An elementary approach of the connection formula for WKB solutions to the Pearcey system with a large parameter

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1. Introduction

The Pearcey integral with a positive large parameter η:

(1.1) $$v(x_1, x_2; \eta) = \int \exp\left\{\eta\left(t^4 + x_2 t^2 + x_1 t\right)\right\} dt.$$

Here the path of integration is taken as an infinite curve connecting distinct two valleys of the integrand.

- There are three independent such paths and hence (1.1) yields three linearly independent entire functions of the variable $x = (x_1, x_2) \in \mathbb{C}^2$.
- The integral is often used in the wave propagation theory and diffraction problems, especially in a model of diffraction effects at a cusp caustic.

It is easy to see that the Pearcey integral

$$v(x_1, x_2; \eta) = \int \exp\left\{\eta\left(t^4 + x_2 t^2 + x_1 t\right)\right\} dt.$$

is a solution to

$$\text{(1.2)} \qquad \begin{cases} Q_1 v := \left(4\partial_1^3 + 2x_2\eta^2\partial_1 + x_1\eta^3\right) v = 0, \\ Q_2 v := \left(\eta\partial_2 - \partial_1^2\right) v = 0. \end{cases}$$

Here we set $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$.

- This is a holonomic system of rank 3 in $\mathbb{C}^2$ and the Pearcey integral gives a basis of the analytic solution space.
- WKB solutions $\exp\left(\int \omega\right)$ to this system can be considered by Aoki [A] and by Hirose [H]. Here $\omega = S^{(1)}dx_1 + S^{(2)}dx_2$.
- Connection problems for WKB solutions are studied intensively by Hirose [H] and by Honda-Kawai-Takei [HKT].

We study (1.2) from a viewpoint slightly different from that of [H] and [HKT].

Let us consider η as an independent variable, not a large parameter. Then the Pearcey integral satisfies not only (1.2) but

$$Q_3 v := (3x_1\partial_1 + 2x_2\partial_2 - 4\eta\partial_\eta - 1)\, v = 0 \quad \left(\partial_\eta = \frac{\partial}{\partial\eta}\right).$$

This equation comes from the weighted homogeneity of the Pearcey integral with respect to (x_1, x_2, η) and it gives natural primitives $\displaystyle\int \omega$. Hence we may consider WKB solutions

$$\psi = \eta^{-1/2} \exp\left(\int \omega\right)$$

by using such primitives.

In this talk, we will show

- The Borel transform ψ_B of thus constructed WKB solutions ψ are algebraic, hence ψ is resurgent.
- The Stokes set of the holonomic system (1.2) is semialgebraic.
- The singularity structure of ψ_B can be analyzed by using algebraic functions.

2. The Pearcey system with a large paremeter

We start from the system of differential equations ([OK]):

$$
\begin{cases} P_1\psi = 0, \\ P_2\psi = 0, \\ P_3\psi = 0 \end{cases} \tag{2.1}
$$

with

$$
\begin{aligned}
P_1 &= 4\partial_1\partial_2 + 2\eta x_2\partial_1 + \eta^2 x_1, \\
P_2 &= 4\partial_2^2 + \eta x_1\partial_1 + 2\eta x_2\partial_2 + \eta, \\
P_3 &= \eta\partial_2 - \partial_1^2 (= Q_2).
\end{aligned}
$$

This is equivalent to (1.2) because

$$
\begin{aligned}
P_1 &= \eta^{-1}(Q_1 + 4\partial_1 Q_2), \\
P_2 &= \eta^{-2}\partial_1 Q_1 + (4\eta^{-2}Q_2 + 8\eta^{-2}\partial_1^2 + 2x_2)Q_2, \\
Q_1 &= \eta P_1 - 4\partial_1 P_3.
\end{aligned}
$$

Note that

$$
P_2 = \eta^{-1}\partial_1 P_1 + 2(2\eta^{-1}\partial_2 + x_2)P_3.
$$

Next we consider η as an independent complex variable.

