Voros coefficients and the topological recursion for the hypergeometric differential equations

Yumiko Takei

National Institute of Technology(KOSEN), Ibaraki College

Conference Complex Differential and Difference Equations II
Mathematical Research and Conference Center, Będlewo
29th August, 2023

1. Introduction
2. Exact WKB analysis
3. Topological recursion
4. Main results

Introduction, I – case of the Weber equation –

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0$$

Topological recursion
([EO1], [CEO])
$\longrightarrow$

For $g \geq 0$, $n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$
  : correlation function
- $F_g \in \mathbb{C}$
  : g-th free energy

Introduction, I – case of the Weber equation –

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0$$

Topological recursion ([EO1], [CEO]) $\longrightarrow$

For $g \geq 0$, $n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$: correlation function
- $F_g \in \mathbb{C}$: g-th free energy

Quantization ([EO1], [DM], [BE]) $\downarrow$

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda + \frac{\nu}{2}\hbar\right)\right] \psi(x, \hbar) = 0$$

$$\psi(x, \hbar) = \exp\left[\sum_{g \geq 0,\, n \geq 1} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n)\right]\Bigg|_{z=z(x)}$$

Introduction, I – case of the Weber equation –

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0$$

Topological recursion ([EO1], [CEO]) $\longrightarrow$

For $g \geq 0$, $n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$: correlation function
- $F_g \in \mathbb{C}$: g-th free energy

Quantization ([EO1], [DM], [BE]) $\downarrow\uparrow$ Classical limit

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda + \frac{\nu}{2}\hbar\right)\right] \psi(x,\hbar) = 0$$

$$\psi(x,\hbar) = \exp\left[\sum_{g\geq 0,\, n\geq 1} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n)\right]\Bigg|_{z=z(x)}$$

Introduction, I – case of the Weber equation –

$$y^2 - \left(\frac{x^2}{4} - \lambda\right) = 0$$

Topological recursion ([EO1], [CEO]) $\longrightarrow$

For $g \geq 0$, $n \geq 1$

- $W_{g,n}(z_1, \ldots, z_n)$: correlation function
- $F_g \in \mathbb{C}$: g-th free energy

Quantization ([EO1], [DM], [BE]) $\downarrow$ $\uparrow$ Classical limit

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda + \frac{\nu}{2}\hbar\right)\right] \psi(x, \hbar) = 0$$

$$\psi(x, \hbar) = \exp\left[\sum_{g \geq 0,\, n \geq 1} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n)\right]\Bigg|_{z=z(x)}$$

Introduction, II – case of the Weber equation –

Quantization ([EO1], [DM], [BE])

$$\psi(x,\hbar) = \exp\left[\sum_{g\geq 0,\, n\geq 1} \frac{\hbar^{2g+n-2}}{n!} \int_D \cdots \int_D W_{g,n}(z_1,\ldots,z_n)\right]\Bigg|_{z=z(x)} \tag{1}$$

is a WKB solution of the Weber equation

$$\left[\hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda + \frac{\nu}{2}\hbar\right)\right]\psi(x,\hbar) = 0.$$

By using this WKB solution, we obtain the following relation.

Relationship between Voros coefficients and the free energy

The Voros coefficient V used in the exact WKB analysis (whose definition will be given later) and $F = \sum F_g \hbar^{2g-2}$ are related as follows:

$$V(\lambda,\nu;\hbar) = F\left(\lambda - \frac{(\nu-1)\hbar}{2};\hbar\right) - F\left(\lambda - \frac{(\nu+1)\hbar}{2};\hbar\right) - \frac{1}{\hbar}\frac{\partial F_0}{\partial \lambda} + \frac{\nu}{2}\frac{\partial^2 F_0}{\partial \lambda^2}.$$

Introduction, III – case of the Weber equation –

Furthermore, we get the explicit forms of free energies and the Voros coefficient.

Explicit forms of free energies (cf. [HZ])

$$F_g(\lambda) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{\lambda^{2g-2}} \quad (g \geq 2),$$

where B_m designates the m-th Bernoulli number defined by

$$\frac{w}{e^w - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} w^m.$$

Explicit form of the Voros coefficient

$$V(\lambda, \nu; \hbar) = \sum_{m=1}^{\infty} \frac{B_{m+1}\left((\nu+1)/2\right)}{m(m+1)} \left(\frac{\hbar}{\lambda}\right)^m,$$

where $B_m(X)$ designates the m-th Bernoulli polynomial defined by

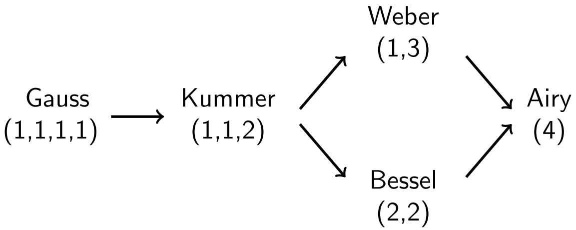
$$\frac{we^{Xw}}{e^w - 1} = \sum_{m=0}^{\infty} \frac{B_m(X)}{m!} w^m.$$

Introduction, IV

Theorem (Iwaki - Koike - T.)

