On Gevrey regularity of solutions for inhomogeneous nonlinear moment partial differential equations

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Introduction

We study the Gevrey regularity of formal solutions for a certain class of inhomogeneous nonlinear moment PDEs of the form

$$(1) \qquad \begin{cases} \partial_{m_0;t}^{\kappa} u - P(t,x,(\partial_{m_0;t}^{i}\partial_{m;x}^{q}u)_{(i,q)\in\Lambda}) = \widetilde{f}(t,x) \\ \partial_{m_0;t}^{j} u(t,x)|_{t=0} = \varphi_j(x) \text{ for } 0 \leq j < \kappa, \end{cases},$$

where P is a polynomial with analytic coefficients , the initial conditions are also analytic at a neighbourhood of the origin and the inhomogeneity $\widetilde{f}(t,x)$ is σ-Gevrey for some $\sigma \geq 0$.

Our aim is to show the connection between the Gevrey order of $\widetilde{f}(t,x)$ and the shape of the Newton polygon for Eq. (**??**), and the Gevrey order of its unique formal solution of (**??**).

Notation

- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ stands for the set of all positive integers.
- $\mathbb{R}^+$ stands for the set of all the nonnegative real numbers and $\mathbb{R}_+^*$ for the set of all the positive real numbers.
- For any $\rho_1, \ldots, \rho_N > 0$ we denote by $D_{\rho_1,\ldots,\rho_N}$ the polydisc $D_{\rho_1} \times \ldots \times D_{\rho_N} \subset \mathbb{C}^N$, where $D_\rho = \{z \in \mathbb{C} : \ |z| < \rho\}$ for any $\rho > 0$.
- For any $d \in \mathbb{R}$ and $\alpha, R > 0$, an open sector in direction d with an opening α and a radius R is a set

$$S_d(\alpha, R) = \left\{x \in \mathbb{C} : \ 0 < |x| < R, \ |\arg x - d| < \frac{\alpha}{2}\right\}.$$

- If $U \subset \mathbb{C}^N$, $N \in \mathbb{N}^*$, is an open set then we denote by $\mathcal{O}(U)$ the set of all holomorphic functions defined in U.
- The set of all formal power series in variable t with coefficients from $F \neq \emptyset$ is denoted by $F[[t]]$.
- By $\mathcal{O}[[t]]$ we denote the set of all formal power series in variable t with analytic coefficients in some common neighborhood of the origin.

Moment functions and operators

Definition 1

A pair (e, E) of $\mathbb{C}$-valued functions is called kernel functions of order $s < 2$ if the three following conditions hold:

1. The function e satisfies the following points:
    1.1 e is holomorphic on the sector $S_0(\pi s)$;
    1.2 $e(t) > 0$ for all $t > 0$;
    1.3 the function $t^{-1}e(t)$ is integrable at zero;
    1.4 e is k-exponentially flat at infinity for $k = 1/s$, that is, for every $\varepsilon > 0$, there exist two positive constants $A, B > 0$ such that $|e(x)| \leq A \exp(-(|x|/B)^k)$ for all $x \in S_0(\pi s - \varepsilon)$.
2. The function E satisfies the following points:
    2.1 E is entire on $\mathbb{C}$ with a global exponential growth of order at most $k = 1/s$ at infinity;
    2.2 the function $t^{-1}E(t)$ is integrable at zero in $S_\pi(\pi(2 - s))$.

Kernel functions

3. The functions e and E are connected by a corresponding moment function m of order s as follows:
   3.1 the function m is defined by the Mellin transform of e:

   $$m(\lambda) = \int_0^{+\infty} t^{\lambda-1} e(t) dt \quad \text{for all } \mathrm{Re}(\lambda) \geq 0; \tag{2}$$

   3.2 the function E has the power series expansion

   $$E(t) = \sum_{j \geq 0} \frac{t^j}{m(j)} \quad \text{for all } t \in \mathbb{C}. \tag{3}$$

4. We assume that $m(0) = 1$.

Remarks on kernel and moment functions

- For any moment function m of order s we call a sequence $(m(j))_{j\geq 0}$ a moment sequence of order s.
- Kernel functions of orders $s \geq 2$ can also be considered after some adjustments to the definition.
- We have $m(\lambda) > 0$ for every $\lambda \geq 0$.
- For any moment function m of order s there exist four positive constants $c, C, a, A > 0$ such that for all $j \geq 0$ we have:

$$ca^j\Gamma(1+(s+1)j) \leq m(j) \leq CA^j\Gamma(1+(s+1)j). \tag{4}$$

Example
The following is a classical example of kernel functions and their corresponding moment function:

- $e(t) = kt^k e^{-t^k}$,
- $E(t) = \sum_{j\geq 0} \frac{t^j}{\Gamma(1+j/k)}$,
- $m(\lambda) = \Gamma(1+\lambda/k)$.

Regular moment functions and moment differentiation

Definition 2

A moment function m of order $s > 0$ is called regular if there exist constants $a, A > 0$ such that

$$a(j+1)^s \leq \frac{m(j+1)}{m(j)} \leq A(j+1)^s \quad \text{for every} \quad j \in \mathbb{N}.$$

Definition 3

Let m_0 be a moment function of order $s_0 > 0$ and

$$\widetilde{u}(t,x) = \sum_{j \geq 0} u_{j,*}(x) \frac{t^j}{m_0(j)}$$

be a formal power series. Then, the moment derivative $\partial_{m_0;t}\widetilde{u}$ of $\widetilde{u}(t,x)$ with respect to t is the formal power series in $\mathcal{O}(D_{\rho_1,\ldots,\rho_N})[[t]]$ defined by

$$\partial_{m_0;t}\widetilde{u}(t,x) = \sum_{j \geq 0} u_{j+1,*}(x) \frac{t^j}{m_0(j)}.$$