- Then the systems (1.2) and (2.1) are subholonomic.
- To get a holonomic system, we add the following equation to (2.1):

$$(3x_1\partial_1 + 2x_2\partial_2 - 4\eta\partial_\eta - 1)\,\psi = 0 \quad (Q_3\psi = 0).$$

- This comes from the weighted homogeneity of the Pearcey integral with respect to (x_1, x_2, η).

We set

$$P_4 = 3x_1\partial_1 + 2x_2\partial_2 - 4\eta\partial_\eta - 1 \ (= Q_3).$$

Let D be the Weyl algebra of the variables (x_1, x_2, η) and I the left ideal in D generated by P_i $(i = 1, 2, 3, 4)$. We denote by M the left D-module defined by I, that is,

$$M = D/I.$$

We call M the Pearcey system with a large parameter.

The proof of Theorem 2.1 follows Oaku's work ([Oaku]).

Theorem 2.1

Let I be the left ideal of D generated by P_j $(j = 1, 2, 3, 4)$ and M the left D-module defined by I:

$$M = D/I.$$

Then M is a holonomic system of rank 3.

- The system M characterizes the 3-dimensional linear subspace spanned by the Pearcey integral in the space of analytic functions.
- There are four valleys of the Pearcey integral and hence we have six infinite paths of integration connecting distinct two valleys. Any three of them are independent, which give a basis of the solution space.

3. WKB solutions

We construct WKB solutions to M. The logarithmic derivatives of the unknown function with respect to x_1 and x_2 are denoted respectively by $S^{(1)}$ and $S^{(2)}$ ([A], [H]):

$$S^{(1)} = \frac{\partial_1 \psi}{\psi}, \quad S^{(2)} = \frac{\partial_2 \psi}{\psi}.$$

We can find $S^{(1)}$ and $S^{(2)}$ by using $Q_j \psi = 0$ $(j = 1, 2)$, which are equivalent to $P_j \psi = 0$ $(j = 1, 2, 3)$:

(3.1)
$$\begin{cases} 4(S^{(1)})^3 + 2\eta^2 x_2 S^{(1)} + \eta^3 x_1 + 12 S^{(1)} \partial_1 S^{(1)} + 4\partial_1^2 S^{(1)} = 0, \\ \eta S^{(2)} - \partial_1 S^{(1)} - (S^{(1)})^2 = 0. \end{cases}$$

We seek formal solutions of the forms

$$S^{(1)} = \sum_{k=-1}^{\infty} \eta^{-k} S_k^{(1)}, \quad S^{(2)} = \sum_{k=-1}^{\infty} \eta^{-k} S_k^{(2)}.$$

Leading terms:

(3.2)
$$4(S_{-1}^{(1)})^3 + 2x_2 S_{-1}^{(1)} + x_1 = 0,$$

(3.3)
$$S_{-1}^{(2)} = (S_{-1}^{(1)})^2.$$

Recurrence relations:

$$S_0^{(1)} = -\frac{1}{2}\partial_1 \log(6(S_{-1}^{(1)})^2 + x_2),$$

$$S_k^{(1)} = -\frac{2}{6(S_{-1}^{(1)})^2 + x_2} \left(\sum_{\substack{k_1+k_2+k_3=k-2 \\ -1\le k_1,\, k_2,\, k_3<k}} S_{k_1}^{(1)} S_{k_2}^{(1)} S_{k_3}^{(1)} \right.$$

$$\left. + 3 \sum_{\substack{k_1+\,k_2=k-2 \\ -1\le k_1,\, k_2<k}} S_{k_1}^{(1)} \partial_1 S_{k_2}^{(1)} + \partial_1^2 S_{k-2}^{(1)} \right) \quad (k \geq 1),$$

$$S_k^{(2)} = \partial_1 S_{k-1}^{(1)} + \sum_{j=-1}^{k} S_j^{(1)} S_{k-j-1}^{(1)} \quad (k \geq 0).$$

Lemma 3.1

Let $\omega = S^{(1)} dx_1 + S^{(2)} dx_2$ denote the 1-form of formal series defined by $S^{(1)}$ and $S^{(2)}$ constructed as above. Then ω is closed.