For the family of the Gauss hypergeometric differential equations, we obtain the above results.

- The confluence diagram for the family of the Gauss hypergeometric differential equations

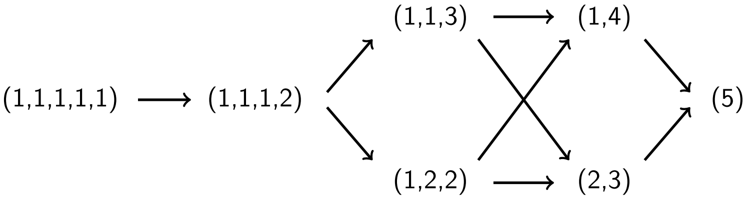


Purpose

Purpose of this talk

To generalize these results to the following hypergeometric differential equations.

- The confluence diagram for the family of the hypergeometric differential systems associated with the 2-dimensional Garnier system ([OK])

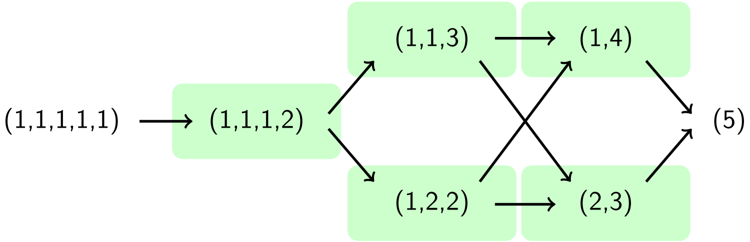


Purpose

Purpose of this talk

To generalize these results to the following hypergeometric differential equations.

- The confluence diagram for the family of the hypergeometric differential systems associated with the 2-dimensional Garnier system ([OK])



(1,1,1,2) equation

The (1,1,1,2) hypergeometric system with a small parameter $\hbar > 0$:

$$\Big[x_1 x_2 \hbar^3 \frac{\partial^3}{\partial {x_1}^3} - \{x_1 + (x_1 - \lambda_0 - \lambda_1 - \lambda_2)x_2\}\,\hbar^2 \frac{\partial^2}{\partial {x_1}^2} + \{x_1 - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)x_2\}\,\hbar \frac{\partial}{\partial x_1} + \lambda_0\Big]\psi = 0, \tag{2}$$

$$\left\{\left(1 - x_2 \hbar \frac{\partial}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \lambda_2 \frac{\partial}{\partial x_1}\right\}\psi = 0.$$

In what follows, setting $x_1 = x$ and $x_2 = t$ (fixed), we consider

$$\Big[tx\hbar^3 \frac{d^3}{dx^3} - \{x + (x - \lambda_0 - \lambda_1 - \lambda_2)t\}\,\hbar^2 \frac{d^2}{dx^2} + \{x - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)t\}\,\hbar \frac{d}{dx} + \lambda_0\Big]\psi = 0. \tag{3}$$

We call (3) the (1,1,1,2) equation.

1. Introduction
2. Exact WKB analysis
3. Topological recursion
4. Main results

Exact WKB analysis, I

Let us consider the following differential equation with a small parameter $\hbar$

$$P\left(x, \hbar\frac{d}{dx}\right)\psi = \left[p_0(x)\hbar^3\frac{d^3}{dx^3} + p_1(x)\hbar^2\frac{d^2}{dx^2} + p_2(x)\hbar\frac{d}{dx} + p_3(x)\right]\psi = 0 \tag{4}$$

and its WKB solutions

$$\psi(x,\hbar) = \exp\left(\int^x S(x,\hbar)\,dx\right), \tag{5}$$

where

$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{j \geqq -1} \hbar^j S_j(x) \tag{6}$$

is a solution of

$$\begin{aligned} &p_0(x)\hbar^3\left(\frac{d^2}{dx^2}S(x,\hbar) + 3S(x,\hbar)\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^3\right) \\ &+ p_1(x)\hbar^2\left(\frac{d}{dx}S(x,\hbar) + S(x,\hbar)^2\right) + \hbar p_2(x)S(x,\hbar) + p_3(x) = 0. \end{aligned} \tag{7}$$