The nonlinear Cauchy problem

The main problem

Let $m_0, m_1, \ldots, m_N$ be regular moment functions of resp. orders $s_0 > 0$ and $s_1, \ldots, s_N \geq 1$. We consider the inhomogeneous nonlinear moment partial differential equations of the form

$$(5) \qquad \begin{cases} \partial^{\kappa}_{m_0;t} u - P(t,x,(\partial^i_{m_0;t}\partial^q_{m;x}u)_{(i,q)\in\Lambda}) = \widetilde{f}(t,x) \\ \partial^j_{m_0;t} u(t,x)_{|}t=0 = \varphi_j(x) \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N}) \text{ for } 0 \leq j < \kappa, \end{cases}$$

where P denotes a polynomial

$$(6) \quad P(t,x,(\partial^i_{m_0;t}\partial^q_{m;x}u)_{(i,q)\in\Lambda}) =$$
$$\sum_{n\in\mathcal{I}} \sum_{(\underline{i},\underline{q},\underline{r})\in\Lambda_n} t^{v_{\underline{i},\underline{q},\underline{r}}} a_{\underline{i},\underline{q},\underline{r}}(t,x) \left(\partial^{i_1}_{m_0;t}\partial^{q_1}_{m;x}u\right)^{r_1} \ldots \left(\partial^{i_n}_{m_0;t}\partial^{q_n}_{m;x}u\right)^{r_n},$$

satisfying a certain set of conditions.

The main problem

- $\kappa \geq 1$ is a positive integer;
- $\partial_{m;x}^{q}$ stands for the moment derivation $\partial_{m_1;x_1}^{q_1}...\partial_{m_N;x_N}^{q_N}$ while $q = (q_1, ..., q_N)$;
- $\widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,...,\rho_N})[[t]]$;
- $\mathcal{I}$ is a non-empty finite subset of $\mathbb{N}^*$ and for any $n \in \mathcal{I}$, the set Λ_n is a non-empty finite subset of n-tuples

$$(\underline{i}, \underline{q}, \underline{r}) = ((i_1, q_1, r_1), ..., (i_n, q_n, r_n)) \in \{0, ..., \kappa - 1\} \times \mathbb{N}^N \times \mathbb{N}^*,$$

  with pairs (i_k, q_k) being two by two distinct;
- $v_{\underline{i},\underline{q},\underline{r}}$ is a nonnegative integer for every $(\underline{i}, \underline{q}, \underline{r}) \in \Lambda_n$;
- $a_{\underline{i},\underline{q},\underline{r}}(t,x) \in \mathcal{O}(D_{\rho_0,\rho_1,...,\rho_N})$ and $a_{\underline{i},\underline{q},\underline{r}}(0,x) \not\equiv 0$ for every $(\underline{i}, \underline{q}, \underline{r}) \in \Lambda_n$.

Remark
Eq. (**??**) is formally well-posed.

Moreover, for $\widetilde{f}(t,x) = \sum_{j\geq 0} f_{j,*}(x)\dfrac{t^j}{m_0(j)}$ and $a_{\underline{i},\underline{q},\underline{r}}(t,x) = \sum_{j\geq 0} a_{\underline{i},\underline{q},\underline{r};j,*}(x)\dfrac{t^j}{m_0(j)}$ the coefficients $u_{j,*}(x)$ of its formal solution $\widetilde{u}(t,x)$ are uniquely determined by

(7)

$$u_{j+\kappa,*}(x) = f_{j,*}(x) + \sum_{n\in\mathcal{I}} \sum_{(\underline{i},\underline{q},\underline{r})\in\Lambda_n} \sum_{j_0+j_1+\ldots+j_{r_1+\ldots+r_n}=j-v_{\underline{i},\underline{q},\underline{r}}} C_{\underline{i},\underline{q},\underline{r},\underline{j},n}(x)$$

together with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0, \ldots, \kappa - 1$, where

$$C_{\underline{i},\underline{q},\underline{r},\underline{j},n}(x) = \frac{m_0(j)}{m_0(j_0)\ldots m_0(j_{r_1+\ldots+r_n})} a_{\underline{i},\underline{q},\underline{r};j_0,*}(x) \times \prod_{\ell=1}^{n} \prod_{h=j_{r_1+\ldots+r_{\ell-1}+1}}^{j_{r_1+\ldots+r_\ell}} \partial_{m;x}^{q_\ell} u_{h+i_\ell,*}(x). \tag{8}$$

Let us denote by $C(a,b)=\{(x,y)\in\mathbb{R}^2; x\leq a \text{ and } y\geq b\}$ for all $a,b\in\mathbb{R}$ and consider an operator $\Delta_{\kappa,P}:=\partial_{m_0;t}^{\kappa}-P(t,x,(\partial_{m_0;t}^{i}\partial_{m;x}^{q})_{(i,q)\in\Lambda})$ associated with Eq. (**??**).

Definition 4

We call moment Newton polygon of $\Delta_{\kappa,P}$, and we denote it by $\mathcal{N}(\Delta_{\kappa,P})$, the convex hull of

$$C(s_0\kappa,-\kappa)\cup\bigcup_{n\in\mathcal{I}}\bigcup_{(\underline{i},\underline{q},\underline{r})\in\Lambda_n}C\left(\sum_{\ell=1}^{n}(s_0r_\ell i_\ell+r_\ell\lambda(sq_\ell)),v_{\underline{i},\underline{q},\underline{r}}-\sum_{\ell=1}^{n}r_\ell i_\ell\right)$$

with $\lambda(sq_\ell)=\displaystyle\sum_{d=1}^{N}s_dq_{\ell,d}$.

