type for small openings, but no global extension operator is possible.

Theorem (J. Jiménez-Garrido, J. S., G. Schindl (2021))

Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

(i) $\gamma(\mathbb{M}) = \infty$.

(ii) *For every $r > 0$, there exists a global extension operator* $V_{\mathbb{M},r} : \mathbb{C}[[z]]_{\{\mathbb{M}\}} \to \widetilde{\mathcal{A}}^u_{\{\mathbb{M}\}}(S_r)$.

(iii) $\widetilde{S}^u_{\{\mathbb{M}\}} = (0, \infty)$.

Examples: For $q > 1$ and $\sigma > 1$, $\mathbb{M}_{q,\sigma} := (q^{n^\sigma})_{n\in\mathbb{N}_0}$.
For $\tau > 0$ and $\sigma > 1$, $\mathbb{M}^{\tau,\sigma} = (n^{\tau n^\sigma})_{n\in\mathbb{N}_0}$.

Global extension operators guarantee local ones (a consequence of Grothendieck's factorization theorem), but no common scaling of the type is assured unless (dc) is satisfied.

A pathological situation

If a weight sequence $\mathbb{M}$ has (dc), (β_2) and $0 < \gamma(\mathbb{M}) < \infty$, then:

- $\omega(\mathbb{M}) = \infty$, so the sequence $\boldsymbol{m}$ is not O-regularly varying (pathological in a sense),
- For sectors S_γ with $0 < \gamma < \gamma(\mathbb{M})$ we have optimal $\mathbb{M}$-flat functions, surjectivity of the Borel map, global inverses and local inverses with type scaling, while for $\gamma > \gamma(\mathbb{M})$ we do not have any of these properties nor injectivity of the Borel map (there do exist nontrivial flat functions, but none is optimal).

THANK YOU VERY MUCH FOR YOUR ATTENTION!

DZIEKUJE!