Exact WKB analysis, II

By substituting (6) into (7) and comparing like powers of both sides with respect to $\hbar$, we obtain

$$p_0(x)S_{-1}^3 + p_1(x)S_{-1}^2 + p_2(x)S_{-1} + p_3(x) = 0 \tag{8}$$

and

$$\begin{aligned}\left(3p_0(x){S_{-1}}^2 + 2p_1(x)S_{-1} + p_2(x)\right) S_{m+1} + \sum_{\substack{i+j+k=m-1\\ i,j,k\geq 0}} S_iS_jS_k + 3\sum_{j=0}^{m-1} S_{m-j-1}S_j \\ +3p_0(x)S_m\frac{dS_{-1}}{dx} + 3p_0(x)S_{-1}\frac{dS_m}{dx} + p_0(x)\frac{d^2S_{m-1}}{dx^2} + p_1(x)\sum_{j=0}^{m} S_{m-j}S_j \\ +p_1(x)\frac{dS_m}{dx} = 0 \quad (m \geq -1).\end{aligned} \tag{9}$$

Eq. (8) has three solutions, and once we fix one of them, we can determine S_m for $m \geq 0$ uniquely and recursively by (9).

Voros coefficients

Then, the Voros coefficient is defined by

$$
\begin{aligned}
V \text{``}=\text{''} & \int_\gamma S(x,\hbar)dx \\
&= \int_\gamma \left(S(x,\hbar) - \hbar^{-1}S_{-1}(x) - S_0(x)\right)dx = \sum_{m=1}^{\infty} \hbar^m \int_\gamma S_m(x)dx,
\end{aligned} \tag{10}
$$

where γ is a path from a singular point to a singular point.

Remark: The Voros coefficient is an important ingredient to describe the global behavior of Borel resummed WKB solutions (5).

Spectral curve

$$\left[tx\hbar^3\frac{d^3}{dx^3} - \{x + (x - \lambda_0 - \lambda_1 - \lambda_2)t\}\,\hbar^2\frac{d^2}{dx^2} + \{x - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)t\}\,\hbar\frac{d}{dx} + \lambda_0\right]\psi = 0.$$

: The (1,1,1,2) equation

Spectral curve

$$\left[tx\hbar^3\frac{d^3}{dx^3} - \{x + (x - \lambda_0 - \lambda_1 - \lambda_2)t\}\,\hbar^2\frac{d^2}{dx^2}\right.$$

$$\left. + \{x - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)t\}\,\hbar\frac{d}{dx} + \lambda_0\right]\psi = 0.$$: The (1,1,1,2) equation

$$\Big\downarrow\ \hbar\frac{d}{dx} \to y,\ \hbar \to 0$$

$$P(x,y) = txy^3 - \{x + (x - \lambda_0 - \lambda_1 - \lambda_2)t\}\,y^2 + \{x - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)t\}\,y + \lambda_0 = 0. \tag{11}$$

Spectral curve

$$\left[tx\hbar^3\frac{d^3}{dx^3} - \{x + (x - \lambda_0 - \lambda_1 - \lambda_2)t\}\,\hbar^2\frac{d^2}{dx^2} + \{x - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)t\}\,\hbar\frac{d}{dx} + \lambda_0\right]\psi = 0.$$

: The (1,1,1,2) equation

Let us consider the following algebraic curve

$$P(x,y) = txy^3 - \{x + (x - \lambda_0 - \lambda_1 - \lambda_2)t\}\,y^2 + \{x - \lambda_0 - \lambda_1 - (\lambda_0 + \lambda_2)t\}\,y + \lambda_0 = 0. \qquad (11)$$

For $z \in \mathbb{P}^1$ we choose

$$\begin{cases} x = x(z) = \dfrac{-(\lambda_0 + \lambda_1 + \lambda_2)tz^2 + \{\lambda_0 + \lambda_1 + (\lambda_0 + \lambda_2)t\}z - \lambda_0}{z(z-1)(tz-1)}, \\ y = y(z) = z. \end{cases} \qquad (12)$$

We call a pair $(x(z), y(z))$ a spectral curve and (12) the (1,1,1,2) curve.