In [A] and [H], a formal solution of the form $\exp\left(\int_{(a_1,a_2)}^{(x_1,x_2)} \omega\right)$ is called a WKB solution to (1.2). Here (a_1, a_2) is a suitably fixed point.

The above construction of $S^{(1)}$ and $S^{(2)}$ does not use

(3.4) $$P_4\psi = 0 \quad (Q_3\psi = 0).$$

That is, η is considered to be a parameter. Thus the WKB solutions $\exp\left(\int_{(a_1,a_2)}^{(x_1,x_2)} \omega\right)$ have ambiguity of multiplicative constants that may depend on η.

Next we take (3.4) into account. We consider a formal solution of the form

$$\eta^{-1/2} \exp\left(\int \omega\right).$$

Then (3.4) makes a constraint for the choice of the primitive $\int \omega$, namely,

(3.5) $$\begin{cases} \displaystyle\int \omega_0 = -\frac{1}{2}\log(6(S_{-1}^{(1)})^2 + x_2), \\ \displaystyle\int \omega_k = -\frac{1}{4k}(3x_1 S_k^{(1)} + 2x_2 S_k^{(2)}) \quad (k \neq 0) \end{cases}$$

up to genuine additive constants. Here we set $\omega = \sum_{j=-1}^{\infty} \eta^{-j}\omega_j$.

From now on, we consider the WKB solutions to M of the form

$$\psi = \eta^{-1/2} \exp\left(\int \omega\right)$$

with the primitive $\displaystyle\int \omega$ taken as (3.5). Explicitly,

$$\begin{aligned}\psi = \frac{1}{\left(\eta\left(6\left(S_{-1}^{(1)}\right)^2 + x_2\right)\right)^{1/2}} \exp\Big(\frac{\eta}{4}(3x_1 S_{-1}^{(1)} + 2x_2 S_{-1}^{(2)}) \\ - \sum_{k=1}^{\infty} \eta^{-k} \frac{1}{4k}(3x_1 S_k^{(1)} + 2x_2 S_k^{(2)})\Big).\end{aligned} \tag{3.6}$$

Let $S_{-1}^{(1),j}$ $(j = 1, 2, 3)$ denote the three roots of (3.2) and set $S_{-1}^{(2),j} = (S_{-1}^{(1),j})^2$. Accordingly, we have three formal solutions $(S^{(1),j}, S^{(2),j})$ $(j = 1, 2, 3)$ to (3.1).

Then we have three 1-forms $\omega^{(j)} = S^{(1),j} dx_1 + S^{(2),j} dx_2$ and WKB solutions ψ_j $(j = 1, 2, 3)$ of the form (3.6).

The branch of $S_{-1}^{(1),j}$ will be specified later.

The turning point set and the Stokes set of M are the same as those of (1.2) which are introduced by [A], [H]. Let $j, k \in \{1, 2, 3\}$ and $j \neq k$.

- A point $x = (x_1, x_2) \in \mathbb{C}^2$ is called a turning point of type (j, k) if

$$\omega_{-1}^{(j)} = \omega_{-1}^{(k)}$$

  holds. The turning point set T is the set of all turning points of some type. Hence it coincides with the zeros of the discriminant:

$$T = \{(x_1, x_2) \,|\, 27x_1^2 + 8x_2^3 = 0\,\}.$$

- The Stokes set $\mathcal{S}$ of the Pearcey system M is defined to be the union for all $j, k = 1, 2, 3; j \neq k$ of the sets

$$\left\{ x = (x_1, x_2) \in \mathbb{C}^2 \,\middle|\, \mathrm{Im} \int_\tau^x (\omega_{-1}^{(j)} - \omega_{-1}^{(k)}) = 0 \right\},$$

  where τ is a turning point of type (j, k). Note that we have to consider all of analytic continuation of $\displaystyle\int_\tau^x (\omega_{-1}^{(j)} - \omega_{-1}^{(k)})$ with respect to x.