For all $n\in\mathcal{I}$ and all $(\underline{i},\underline{q},\underline{r})\in\Lambda_n$ we assume that $\displaystyle\sum_{\ell=1}^{n}r_\ell i_\ell-v_{\underline{i},\underline{q},\underline{r}}<\kappa$.

The Newton polygon

For any $n \in \mathcal{I}$, let us denote by $\mathcal{S}_n$ the set of all the the tuples $(\underline{i}, \underline{q}, \underline{r}) \in \Lambda_n$ such that

$$\sum_{\ell=1}^{n} (s_0 r_\ell i_\ell + r_\ell \lambda(s q_\ell)) > s_0 \kappa.$$

Let $\mathcal{S} = \bigcup_{n \in \mathcal{I}} \mathcal{S}_n$.

1. Assume $\mathcal{S} = \emptyset$. Then, the moment Newton polygon is reduced to $C(s_0\kappa, -\kappa)$ (see Fig. **??**).
2. If $\mathcal{S} \neq \emptyset$, the moment Newton polygon has at least one side with a positive slope. Moreover, its smallest positive slope k is given by

$$k = \min_{\substack{n \in \mathcal{I} \\ (\underline{i},\underline{q},\underline{r}) \in \mathcal{S}_n}} \left(\frac{\kappa + v_{\underline{i},\underline{q},\underline{r}} - \sum_{\ell=1}^{n} r_\ell i_\ell}{\sum_{\ell=1}^{n} (s_0 r_\ell i_\ell + r_\ell \lambda(s q_\ell)) - s_0 \kappa} \right)$$

$$= \frac{\kappa + v_{\underline{i}^*,\underline{q}^*,\underline{r}^*} - \sum_{\ell=1}^{n^*} r_\ell^* i_\ell^*}{\sum_{\ell=1}^{n^*} (s_0 r_\ell^* i_\ell^* + r_\ell^* \lambda(s q_\ell^*)) - s_0 \kappa}.$$

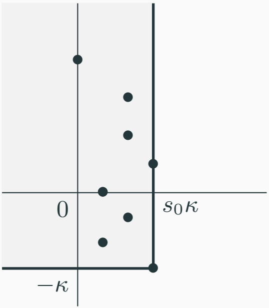


Figure 1: Case $\mathcal{S} = \emptyset$

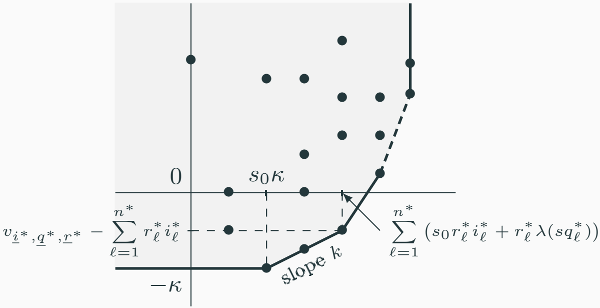


Figure 2: Case $\mathcal{S} \neq \emptyset$

Definition 5

Let $\sigma \geq 0$. Then, a formal power series

$$\widetilde{u}(t,x) = \sum_{j\geq 0} u_{j,*}(x) t^j \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N})[[t]]$$

is said to be Gevrey of order σ (or, for short, σ-Gevrey) if there exist a radius $0 < r < \min\{\rho_1, \ldots, \rho_N\}$ and constants $C, K > 0$ such that

$$|u_{j,*}(x)| \leq CK^j \Gamma(1 + \sigma j)$$

for all $x \in D_{r,\ldots,r}$ and all $j \geq 0$.

The Gevrey regularity theorem

Theorem 1

Let

$$\sigma_c = \frac{1}{k} = \begin{cases} \frac{\sum_{\ell=1}^{n^*} \left(s_0 r_\ell^* i_\ell^* + r_\ell^* \lambda(s q_\ell^*)\right) - s_0 \kappa}{\kappa + v_{\underline{i}^*, \underline{q}^*, \underline{r}^*} - \sum_{\ell=1}^{n^*} r_\ell^* i_\ell^*} & \textit{when } \mathcal{S} \neq \emptyset \\ 0 & \textit{when } \mathcal{S} = \emptyset \end{cases}$$

Then,

1. $\widetilde{u}(t,x)$ *and* $\widetilde{f}(t,x)$ *are simultaneously* σ*-Gevrey for any* $\sigma \geq \sigma_c$*;*
2. $\widetilde{u}(t,x)$ *is generically* σ_c*-Gevrey while* $\widetilde{f}(t,x)$ *is* σ*-Gevrey with* $\sigma < \sigma_c$*.*

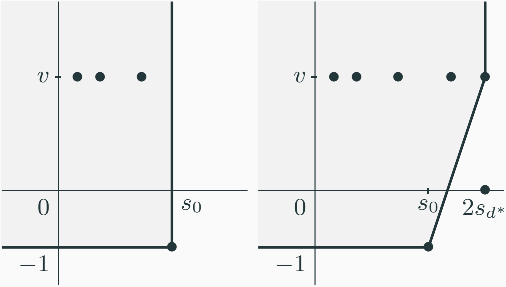
Example
Let us consider the semilinear regular moment heat equation

$$(9)\qquad \begin{cases} \partial_{m_0;t}u - t^v a(t,x)\Delta_{m;x}u + b(t,x)u^r = \widetilde{f}(t,x) \\ u(0,x) = \varphi(x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_N}) \end{cases}$$

where

- $\Delta_{m;x} = \partial^2_{m_1;x_1} + \dots + \partial^2_{m_N;x_N}$ is the moment Laplace operator;
- the degree r is an integer at least 2;
- v is a nonnegative integer;
- the coefficients $a(t,x)$ and $b(t,x)$ are analytic on a polydisc $D_{\rho_0,\rho_1,\dots,\rho_N}$ and $a(0,x) \not\equiv 0$;
- $\widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_N})[[t]]$.