Topological recursion (cf. [EO1])

Let $(x(z), y(z))$ be a spectral curve. We first define

$$W_{0,1}(z) = y(z)\frac{dx}{dz}(z)dz, \quad W_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1 - z_2)^2}.$$

For $g \geqq 0$, $n \geqq 0$ and $2g - 2 + n \geqq 0$, we construct meromorphic differentials $W_{g,n}(z_1, \ldots, z_n)$ on $(\mathbb{P}^1)^n$ by the following recursive formulas.

$$W_{g,n+1}(z_0, z_1, \ldots, z_n) = \sum_{a\,:\ \text{branch point}} \operatorname*{Res}_{z=a} \frac{\left(\frac{1}{z_0 - z}\right) dz_0}{(y(z) - y(\bar{z}))\, dx(z)}$$

$$\times \left\{ W_{g-1,n+2}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1,2,\ldots,n\}}}' W_{g_1,1+|I|}(z, z_I) W_{g_2,1+|J|}(\bar{z}, z_J) \right\}.$$

- Branch points are zeros of $dx(z)$ (assume that all branch points are simple);
- $\bar{z}$ is a local conjugate point of z near a branch point (i.e. $x(\bar{z}) = x(z)$);
- $z_I = (z_{i_1}, \ldots, z_{i_r})$ for $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$.

Free energy

We define $F_g = W_{g,0}$, called free energies, by the following ([EO1], [CEO]):

$$F_g = \frac{1}{2-2g} \sum_{a\,:\ \text{branch point}} \operatorname*{Res}_{z=a} \Phi(z) W_{g,1}(z) \quad (g \geqq 2),$$

where $\Phi(z)$ is any function satisfying $\frac{d\Phi}{dz}(z) = y(z)\,\frac{dx}{dz}(z)$.

Variational formula (cf. [EO2])

From the variational formula, $W_{g,n}(z_1, \ldots, z_n)$ and F_g satisfy

$$\frac{\partial W_{g,n}}{\partial \lambda_j}(z_1, \ldots, z_n) = \int_{\zeta \in \gamma_j} W_{g,n+1}(z_1, \ldots, z_n, \zeta) \quad (2g + n \geq 2), \tag{13}$$

$$\frac{\partial F_g}{\partial \lambda_j} = \int_{\zeta \in \gamma_j} W_{g,1}(\zeta) \quad (g \geq 1), \tag{14}$$

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} W_{g,1}(z) \quad (g \geq 1). \tag{15}$$

Here, γ_j $(j = 0, 1)$ is a path from $z = \infty$ to $z = j$ and γ_2 is a path from $z = \infty$ to $z = \frac{1}{t}$.

x	0_2	∞_0	∞_1	∞_2
z	∞	0	1	$\frac{1}{t}$

Table: Correspondence of points for the (1,1,1,2) curve

Quantization ([BE])

We define

$$\psi(x,\hbar) = \exp\left[\hbar^{-1}\int_D W_{0,1}(z) + \frac{1}{2!}\int_D\int_D \left\{W_{0,2}(z_1,z_2) - \frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2}\right\}\right.$$
$$\left.\left. + \sum_{m=1}^{\infty}\hbar^m \left\{\sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \frac{1}{n!}\int_D\cdots\int_D W_{g,n}(z_1,\ldots,z_n)\right\}\right]\right|_{z=z(x)}, \qquad (16)$$

where $z = z(x)$ is an inverse function of $x = x(z)$ and

$$\int_D = \nu_0\int_0^z + \nu_1\int_1^z + \nu_2\int_{1/t}^z \qquad (\nu_0+\nu_1+\nu_2 = 1).$$

Then, $\psi(x,\hbar)$ is a WKB solution of

$$\left[tx\hbar^3\frac{d^3}{dx^3} - \{x + (x - \tilde{\lambda}_0 - \tilde{\lambda}_1 - \tilde{\lambda}_2 - 3\hbar)t\}\,\hbar^2\frac{d^2}{dx^2}\right.$$
$$\left. + \{x - \tilde{\lambda}_0 - \tilde{\lambda}_1 - 2\hbar - (\tilde{\lambda}_0 + \tilde{\lambda}_2 + 2\hbar)t\}\,\hbar\frac{d}{dx} + \tilde{\lambda}_0 + \hbar\right]\psi = 0, \qquad (17)$$

where $\tilde{\lambda}_j = \lambda_j - \nu_j\hbar$ $(j = 0,1,2)$. We call (17) the quantum (1,1,1,2) curve.