Using the primitive $\int \omega_{-1}$ given by (3.5), we see

$$\int_{\tau}^{x} (\omega_{-1}^{(j)} - \omega_{-1}^{(k)}) = \frac{1}{4}(S_{-1}^{(1),j} - S_{-1}^{(1),k})(3x_1 + 2x_2(S_{-1}^{(1),j} + S_{-1}^{(1),k})) =: F(x_1, x_2),$$

where τ is a turning point of type (j,k). Since $S_{-1}^{(1),j}, S_{-1}^{(1),k}$ are roots of the cubic equation $4\zeta^3 + 2x_2\zeta + x_1 = 0$, F is an algebraic function. More explicitly, F is defined by

$$16F^6 + 32x_2\left(27x_1^2 - x_2^3\right)F^4 + 16x_2^2\left(27x_1^2 - x_2^3\right)^2 F^2 + x_1^2\left(27x_1^2 + 8x_2^3\right)^3 = 0.$$

Thus we have

Theorem 4.1

The Stokes set $\mathcal{S}$ of the Pearcey system M is described as

$$\mathcal{S} = \{(x_1, x_2) \in \mathbb{C}^2 \mid \operatorname{Im} F(x_1, x_2) = 0\}.$$

Hence it is a semialgebraic set as a subset of $\mathbb{C}^2 \simeq \mathbb{R}^4$.

- The set of "crossing points" of Stokes surfaces is also semialgebraic.
- We may draw the figure of (a section of) Stokes set without numerical integration.

5. Borel transform of WKB solutions

Let $\psi_{j,B}$ be the Borel transform of the WKB solution

$$\psi_j = \frac{1}{\left(\eta\left(6\left(S_{-1}^{(1),j}\right)^2 + x_2\right)\right)^{1/2}} \exp\Big(\frac{\eta}{4}(3x_1 S_{-1}^{(1),j} + 2x_2 S_{-1}^{(2),j}) - \sum_{k=1}^{\infty} \eta^{-k} \frac{1}{4k}(3x_1 S_k^{(1),j} + 2x_2 S_k^{(2),j})\Big).$$

for $j = 1, 2, 3$ and $P_{k,B}$ the formal Borel transform of P_k $(k = 1, 2, 3, 4)$. The explicit forms of $P_{k,B}$'s are given as follows:

$$\begin{aligned} P_{1,B} &= 4\partial_1\partial_2 + 2x_2\partial_y\partial_1 + x_1\partial_y^2, \\ P_{2,B} &= 4\partial_2^2 + x_1\partial_y\partial_1 + 2x_2\partial_y\partial_2 + \partial_y, \\ P_{3,B} &= \partial_y\partial_2 - \partial_1^2, \\ P_{4,B} &= 3x_1\partial_1 + 2x_2\partial_2 - 4\partial_y(-y) - 1 \\ &\quad (= 3x_1\partial_1 + 2x_2\partial_2 + 4y\partial_y + 3). \end{aligned}$$

Here y denotes the variable of the Borel plane. By the definition,

$$P_{k,B}\psi_{j,B} = 0$$

holds for $j = 1, 2, 3; k = 1, 2, 3, 4$.

Since $S_{-1}^{(1),j}\Big|_{x_2=0}$ is a root of the cubic equation $4\zeta^3 + x_1 = 0$ of ζ, we can specify the branch of $S_{-1}^{(1),j}$ and hence ψ_j by $S_{-1}^{(1),j}\Big|_{x_2=0} = -\dfrac{x_1^{1/3}}{4^{1/3}} e^{2\pi i j/3}$.