The moment Newton polygon associated with Eq. (??) is as shown on Fig ?? below. If any exists, we define d^* by
$d^* = \max\{d \in \{1, ..., N\} : \ 2s_d > s_0\}$.



(a) Case $2s_d \leq s_0$ for all $d \in \{1, ..., N\}$

(b) Case $2s_d > s_0$ for some $d \in \{1, ..., N\}$.

Figure 3: The moment Newton polygon associated with Eq. (??)

Then we have

$$\sigma_c = \begin{cases} 0 & \text{if } 2s_d \leq s_0 \text{ for all } d \in \{1, ..., N\} \\ \dfrac{2s_{d^*} - s_0}{1+v} & \text{otherwise} \end{cases}$$

and the Gevrey regularity of the unique formal solution $\widetilde{u}(t,x)$ of Eq. (**??**) follows from Theorem **??**.

Sketch of the proof

Sketch of the proof

The proof of the main theorem is devided into two parts.

- Proof of the first point is based on the modified Nagumo norms, the technique of majorant series and the fixed-point procedure.
- To prove the second point of the theorem we shall present an explicit example for which $\widetilde{u}(t,x)$ is σ'-Gevrey for no $\sigma' < \sigma_c$ while $\widetilde{f}(t,x)$ is σ-Gevrey with $\sigma < \sigma_c$.

Modified Nagumo norms

For any $\alpha \geq 0$ and $s > 0$, we consider the formal power series

$$\Theta_{\alpha,s}(x) = \sum_{j\geq 0} \binom{\alpha+j-1}{j}^s x^j$$

with

$$\binom{\alpha+j-1}{j} = \frac{\Gamma(\alpha+j)}{\Gamma(1+j)\Gamma(\alpha)} = \begin{cases} 1 & \text{if } j=0 \\ \dfrac{\alpha(\alpha+1)...(\alpha+j-1)}{j!} & \text{if } j\geq 1 \end{cases}.$$

Definition 6

Let $f(x) = \sum_{j_1,\ldots,j_N \geq 0} f_{j_1,\ldots,j_N} x_1^{j_1} \ldots x_N^{j_N} \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N})$ be an analytic function on $D_{\rho_1,\ldots,\rho_N}$. Let $s = (s_1, \ldots, s_N) \in (\mathbb{R}_+^*)^N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in [1, +\infty[^N \cup \{0\}$, and suppose that $0 < r < \min(\rho_1, \ldots, \rho_N)$. Then, the modified Nagumo norm $\|f\|_{\alpha,r,s}$ of f with indices (α, r, s) is defined by:

$$\|f\|_{\alpha,r,s} = \begin{cases} \displaystyle\sum_{j_1,\ldots,j_N \geq 0} |f_{j_1,\ldots,j_N}| \, r^{j_1+\ldots+j_N} & \text{if } \alpha = 0 \\ \displaystyle\inf\left(A \geq 0 : f(x) \ll A \prod_{d=1}^{N} \frac{1}{r^{\alpha_d}} \Theta_{\alpha_d,s_d}\left(\frac{x_d}{r}\right)\right) & \text{otherwise} \end{cases}.$$

Modified Nagumo norms

Remark
The modified Nagumo norms are well defined for $\alpha \in [1, +\infty[^N$.

Proposition 1

For fixed (α, r, s)*, the function* $\|f\|_{\alpha,r,s} : \mathcal{O}(D_{\rho_1,\ldots,\rho_N}) \to \mathbb{R}^+$ *defines a norm on* $\mathcal{O}(D_{\rho_1,\ldots,\rho_N})$*.*

Proposition 2

Let $f(x), g(x) \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N})$*,* $s \in [1, +\infty[^N$ *and* $\alpha, \beta \in [1, +\infty[^N \cup \{0\}$ *and* $0 < r < \min(\rho_1, ..., \rho_N)$*. Then,* $\|fg\|_{\alpha+\beta,r,s} \leq \|f\|_{\alpha,r,s} \|g\|_{\beta,r,s}$*.*

Proposition 3

Assume that $m_1, ..., m_N$ *are all regular moment functions. Then, for all* $\alpha \in [1, +\infty[^N$ *and all* $q \in \mathbb{N}^N$*, there exists* $C > 0$ *such that*

$$\left\|\partial_{m;x}^q f\right\|_{\alpha+q,r,s} \leq C^{\lambda(q)} \left(\prod_{d=1}^{N} q_d!^{s_d} \binom{\alpha_d + q_d - 1}{q_d}^{s_d} \right) \|f\|_{\alpha,r,s} .$$

Proposition 4

Let $\widetilde{u}(t,x) = \sum_{j\geq 0} u_{j,*}(x)t^j \in \mathcal{O}(D_{\rho_1,\dots,\rho_N})[[t]]_\sigma$ *be a* σ*-Gevrey formal power series. Let* $0 < r < \min\{\rho_1,\dots,\rho_N\}$*. Then, for all* $\alpha \in [1,+\infty[^N \cup \{0\}$ *and all* $s \in (\mathbb{R}_+^*)^N$*, there exist* $A, B > 0$ *such that the following inequality holds for all* $j \geq 0$*:*

$$\|u_{j,*}\|_{j\alpha,r,s} \leq AB^j\Gamma(1+\sigma j).$$

Proposition 5

Let $0 < \rho < r < \min(\rho_1,\dots,\rho_N)$*. Then, there exists* $A > 0$ *such that, for all* $f(x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_N})$ *and all* $\alpha \in [1,+\infty[^N \cup \{0\}$*, the following inequality holds for all* $x \in D_{\rho,\dots,\rho}$*:*

$$|f(x)| \leq A^{\lambda(\alpha)} \|f\|_{\alpha,r,s}.$$

Sketch of the proof - point 1

It is clear that $\widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_N})[[t]]_\sigma \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\dots,\rho_N})[[t]]_\sigma$.