Theorem 1

Let $F_g(\lambda_0, \lambda_1, \lambda_2, t)$ be free energies for the spectral (1,1,1,2) curve and

$$F(\underline{\lambda}, t; \hbar) = F(\lambda_0, \lambda_1, \lambda_2, t; \hbar) = \sum_{g=0}^{\infty} F_g(\lambda_0, \lambda_1, \lambda_2, t)\hbar^{2g-2}$$

be the generating function of $F_g(\lambda_0, \lambda_1, \lambda_2, t)$. Then, we obtain

$$\begin{aligned} V^{(0,\infty)} &= F\left(\tilde{\lambda}_0 + \hbar, \tilde{\lambda}_1, \tilde{\lambda}_2, t; \hbar\right) - F\left(\tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, t; \hbar\right) \\ &\quad - \frac{\partial F_0}{\partial \lambda_0}\hbar^{-1} + \nu_0\nu_1 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} + \nu_0\nu_2 \frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_2} + \frac{2\nu_0 - 1}{2} \frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \end{aligned}$$

where $V^{(j,\infty)}$ are Voros coefficients for the quantum (1,1,1,2) curve whose path is from 0_2 to ∞_j ($j = 0, 1, 2$).

x	0_2	∞_0	∞_1	∞_2
z	∞	0	1	$\frac{1}{t}$

The other Voros coefficients $V^{(1,\infty)}$ and $V^{(2,\infty)}$ can be expressed similarly.

Sketch of Proof of Theorem 1, I

We can express $V^{(0,\infty)}$ in terms of $W_{g,n}$ as follows:

$$
\begin{aligned}
V^{(0,\infty)} &= \sum_{m=1}^{\infty} \hbar^m \int_{\infty}^{0} \left\{ \sum_{2g+n-2=m} \frac{1}{n!} \frac{d}{dz} \int_D \cdots \int_D W_{g,n}(z_1, \ldots, z_n) \right\} dz \\
&= \sum_{m=1}^{\infty} \hbar^m \sum_{2g+n-2=m} \frac{1}{n!} \left\{ \left((1-\nu_0) \int_{\gamma_0} - \nu_1 \int_{\gamma_1} - \nu_2 \int_{\gamma_2} \right)^n W_{g,n} \right. \\
&\qquad \left. - \left(-\nu_0 \int_{\gamma_0} - \nu_1 \int_{\gamma_1} - \nu_2 \int_{\gamma_2} \right)^n W_{g,n} \right\} \\
&= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g \geqq 0, n \geqq 1}} \left[\sum_{k_0+k_1+k_2=n} \frac{\left\{ (1-\nu_0)^{k_0} - (-\nu_0)^{k_0} \right\} (-\nu_1)^{k_1} (-\nu_2)^{k_2}}{k_0! k_1! k_2!} \right. \\
&\qquad \left. \times \left(\int_{\gamma_0} \right)^{k_0} \left(\int_{\gamma_1} \right)^{k_1} \left(\int_{\gamma_2} \right)^{k_2} W_{g,n} \right],
\end{aligned}
$$

where we use the notation $\left(\int_\gamma \right)^n W_{g,n} = \int_{\zeta_1 \in \gamma} \cdots \int_{\zeta_n \in \gamma} W_{g,n}(\zeta_1, \ldots, \zeta_n)$.

Sketch of Proof of Theorem 1, II

$$
\begin{aligned}
&V^{(0,\infty)} \\
&= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g\geqq 0, n\geqq 1}} \left[\sum_{k_0+k_1+k_2=n} \frac{\left\{(1-\nu_0)^{k_0} - (-\nu_0)^{k_0}\right\}(-\nu_1)^{k_1}(-\nu_2)^{k_2}}{k_0!k_1!k_2!} \left(\int_{\gamma_0}\right)^{k_0} \left(\int_{\gamma_1}\right)^{k_1} \left(\int_{\gamma_2}\right)^{k_2} W_{g,n} \right] \\
&= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g+n-2=m \\ g\geqq 0, n\geqq 1}} \left[\sum_{k_0+k_1+k_2=n} \frac{\left\{(1-\nu_0)^{k_0} - (-\nu_0)^{k_0}\right\}(-\nu_1)^{k_1}(-\nu_2)^{k_2}}{k_0!k_1!k_2!} \frac{\partial^n}{\partial\lambda_0^{k_0}\partial\lambda_1^{k_1}\partial\lambda_2^{k_2}} \left\{ \sum_{g\geq 0} \hbar^{2g-2} F_g \right\} \right] \\
&\quad - \frac{1}{\hbar}\frac{\partial F_0}{\partial\lambda_0} + \nu_0\nu_1 \frac{\partial^2 F_0}{\partial\lambda_0\partial\lambda_1} + \nu_0\nu_2 \frac{\partial^2 F_0}{\partial\lambda_0\partial\lambda_2} + \frac{2\nu_0-1}{2}\frac{\partial^2 F_0}{\partial{\lambda_0}^2} \\
&= F\left(\tilde\lambda_0+\hbar, \tilde\lambda_1, \tilde\lambda_2, t; \hbar\right) - F\left(\tilde\lambda_0, \tilde\lambda_1, \tilde\lambda_2, t; \hbar\right) - \frac{\partial F_0}{\partial\lambda_0}\hbar^{-1} + \nu_0\nu_1 \frac{\partial^2 F_0}{\partial\lambda_0\partial\lambda_1} + \nu_0\nu_2 \frac{\partial^2 F_0}{\partial\lambda_0\partial\lambda_2} + \frac{2\nu_0-1}{2}\frac{\partial^2 F_0}{\partial{\lambda_0}^2},
\end{aligned}
$$