The Borel transform:

$$\exp(\eta\varpi_j) \sum_{\ell=0}^{\infty} \eta^{-\frac{1}{2}-\ell} f_{\ell,j}(x_1, x_2),$$

where $\varpi_j = \dfrac{1}{4}(3x_1 S_{-1}^{(1),j} + 2x_2 S_{-1}^{(2),j})$ and $f_{0,j}(x_1, x_2) = \Big(6(S_{-1}^{(1),j})^2 + x_2\Big)^{-\frac{1}{2}}$ $(j = 1, 2, 3)$.

$\psi_{j,B}$ has a singularity at $u_j := -\varpi_j$ and $u = u_j$ satisfies

$$256u^3 - 128x_2^2 u^2 + 16x_2(9x_1^2 + x_2^3)u - x_1^2(27x_1^2 + 4x_2^3) = 0 \tag{5.1}$$

$(j = 1, 2, 3)$ and the Stokes set is also expressed in the form

$$\bigcup_{j \neq k} \{(x_1, x_2) \,|\, \mathrm{Im}(u_j - u_k) = 0\,\}$$

in terms of the roots u_j $(j = 1, 2, 3)$ of (5.1).

Let D_B be the Weyl algebra of the variable (x_1, x_2, y) and I_B the left D_B-ideal generated by $P_{k,B}$ $(k = 1, 2, 3, 4)$.

Theorem 5.1

Let M_B denote the left D_B-module defined by I_B :

$$M_B = D_B / I_B.$$

Then M_B is a holonomic system of rank 3.

- P_j $(j = 1, 2, 3, 4)$ are obtained from Q_k $(k = 1, 2, 3)$ by applying the algorithm of Gröbner basis with respect the monomial order $\eta \succ x_1 \succ x_2$ and by dividing some powers of η.
- The system $Q_k \psi = 0$ $(k = 1, 2, 3)$ is holonomic of rank 3, however, the system $Q_{k,B} \varphi = 0$ $(k = 1, 2, 3)$ has rank 6 (pointed out by Hirose) whereas it is holonomic.
- In addition to 3 dimensional analytic solution space, it has 3 dimensional redundant solutions expressed in terms of the delta function:

$$\psi = c_0 x_2^{-3/2} \delta(\eta) + c_1 x_1 x_2^{-3} \delta(\eta) + c_2 \left(x_1^2 x_2^{-9/2} \delta(\eta) + \frac{4}{7} x_2^{-7/2} \delta'(\eta) \right),$$

  where c_0, c_1, c_2 are arbitrary constants.

Thus M_B characterizes the subspace of analytic functions spanned by $\psi_{j,B}$ $(j = 1, 2, 3)$.

We go back to the Pearcey integral

$$v = \int \exp\left\{\eta\left(t^4 + x_2 t^2 + x_1 t\right)\right\} dt$$

and rewrite the right-hand side by setting $t^4 + x_2 t^2 + x_1 t = -y$:

$$v = \int \exp(-\eta y) g(x_1, x_2, y) dy.$$

Here g is defined by

$$g(x_1, x_2, y) = \left.\frac{1}{4t^3 + 2x_2 t + x_1}\right|_{t=t(x_1,x_2)}.$$

The path of integration is suitably modified and $t = t(x_1, x_2)$ is a root of the quartic equation $t^4 + x_2 t^2 + x_1 t = -y$.

Lemma 5.2

The function g defined as above satisfies the quartic equation

$$(4x_1^2 x_2(36y - x_2^2) + 16y(x_2^2 - 4y)^2 - 27x_1^4)\, g^4 + 2(-8x_2 y + 2x_2^3 + 9x_1^2)\, g^2 - 8x_1 g + 1 = 0$$

and it is a solution to the holonomic system M_B.