Let us fix $\sigma \geq \sigma_c$ and assume that $\widetilde{f}(t,x) = \sum_{j\geq 0} f_{j,*}(x)\dfrac{t^j}{m_0(j)}$ is σ-Gevrey.

Then, there exist $0 < r < \min(\rho_1, ..., \rho_N)$ and $C, K > 0$ such that $|f_{j,*}(x)| \leq CK^j m_0(j)\Gamma(1+\sigma j)$ for all $x \in D_{r,\dots,r}$ and all $j \geq 0$.

In order to prove that $u_{j,*}(x)$ satisfy similar inequalities, we use modified Nagumo norms with indices $((j+\kappa)\alpha_\sigma, r, s)$, where $\alpha_\sigma \in (\mathbb{R}^+)^N$ is the multi-index with all components equal to $(\sigma+s_0)(\kappa+v)$, with $v = \varsigma + \max v_{\underline{i},\underline{q},\underline{r}}$ and

$$\varsigma = \max\left(\frac{1-(\sigma+s_0)(\kappa+\max v_{\underline{i},\underline{q},\underline{r}})}{\sigma+s_0},\right.$$
$$\left.\max_{(\underline{i},\underline{q},\underline{r})\in\bigcup_{n\in\mathcal{I}}\Lambda_n}\left(\frac{1}{(\sigma+s_0)\left(\kappa-\sum_{\ell=1}^{n} r_\ell i_\ell + v_{\underline{i},\underline{q},\underline{r}}\right)}\right)\right).$$

Sketch of the proof - point 1

After applying this norm to both sides of (??) and using the propositions from before, we receive:

$$\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\alpha_\sigma,r,s}}{m_0(j+\kappa)\Gamma(1+\sigma(j+\kappa))} \leq \frac{\|f_{j,*}\|_{(j+\kappa)\alpha_\sigma,r,s}}{m_0(j+\kappa)\Gamma(1+\sigma(j+\kappa))} + \sum_{n\in\mathcal{I}} \sum_{(\underline{i},\underline{q},\underline{r})\in\Lambda_n} \sum_{\substack{j_0+j_1+\ldots+ \\ j_{r_1+\ldots+r_n}=j-v_{\underline{i},\underline{q},\underline{r}}}} B_{\underline{i},\underline{q},\underline{r},\underline{j},n}(x)$$

with

$$B_{\underline{i},\underline{q},\underline{r},\underline{j},n}(x) = \frac{\tilde{C}\left\|a_{\underline{i},\underline{q},\underline{r};j_0,*}\right\|_{\alpha'_\sigma(j_0),r,s}}{\Gamma(1+\sigma j_0)m_0(j_0)} \times \prod_{\ell=1}^{n} \prod_{h=j_{r_1+\ldots+r_{\ell-1}+1}}^{j_{r_1+\ldots+r_\ell}} \frac{\|u_{h+i_\ell,*}\|_{(h+i_\ell)\alpha_\sigma,r,s}}{m_0(h+i_\ell)\Gamma(1+\sigma(h+i_\ell))}$$

for all $j \geq v_{\underline{i},\underline{q},\underline{r}}$ with $\alpha'_\sigma(j_0) = \left(j_0+\kappa-\sum_{\ell=1}^{n} r_\ell i_\ell + v_{\underline{i},\underline{q},\underline{r}}\right)\alpha_\sigma - \sum_{\ell=1}^{n} r_\ell q_\ell.$

Sketch of the proof - point 1

The next step is to bound $\|u_{h+i_\ell,*}\|_{(h+i_\ell)\alpha_\sigma,r,s}$ using the majorant series method.

Let us set

$$g_{j,s} = \frac{\|f_{j,*}\|_{(j+\kappa)\alpha_\sigma,r,s}}{m_0(j+\kappa)\Gamma(1+\sigma(j+\kappa))} \quad \text{and} \quad \alpha_{\underline{i},\underline{q},\underline{r},j,s} = \frac{\tilde{C}\left\|a_{\underline{i},\underline{q},\underline{r};j,*}\right\|_{\alpha'_\sigma(j),r,s}}{\Gamma(1+\sigma j)m_0(j)},$$

Lemma 1

There exist four positive constants $B', B'', C', C'' > 0$ such that the following inequalities hold for all $j \geq 0$:

$$g_{j,s} \leq C' B'^j \quad \textit{and} \quad \alpha_{\underline{i},\underline{q},\underline{r},j,s} \leq C'' B''^j.$$

Sketch of the proof - point 1

Let us now consider the formal power series $v(X) = \sum_{j \geq 0} v_j X^j$, the coefficients of which are recursively determined for all $j \geq 0$ by the relations

$$(10) \qquad v_{j+\kappa} = g_{j,s} + \sum_{n \in \mathcal{I}} \sum_{(\underline{i},\underline{q},\underline{r}) \in \Lambda_n} \sum_{\substack{j_0+j_1+\ldots+j_{\widetilde{r}} \\ = j + \sum_{\ell=1}^{n} r_\ell i_\ell - v_{\underline{i},\underline{q},\underline{r}}}} \alpha_{\underline{i},\underline{q},\underline{r},j_0,s} v_{j_1} \ldots v_{j_{\widetilde{r}}}$$

starting with the initial conditions

$$v_0 = 1 + \frac{\|\varphi_0\|_{0,r,s}}{m_0(0)}, \text{ and, for } j = 1, \ldots, \kappa - 1 \text{ (if } \kappa \geq 2\text{):}$$