where we use

$$
\frac{\partial W_{g,n}}{\partial\lambda_j}(z_1,\ldots,z_n) = \int_{\zeta\in\gamma_j} W_{g,n+1}(z_1,\ldots,z_n,\zeta) \quad (2g+n\geq 2), \tag{13}
$$

$$
\frac{\partial F_g}{\partial\lambda_j} = \int_{\zeta\in\gamma_j} W_{g,1}(\zeta) \quad (g\geq 1). \tag{14}
$$

Explicit form of free energies, I

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \quad (18)$$

$$F(\lambda_0,\lambda_1+\hbar,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_1}^2}, \quad (19)$$

$$F(\lambda_0,\lambda_1,\lambda_2+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1,\lambda_2-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_2}^2}. \quad (20)$$

(18) is rewritten as

$$\left\{e^{\hbar\frac{\partial}{\partial\lambda_0}}-2+e^{-\hbar\frac{\partial}{\partial\lambda_0}}\right\}F(\underline{\lambda},t,\hbar)=e^{-\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^2F(\underline{\lambda},t,\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

Explicit form of free energies, I

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \quad (18)$$

$$F(\lambda_0,\lambda_1+\hbar,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_1}^2}, \quad (19)$$

$$F(\lambda_0,\lambda_1,\lambda_2+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1,\lambda_2-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_2}^2}. \quad (20)$$

(18) is rewritten as

$$\left\{e^{\hbar\frac{\partial}{\partial\lambda_0}}-2+e^{-\hbar\frac{\partial}{\partial\lambda_0}}\right\}F(\underline{\lambda},t,\hbar)=e^{-\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^2F(\underline{\lambda},t,\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

Explicit form of free energies, I

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \quad (18)$$

$$F(\lambda_0,\lambda_1+\hbar,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_1}^2}, \quad (19)$$

$$F(\lambda_0,\lambda_1,\lambda_2+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1,\lambda_2-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_2}^2}. \quad (20)$$

(18) is rewritten as

$$F(\underline{\lambda},t,\hbar)=e^{\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^{-2}\frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

$$\frac{e^w}{(e^w-1)^2}=\frac{1}{w^2}-\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g-2)!}w^{2g-2} \qquad \downarrow \qquad \frac{\partial^2 F_0}{\partial {\lambda_0}^2}=\log\lambda_0-\log(\lambda_0+\lambda_1+\lambda_2)$$

$$F(\underline{\lambda},t,\hbar)=F_0(\underline{\lambda},t)\hbar^{-2}+F_1(\underline{\lambda},t)$$
$$+\sum_{g=2}^{\infty}\left[\frac{B_{2g}}{2g(2g-2)}\left\{{\lambda_0}^{2-2g}-(\lambda_0+\lambda_1+\lambda_2)^{2-2g}\right\}+\tilde{G}(\lambda_1,\lambda_2,t)\right]\hbar^{2g-2},$$

Explicit form of free energies, I

From Theorem 1 and contiguity relations we obtain

$$F(\lambda_0+\hbar,\lambda_1,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0-\hbar,\lambda_1,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_0}^2}, \quad (18)$$

$$F(\lambda_0,\lambda_1+\hbar,\lambda_2,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1-\hbar,\lambda_2,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_1}^2}, \quad (19)$$

$$F(\lambda_0,\lambda_1,\lambda_2+\hbar,t;\hbar)-2F(\lambda_0,\lambda_1,\lambda_2,t;\hbar)+F(\lambda_0,\lambda_1,\lambda_2-\hbar,t;\hbar)=\frac{\partial^2 F_0}{\partial {\lambda_2}^2}. \quad (20)$$