For general (x_1, x_2, y), there are four roots g_k $(k = 1, 2, 3, 4)$ of the quartic equation, which satisfy $g_1 + g_2 + g_3 + g_4 = 0$. Looking at the singularity of g_k, we find that any three of g_k's are linearly independent. Thus we have

Theorem 5.3

The Borel transform $\psi_{j,B}$ of the WKB solution ψ_j $(j = 1, 2, 3)$ can be written as a linear combination of any three of g_k's. In particular, $\psi_{j,B}$'s are algebraic. Hence ψ_j's are Borel summable and resurgent.

We will see the explicit forms of $\psi_{j,B}$ in terms of g_k's.

Since $P_{4,B}\psi_{j,B} = 0$, $\psi_{j,B}$ has the weighted homogeneity

$$\psi_{j,B}(\lambda^3 x_1, \lambda^2 x_2, \lambda^4 y) = \lambda^{-3}\psi_{j,B}(x_1, x_2, y).$$

If $x_1 \neq 0$, we have

$$\psi_{j,B}\left(1, \frac{x_2}{x_1^{2/3}}, \frac{y}{x_1^{4/3}}\right) = x_1\psi_{j,B}(x_1, x_2, y).$$

We introduce new variables s, t by setting

$$s = \frac{y}{x_1^{4/3}}, \quad t = \frac{x_2}{x_1^{2/3}}.$$

Then $x_1\psi_{j,B}$ can be considered as a function of (s, t).

We set $p_\ell = 3/4^{4/3} e^{2\pi i \ell/3}$ ($\ell = 1, 2, 3$). Expansion of $x_1 \psi_{j,B}\big|_{t=0}$ at $s = p_j$:

$$x_1 \psi_{j,B}\big|_{t=0} = -\frac{4^{1/3}}{\sqrt{6\pi}} e^{-2\pi i j/3} (s - p_j)^{-1/2} \times \left(1 - \frac{7}{9 \cdot 2^{1/3}} e^{-2\pi i j/3} (s - p_j) + O\big((s - p_j)^2\big)\right).$$

The branch is chosen as $(s - p_j)^{1/2} > 0$ if $\mathrm{Im}(s - p_j) = 0$ and $s - p_j > 0$. The branch cut for the function $(s - p_j)^{1/2}$ is taken as a half line with the negative real direction starting at p_j and the argument is taken as

$$-\pi < \arg(s - p_j) \leq \pi$$

for general s. Hence we have

(5.2) $$(p_1 - s)^{-1/2} = e^{-\pi i/2} (s - p_1)^{-1/2} = -i (s - p_1)^{-1/2}$$

for $s = e^{2\pi i/3} \sigma$ ($0 < \sigma < p_3$),

(5.3) $$(p_2 - s)^{-1/2} = e^{\pi i/2} (s - p_2)^{-1/2} = i (s - p_2)^{-1/2}$$

for $s = e^{4\pi i/3} \sigma$ ($0 < \sigma < p_3$) and

(5.4) $$(p_3 - s)^{-1/2} = e^{\pi i/2} (s - p_3)^{-1/2} = i (s - p_3)^{-1/2}$$

for $0 < s < p_3$.

We recall that the algebraic function g is defined by

$$g(x_1, x_2, y) = \left. \frac{1}{4t^3 + 2x_2 t + x_1} \right|_{t=t(x_1,x_2)},$$

where $t = t(x_1, x_2)$ is a root of $t^4 + x_2 t^2 + x_1 t = y$. Since this has the same weighted homogeneity as $\psi_{j,B}$, we can regard $h = x_1 g$ as a function of (s, t). It follows from Lemma 5.2 that h is a root of

$$(256s^3 - 128s^2t^2 + 16st(t^3 + 9) - 4t^3 - 27)h^4 + (4t^3 - 16st + 18)h^2 - 8h + 1 = 0.$$