$$v_j = \frac{\|\varphi_j\|_{j\alpha_\sigma,r,s}}{m_0(j)\Gamma(1+\sigma j)} + \sum_{(\underline{i},\underline{q},\underline{r}) \in V_j} \sum_{\substack{j_0+j_1+\ldots+j_{\widetilde{r}} \\ = j - \kappa + \sum_{\ell=1}^{n} r_\ell i_\ell - v_{\underline{i},\underline{q},\underline{r}}}} \alpha_{\underline{i},\underline{q},\underline{r},j_0,s} v_{j_1} \ldots v_{j_{\widetilde{r}}},$$

where $\widetilde{r} = \max_{(\underline{i},\underline{q},\underline{r}) \in \bigcup_{n \in \mathcal{I}} \Lambda_n} (r_1 + \ldots + r_n)$, and where

$$V_j = \left\{ (\underline{i},\underline{q},\underline{r}) \in \bigcup_{n \in \mathcal{I}} \Lambda_n \text{ such that } j - \kappa + \sum_{\ell=1}^{n} r_\ell i_\ell - v_{\underline{i},\underline{q},\underline{r}} \geq 0 \right\}.$$

Proposition 6

The inequalities

$$0 \leq \frac{\|u_{j,*}\|_{j\alpha_\sigma,r,s}}{m_0(j)\Gamma(1+\sigma j)} \leq v_j \tag{11}$$

hold for all $j \geq 0$.

Proposition 7

The formal series $v(X)$ *is convergent. In particular, there exist two positive constants* $C', K' > 0$ *such that* $v_j \leq C'K'^j$ *for all* $j \geq 0$.

Sketch of the proof - point 1

To prove Proposition **??**, it is necessary to observe that $v(X)$ is the unique formal power series in X solution of the functional equation

(12) $$v(X) = X\alpha(X)(v(X))^{\tilde{r}} + h(X),$$

where $\alpha(X)$ and $h(X)$ are the two formal power series defined by

$$\alpha(X) = \sum_{n\in\mathcal{I}} \sum_{(\underline{i},\underline{q},\underline{r})\in\Lambda_n} X^{\kappa-\sum_{\ell=1}^{n} r_\ell i_\ell - 1 + v_{\underline{i},\underline{q},\underline{r}}} \alpha_{\underline{i},\underline{q},\underline{r},s}(X) \text{ and}$$

$$h(X) = A_0 + A_1 X + ... + A_{\kappa-1}X^{\kappa-1} + X^\kappa \sum_{j\geq 0} g_{j,s}X^j$$

with

$$\alpha_{\underline{i},\underline{q},\underline{r},s}(X) = \sum_{j\geq 0} \alpha_{\underline{i},\underline{q},\underline{r},j,s}X^j, \quad A_0 = 1 + \frac{\|\varphi_0\|_{0,r,s}}{m_0(0)},$$

$$A_j = \frac{\|\varphi_j\|_{j\alpha_\sigma,r,s}}{m_0(j)\Gamma(1+\sigma j)} \text{for } j = 1, ..., \kappa - 1 \text{ (if } \kappa \geq 2).$$

Sketch of the proof - point 1

$\alpha(X)$ and $h(X)$ are convergent power series with nonnegative coefficients, with radii of convergence r_α and r_h, respectively. They both define increasing functions within their respective regions of convergence. Moreover, seeing as $a_{\underline{i},\underline{q},\underline{r};0,*}(x) \not\equiv 0$ and $A_0 \geq 1$, we have $\alpha(r) > 0$ and $h(r) > 0$ for all $r \in]0, r_\alpha[$ and $r \in]0, r_h[$ respectively.

To determine that $v(X)$ is convergent, the fixed point method will be used. Let us define a formal power series $V(X) = \sum_{\mu \geq 0} V_\mu(X)$ and let us choose the solution of the functional equation (**??**) given by the system

$$\begin{cases} V_0(X) = h(X) \\ V_{\mu+1}(X) = X\alpha(X) \sum_{\mu_1+\ldots+\mu_{\tilde{r}}=\mu} V_{\mu_1}(X)\ldots V_{\mu_{\tilde{r}}}(X) \quad \text{for } \mu \geq 0. \end{cases}$$

Sketch of the proof - point 1

By inductive reasoning on $\mu \geq 0$, we establish that

$$V_\mu(x) = \widetilde{C}_{\mu,\widetilde{r}} X^\mu \alpha(X)^\mu h(X)^{(\widetilde{r}-1)\mu+1}$$

with

$$\widetilde{C}_{\mu+1,\widetilde{r}} = \sum_{\mu_1+\ldots+\mu_{\widetilde{r}}=\mu} \widetilde{C}_{\mu_1,\widetilde{r}} \ldots \widetilde{C}_{\mu_{\widetilde{r}},\widetilde{r}}$$

for every $\mu > 0$ and $\widetilde{C}_{0,\widetilde{r}} = 1$.

It follows from the analyticity of $\alpha(X)$ and $h(X)$ that all $V_\mu(X)$ define analytic functions on the disc with center $0 \in \mathbb{C}$ and radius $\min\{r_\alpha, r_h\}$). Moreover $V_\mu(X)$ is of order X^μ for all $\mu \geq 0$. Hence, the series $V(X)$ makes sense as a formal power series in X, and we obtain $V(X) = v(X)$ by unicity.