(18) is rewritten as

$$F(\underline{\lambda},t,\hbar)=e^{\hbar\frac{\partial}{\partial\lambda_0}}\left(e^{\hbar\frac{\partial}{\partial\lambda_0}}-1\right)^{-2}\frac{\partial^2 F_0}{\partial {\lambda_0}^2}.$$

$$\frac{e^w}{(e^w-1)^2}=\frac{1}{w^2}-\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g-2)!}w^{2g-2}$$

$$\downarrow \quad \frac{\partial^2 F_0}{\partial {\lambda_0}^2}=\log\lambda_0-\log(\lambda_0+\lambda_1+\lambda_2)$$

$$F(\underline{\lambda},t,\hbar)=F_0(\underline{\lambda},t)\hbar^{-2}+F_1(\underline{\lambda},t)$$
$$+\sum_{g=2}^{\infty}\left[\frac{B_{2g}}{2g(2g-2)}\left\{{\lambda_0}^{2-2g}+{\lambda_1}^{2-2g}+{\lambda_2}^{2-2g}-(\lambda_0+\lambda_1+\lambda_2)^{2-2g}\right\}+G(t)\right]\hbar^{2g-2},$$

Explicit form of free energies, II

Then we obtain that for $g \geq 2$

$$F_g = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{{\lambda_0}^{2g-2}} + \frac{1}{{\lambda_1}^{2g-2}} + \frac{1}{{\lambda_2}^{2g-2}} + \frac{1}{(\lambda_0+\lambda_1+\lambda_2)^{2g-2}} \right\} + G(t),$$

where $G(t)$ is an unknown function. In the following, we will prove $G(t) = 0$.

By substituting $\nu_0 = \nu_1 = 0$ in (16), we get

$$\sum_{m=-1}^{\infty} \hbar^m \int^{x(z)} S_m dx = \sum_{m=-1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 1}} \frac{1}{n!} \int_{z_1=1/t}^{z} \cdots \int_{z_n=1/t}^{z} W_{g,n}(z_1,\dots,z_n) \right\}, \tag{21}$$

$$\sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z)$$
$$= \sum_{g\geq 0} \hbar^{2g-1} W_{g,1}(z) + \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=1/t}^{z} \cdots \int_{z_n=1/t}^{z} W_{g,n}(z,z_2,\dots,z_n). \tag{22}$$

Explicit form of free energies, III

$$\operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{m=-1}^{\infty} \hbar^m S_m(x(z))dx(z) = \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{g\geq 0} \hbar^{2g-1} W_{g,1}(z)$$
$$+ \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=1/t}^{z} \cdots \int_{z_n=1/t}^{z} W_{g,n}(z, z_2, \ldots, z_n). \tag{23}$$

Explicit form of free energies, III

$$\operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{m=-1}^{\infty} \hbar^m S_m(x(z))dx(z) = \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{g\geq 0} \hbar^{2g-1} W_{g,1}(z)$$
$$+ \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=1/t}^{z} \cdots \int_{z_n=1/t}^{z} W_{g,n}(z,z_2,\ldots,z_n). \tag{23}$$

From the variational formula

$$\frac{\partial F_g}{\partial t} = -\operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} W_{g,1}(z) \quad (g\geq 1), \tag{15}$$

we obtain

$$\sum_{g\geq 1} \hbar^{2g-1} \frac{\partial F_g}{\partial t} = \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \hbar^{-1} W_{0,1} - \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{m=-1}^{\infty} \hbar^m S_m(x(z))dx(z)$$
$$+ \operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{\substack{2g+n-2=m \\ g\geq 0,\, n\geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=1/t}^{z} \cdots \int_{z_n=1/t}^{z} W_{g,n}(z,z_2,\ldots,z_n). \tag{24}$$

Then, we compare the odd degree terms with respect to $\hbar$ of both-sides.

Explicit form of free energies, IV

From $S_m(x) \sim O(1/x^2)$ $(m \geq 1)$, we find that

$$\operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{m=1}^{\infty} \hbar^m S_m(x(z)) dx(z) = 0. \tag{25}$$

Because $W_{g,n}(z_1, z_2, \ldots, z_n)$ is holomorphic at $z_i = 1/t$ $(1 \leq i \leq n)$,

$$W_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{k_1,\ldots,k_n \geq 0} a_{k_1,\ldots,k_n} \left(z_1 - \frac{1}{t}\right)^{k_1} \cdots \left(z_n - \frac{1}{t}\right)^{k_n}, \tag{26}$$

$$\int_{1/t}^{z_2} \cdots \int_{1/t}^{z_n} W_{g,n}(z, z_2, \ldots, z_n)\bigg|_{z_2,\ldots z_n = z} = \left(z - \frac{1}{t}\right)^{n-1} \left\{a_{0,\ldots,0} + O\left(z - \frac{1}{t}\right)\right\}. \tag{27}$$

By using this we obtain

$$\operatorname*{Res}_{z=\frac{1}{t}} \frac{\lambda_2}{t(tz-1)} \sum_{\substack{2g+n-2=m \\ g \geq 0,\, n \geq 2}} \frac{\hbar^{2g+n-2}}{(n-1)!} \int_{z_2=1/t}^{z} \cdots \int_{z_n=1/t}^{z} W_{g,n}(z, z_2, \ldots, z_n) = 0. \tag{28}$$

Therefore, $\dfrac{\partial F_g}{\partial t} = \dfrac{\partial G}{\partial t} = 0$ holds for $g \geq 1$.