We specify the branches h_j $(j = 1, 2, 3, 4)$ of the algebraic function h near the origin by their local behaviors:

$$\begin{aligned}
h_1(s,t) &= \frac{1}{3} + \frac{4}{9} e^{-\frac{2\pi i}{3}} s + \frac{2}{9} e^{\frac{2\pi i}{3}} t + \cdots, \\
h_2(s,t) &= \frac{1}{3} + \frac{4}{9} e^{\frac{2\pi i}{3}} s + \frac{2}{9} e^{-\frac{2\pi i}{3}} t + \cdots, \\
h_3(s,t) &= \frac{1}{3} + \frac{4}{9} s + \frac{2}{9} t + \cdots, \\
h_4(s,t) &= -1 - 2st - 4s^3 + \cdots.
\end{aligned}$$

Now we specify the branches g_j of g by setting

$$g_j = h_j / x_1 \qquad (j = 1, 2, 3, 4).$$

Let us consider the restriction of h to $t = 0$. It satisfies

$$\left(256s^3 - 27\right) h^4 + 18h^2 - 8h + 1 = 0.$$

Here we also use h for $h(s, 0)$. Hence $h(s, 0)$ has a singularity at $s = p_\ell (= 3/4^{4/3} e^{2\pi i\ell/3})$ $(\ell = 1, 2, 3)$. Taking the local expansions of the roots at $s = p_\ell$, we can specify the branches $h_j^{(\ell)}(s, 0) (j = 1, 2, 3, 4)$ near $s = p_\ell$ as

$$\begin{aligned}
h_1^{(\ell)}(s, 0) &= \frac{1}{2^{5/6}\sqrt{3}} e^{-\frac{2\pi i}{3}\ell} (p_\ell - s)^{-1/2} + O(1), \\
h_2^{(\ell)}(s, 0) &= -\frac{1}{2^{5/6}\sqrt{3}} e^{-\frac{2\pi i}{3}\ell} (p_\ell - s)^{-1/2} + O(1), \\
h_3^{(\ell)}(s, 0) &= \frac{4 + i\sqrt{2}}{18} + O(p_\ell - s), \\
h_4^{(\ell)}(s, 0) &= \frac{4 - i\sqrt{2}}{18} + O(p_\ell - s).
\end{aligned}$$

Here the branches of the square roots are chosen as (5.2)–(5.4). Since h is holomorphic near $t = 0$, the branches $h_j^{(\ell)}(s, 0)$ given above also specifies the branches $h_j^{(\ell)}(s, t)$ of $h(s, t)$ near $s = p_\ell$ if $|t|$ is sufficiently small.

The following lemma shows how these branches are related.

Lemma 6.1

The branches $h_j^{(\ell)}$ $(j = 1, 2, 3, 4; \ell = 1, 2, 3)$ and h_j $(j = 1, 2, 3, 4)$ satisfy the relations

(6.1)
$$\begin{aligned}
h_1(s,0) &= h_4^{(3)}(s,0) = h_2^{(1)}(s,0) = h_3^{(2)}(s,0),\\
h_2(s,0) &= h_3^{(3)}(s,0) = h_4^{(1)}(s,0) = h_2^{(2)}(s,0),\\
h_3(s,0) &= h_1^{(3)}(s,0) = h_3^{(1)}(s,0) = h_4^{(2)}(s,0),\\
h_4(s,0) &= h_2^{(3)}(s,0) = h_1^{(1)}(s,0) = h_1^{(2)}(s,0)
\end{aligned}$$

for $|s| < p_3$.