To conclude the proof, it remains to show that $V(X)$ is convergent. To do that, let us fix $0 < r < \min\{r_\alpha, r_h\}$. Then, for all $\mu \geq 0$ and for $|X| \leq r$ we receive

$$|V_\mu(X)| \leq \widetilde{C}_{\mu,\widetilde{r}} |X|^\mu \alpha(r)^\mu h(r)^{(\widetilde{r}-1)\mu+1}.$$

Proposition 8

*Let $C', K' > 0$ be as in Proposition **??**. Then, the following inequality holds for all $j \geq 0$:*

$$\|u_{j,*}\|_{j\alpha_\sigma,r,s} \leq C' K'^j m_0(j)\Gamma(1+\sigma j).$$

Let us now apply Proposition **??**: there exists $A > 0$ such that the following inequality holds for all $j \geq 0$ and all $x \in D_{\rho,\dots,\rho}$:

$$|u_{j,*}(x)| \leq A^{\lambda(j\alpha_\sigma)} \|u_{j,*}\|_{j\alpha_\sigma,r,s}.$$

From the fact that $\lambda(j\alpha_\sigma) = j\lambda(\alpha_\sigma)$ and Proposition **??** it follows that

$$|u_{j,*}(x)| \leq C'(K' A^{\lambda(\alpha_\sigma)})^j m_0(j)\Gamma(1+\sigma j)$$

for all $x \in D_{\rho,\dots,\rho}$ and all $j \geq 0$.

Sketch of the proof - point 2

According to the filtration of the σ-Gevrey spaces $\mathcal{O}(D_{\rho_1,\ldots,\rho_N})[[t]]_s$ and the first point of Theorem **??**, the following implications hold:

$$\widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N})[[t]]_\sigma \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N})[[t]]_{\sigma_c}$$
$$\Rightarrow \widetilde{u}(t,x) \in \mathcal{O}(D_{\rho_1,\ldots,\rho_N})[[t]]_{\sigma_c}.$$

Lemma 2

Let $m_1,\ldots,m_N$ be regular moment functions of respective orders $s_1,\ldots,s_N \geq 1$ Then function

$$\mathcal{E}_m(x) = \prod_{d=1}^{N} \left(\sum_{j_d \geq 0} a_d^{j_d} j_d!^{s_d} \frac{x_d^{j_d}}{m_d(j_d)} \right)$$

defines an analytic function on the polydisc $D_{1,\ldots,1}$.

Sketch of the proof - point 2

Proposition 9

Let us consider the equation

$$\begin{cases} \partial_{m_0;t}^{\kappa} u - \sum_{n\in\mathcal{I}} \sum_{(\underline{i},\underline{q},\underline{r})\in\Lambda_n} t^{v_{\underline{i},\underline{q},\underline{r}}} a_{\underline{i},\underline{q},\underline{r}} \left(\partial_{m_0;t}^{i_1}\partial_{m;x}^{q_1} u\right)^{r_1} \dots \left(\partial_{m_0;t}^{i_n}\partial_{m;x}^{q_n} u\right)^{r_n} = \widetilde{f}(t,x) \\ \partial_{m_0;t}^{j} u(t,x)|_{t=0} = \varphi_j(x),\ j=0,\dots,\kappa-1 \end{cases}$$

where

- *the coefficients* $a_{\underline{i},\underline{q},\underline{r}}$ *are positive real numbers for all* $(\underline{i},\underline{q},\underline{r})\in\Lambda_n$ *and all* $n\in\mathcal{I}$*;*
- $i_\ell^* = 0$ *and* $q_\ell^* = (0,\dots,0)$ *for all* $\ell\in\{1,\dots,n^*-1\}$*, and* $r_{n^*}^* = 1$*;*
- *the initial condition* $\varphi_{i_{n^*}^*}(x)$ *is the analytic function* $\mathcal{E}_m(x)$ *on the disc* $D_{1,\dots 1}$*;*
- *the initial conditions* $\varphi_j(x)$ *for* $j\neq i_{n^*}^*$ *are analytic functions on* $D_{1,\dots,1}$ *satisfying* $\partial_{m;x}^{\ell}\varphi_j(0)>0$ *for all* $\ell\in\mathbb{N}^N$*.*
- $\widetilde{f}(t,x)$ *is* σ*-Gevrey and* $\partial_{m;x}^{\ell} f_{j,*}(0)\geq 0$ *for all* $j\geq 0$ *and all* $\ell\in\mathbb{N}^N$*.*

Sketch of the proof - point 2

Remark
Due to our assumptions the previous equation is reduced to a nonlinear equation of the form

$$\begin{cases} \partial_{m_0;t}^{\kappa} u - \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \left(\sum_{r\in P_{i,q}} a_{i,q,r} t^{v_{i,q,r}} u^r \right) \partial_{m_0;t}^{i} \partial_{m;x}^{q} u = \widetilde{f}(t,x) \\ \partial_{m_0;t}^{j} u(t,x)|_{t=0} = \varphi_j(x),\ j = 0, ..., \kappa - 1 \end{cases}$$

where

- $\mathcal{K}$ is a nonempty subset of $\{0, ..., \kappa - 1\}$;
- Q_i is a nonempty finite subset of $\mathbb{N}^N$ for all $i \in \mathcal{K}$;
- $P_{i,q}$ is a nonempty finite subset of $\mathbb{N}$ for all $i \in \mathcal{K}$ and all $q \in Q_i$.

For the sake of clarity, we retain the notations used before and will not use this simpler form.

Observe in particular that we have $\sigma_c = \dfrac{s_0 i_{n^*}^* + \lambda(sq_{n^*}^*) - s_0\kappa}{\kappa + v_{\underline{i}^*,\underline{q}^*,\underline{r}^*} - i_{n^*}^*}$.

Sketch of the proof - point 2

It is sufficient to prove that $\widetilde{u}(t,x)$ is σ'-Gevrey for no $\sigma' < \sigma_c$.