Explicit forms of Voros coefficients

Using the explicit form of free energies

$$F_g = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{{\lambda_0}^{2g-2}} + \frac{1}{{\lambda_1}^{2g-2}} + \frac{1}{{\lambda_2}^{2g-2}} + \frac{1}{(\lambda_0+\lambda_1+\lambda_2)^{2g-2}} \right\} \tag{29}$$

and Theorem 1, we get the explicit forms of the Voros coefficients.

Explicit form of the Voros coefficient $V^{(0,\infty)}(\lambda_0, \lambda_1, \lambda_2, \nu_0, \nu_1, \nu_2; \hbar)$

$$V^{(0,\infty)} = \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \left\{ \frac{B_{m+1}(\nu_0)}{{\lambda_0}^m} - \frac{(-1)^{m+1} B_{m+1}}{(\lambda_0+\lambda_1+\lambda_2)^m} \right\}. \tag{30}$$

Other Voros coefficients $V^{(1,\infty)}$ and $V^{(2,\infty)}$ can be expressed similarly.

Summary

For (confluent) hypergeometric differential equations of second and third order, we obtain the following results:

- Voros coefficients are expressed as the difference values of the generating function of the free energies with respect to parameters. $\longrightarrow$ It means that the Voros coefficients are controlled by the free energy, in other words, the free energy is more essential quantity.
- As its applications, we get the explicit forms of the Voros coefficients and free energies.

Summary

For (confluent) hypergeometric differential equations of second and third order, we obtain the following results:

- Voros coefficients are expressed as the difference values of the generating function of the free energies with respect to parameters.
  $\longrightarrow$ It means that the Voros coefficients are controlled by the free energy, in other words, the free energy is more essential quantity.

- As its applications, we get the explicit forms of the Voros coefficients and free energies.

Thank you for your attention !

References I

[ATT] T. Aoki, T. Takahashi and M. Tanda, Borel sums of Voros coefficients of Gauss' hypergeometric differential equations with a large parameter and confluence, to appear in RIMS Kôkyûroku Bessatsu.

[BE] V. Bouchard and B. Eynard, Reconstructing WKB from topological recursion, *Journal de l'Ecole polytechnique – Mathematiques*, **4** (2017), 845–908.

[CEO] L. Chekhov, B.Eynard and N. Orantin, Free energy topological expansion for the 2-matrix model, *JHEP12* (2006), 053.

[DM] O. Dumitrescu and M. Mulase, Quantum curves for Hitchin fibrations and the Eynard-Orantin theory, Lett. Math. Phys., **104** (2014), 635–671.

[EO1] B. Eynard and N. Orantin, Invariants of algebraic curves and topological expansion, Comm. in Number Theory and Phys., **1** (2007), 347–452.

[EO2] B. Eynard and N. Orantin, Algebraic methods in random matrices and enumerative geometry, arXiv:0811.3531 [math-ph].

[HZ] J. Harer and D. Zagier : The Euler characteristic of the moduli space of curves, Invent. Math., **85** (1986), 457–485.

References II

[IKoT] K. Iwaki, T. Koike and Y.-M. Takei, Voros coefficients for the hypergeometric differential equations and Eynard-Orantin's topological recursion – Part I : For the Weber Equation –, *Annales Henri Poincaré* **24** (2023), 1305–1353.

[IKoT2] K. Iwaki, T. Koike and Y.-M. Takei, Voros coefficients for the hypergeometric differential equations and Eynard-Orantin's topological recursion – Part II : For Confluent Family of Hypegeometric Equations –, Journal of Integrable Systems **3** (2019), 1–46.

[OK] K. Okamoto and H. Kimura : On particular solutions of the Garnier systems and the hypergeometric functions of several variables, Quarterly J. Math., 37 (1986), 61–80.

[T] Y.-M.Takei, Voros coefficients for a class of the hypergeometric differential equations associated with the degeneration of the 2-dimensional Garnier system and the topological recursion; arXiv:2005.08957.