Let us rewrite the general relations (**??**) as

$$u_{j+\kappa,*}(x) = A_{\underline{i}^*,\underline{q}^*,\underline{r}^*}(x)\frac{m_0(j)}{m_0(j-v_{\underline{i}^*,\underline{q}^*,\underline{r}^*})}\partial_{m;x}^{q^*_{n^*}} u_{j-v_{\underline{i}^*,\underline{q}^*,\underline{r}^*}+i^*_{n^*},*}(x) + R_j(x)$$

with $A_{\underline{i}^*,\underline{q}^*,\underline{r}^*}(x) = a_{\underline{i}^*,\underline{q}^*,\underline{r}^*} \displaystyle\prod_{\ell=1}^{n^*-1} (u_{0,*}(x))^{r^*_\ell}$ and

$$R_j(x) = f_{j,*}(x) + \sum_{\substack{j_1+\ldots+j_{r_1^*+\ldots+r_{n^*}^*}=j-v_{\underline{i^*},\underline{q^*},\underline{r^*}} \\ (j_1,\ldots,j_{r_1^*+\ldots+r_{n^*}^*})\neq(0,\ldots,0,j-v_{\underline{i^*},\underline{q^*},\underline{r^*}})}} C_{\underline{i^*},\underline{q^*},\underline{r^*},\underline{j},n^*}(x) + \sum_{\substack{(\underline{i},\underline{q},\underline{r})\in\bigcup_{n\in\mathcal{I}}\Lambda_n \\ (n,\underline{i},\underline{q},\underline{r})\neq(n^*,\underline{i^*},\underline{q^*},\underline{r^*})}} \sum_{j_0+j_1+\ldots+j_{r_1+\ldots+r_n}=j-v_{\underline{i},\underline{q},\underline{r}}} C_{\underline{i},\underline{q},\underline{r},\underline{j},n}(x)$$

for all $j \geq 0$, with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0, \ldots, \kappa - 1$.

Sketch of the proof - point 2

We easily check that, for all $j \geq 0$:

$$u_{j(v_{\underline{i}^*,\underline{q}^*,\underline{r}^*}+\kappa-i^*_{n^*})+i^*_{n^*},*}(x) = \left(A_{\underline{i}^*,\underline{q}^*,\underline{r}^*}(x)\right)^j \partial_{m;x}^{jq^*_{n^*}} \varphi_{i^*_{n^*}}(x) \times \prod_{k=0}^{j-1} \frac{m_0(k(v_{\underline{i}^*,\underline{q}^*,\underline{r}^*}+\kappa-i^*_{n^*})+v_{\underline{i}^*,\underline{q}^*,\underline{r}^*})}{m_0(k(v_{\underline{i}^*,\underline{q}^*,\underline{r}^*}+\kappa-i^*_{n^*}))} + \mathrm{rem}_j(x)$$

with $A_{\underline{i}^*,\underline{q}^*,\underline{r}^*}(0) > 0$ and $\mathrm{rem}_j(0) \geq 0$.

Observe that

$$\partial_{m;x}^{jq^*_{n^*}} \varphi_{i^*_{n^*}}(0) = \prod_{d=1}^{N} a_d^{jq^*_{n^*,d}} (jq^*_{n^*,d})!^{s_d}.$$

We can also deduce that there exist $C, K > 0$ such that

$$(13) \qquad u_{j(v_{\underline{i}^*,\underline{q}^*,\underline{r}^*}+\kappa-i^*_{n^*})+i^*_{n^*},*}(0) \geq CK^j (jv_{\underline{i}^*,\underline{q}^*,\underline{r}^*})!^{s_0} \prod_{d=1}^{N} (jq^*_{n^*,d})!^{s_d}.$$

Sketch of the proof - point 2

Suppose that $\widetilde{u}(t,x)$ is σ'-Gevrey for some $\sigma' < \sigma_c$. Then, Definition **??**, properties of moment functions and inequality (**??**) imply

$$1 \le C' K'^j \frac{\Gamma(1+(\sigma'+s_0)(j(v_{\underline{i}^*,\underline{q}^*,\underline{r}^*}+\kappa-i^*_{n^*})+i^*_{n^*}))}{(jv_{\underline{i}^*,\underline{q}^*,\underline{r}^*})!^{s_0}\prod_{d=1}^{N}(jq^*_{n^*,d})!^{s_d}} \tag{14}$$

for all $j \ge 0$ and some convenient positive constants $C', K' > 0$ independent of j. Using the Stirling formula we conclude that the right hand-side of (**??**) goes to 0 when j tends to infinity. This ends the proof.

Additional remarks

Additional remarks

- When the moment functions $m_0, m_1, ..., m_N$ are chosen so that $m_0(\lambda) = m_1(\lambda)... = m_N(\lambda) = \Gamma(1+\lambda)$, Eq. (**??**) is reduced to a classical inhomogeneous nonlinear partial differential equation. In particular, Theorem **??** allows to study the Gevrey regularity of its formal power series solution, including the non-Kovalevskaya case.
- In the Kovalevskaya case our result is weaker than the Cauchy-Kovalevskaya Theorem. Let us consider the partial differential equation

$$(15)\qquad \begin{cases} \partial_t^3 u + \partial_t \partial_x u + (\partial_x^2 u)^3 = 0 \\ \partial_t^j u(t,x)|_{t=0} = \varphi_j(x),\ j = 0,1,2 \end{cases} .$$

  in two variables $(t,x) \in \mathbb{C}^2$. From Cauchy-Kovalevskaya Theorem it follows that the formal solution $\widetilde{u}(t,x)$ defines an analytic function at the origin of $\mathbb{C}^2$, whereas Theorem **??** tells us that $\widetilde{u}(t,x)$ is 1-Gevrey. This is not contradictory, but our result is clearly weaker.

THANK YOU FOR YOUR ATTENTION!

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