7. Relationship between Borel tranform of WKB solutions and algebraic functions

Recall

$$\psi_j = \frac{1}{\left(\eta\left(6\left(S_{-1}^{(1),j}\right)^2 + x_2\right)\right)^{1/2}} \exp\Big(\frac{\eta}{4}(3x_1 S_{-1}^{(1),j} + 2x_2 S_{-1}^{(2),j})$$

$$- \sum_{k=1}^{\infty} \eta^{-k} \frac{1}{4k}(3x_1 S_k^{(1),j} + 2x_2 S_k^{(2),j})\Big).$$

This can be expanded in the form

$$\exp(\eta\varpi_j) \sum_{\ell=0}^{\infty} \eta^{-\frac{1}{2}-\ell} f_{\ell,j}(x_1, x_2),$$

where $\varpi_j = \dfrac{1}{4}(3x_1 S_{-1}^{(1),j} + 2x_2 S_{-1}^{(2),j})$ and $f_{0,j}(x_1, x_2) = \Big(6(S_{-1}^{(1),j})^2 + x_2\Big)^{-\frac{1}{2}}$ $(j = 1, 2, 3)$.

By using Theorem 5.3, we obtain

$$x_1\psi_{\ell,B} = \sum_{k=1}^{4} C_k^{(\ell)} h_k^{(\ell)}(s,t),$$

where $C_k^{(\ell)}$ $(k = 1, 2, 3, 4)$ is a constant independent of s and t. Comparing the coefficients for the power of $(s - p_\ell)$ on its both sides, we have

$$x_1\psi_{1,B}\big|_{t=0} = \frac{i}{\sqrt{\pi}}(h_2^{(1)} - h_1^{(1)}),$$

$$x_1\psi_{2,B}\big|_{t=0} = -\frac{i}{\sqrt{\pi}}(h_2^{(2)} - h_1^{(2)}),$$

$$x_1\psi_{3,B}\big|_{t=0} = -\frac{i}{\sqrt{\pi}}(h_2^{(3)} - h_1^{(3)}).$$

Consequently, we have

$$x_1\psi_{1,B}\big|_{t=0} = \frac{i}{\sqrt{\pi}}(h_1 - h_4),$$

$$x_1\psi_{2,B}\big|_{t=0} = -\frac{i}{\sqrt{\pi}}(h_2 - h_4),$$

$$x_1\psi_{3,B}\big|_{t=0} = \frac{i}{\sqrt{\pi}}(h_3 - h_4).$$

Since $x_1\psi_{j,B}$ are holomorphic at $t = 0$, we can obtain the expressions of it in terms of g_k's.

Theorem 7.1

Under the notation given above, if $|x_2|$ is sufficiently small, the Borel transform of the WKB solution ψ_k to the Pearcey system is expressed in the form

$$\psi_{k,B} = (-1)^{k-1} \frac{i}{\sqrt{\pi}} (g_k - g_4)$$

for $k = 1, 2, 3$. If we take a new branch cut of $\psi_{k,B}$ as the half-line starting from u_k with the positive real direction, the above relation is written in the form

$$\psi_{k,B} = \frac{i}{\sqrt{\pi}} \Delta_{u_k} g_4.$$

Here $\Delta_{u_k} g_4$ denotes the discontinuity of g_4 along the branch cut of $\psi_{k,B}$.

- $g = g_k$ $(k = 1, 2, 3, 4)$ are the roots of
$$(4x_1^2 x_2(36y - x_2^2) + 16y(x_2^2 - 4y)^2 - 27x_1^4)\, g^4 + 2(-8x_2 y + 2x_2^3 + 9x_1^2)\, g^2 - 8x_1 g + 1 = 0.$$
- The branch of g_k are specified by
$$x_1 g_1\big|_{t=0} = \frac{1}{3} + \frac{4}{9} e^{-2\pi i/3} s + O(s^2), \quad x_1 g_2\big|_{t=0} = \frac{1}{3} + \frac{4}{9} e^{2\pi i/3} s + O(s^2),$$
$$x_1 g_3\big|_{t=0} = \frac{1}{3} + \frac{4}{9} s + O(s^2), \quad x_1 g_4\big|_{t=0} = -1 + O(s).$$

  Here we set $s = \dfrac{y}{x_1^{4/3}}, \quad t = \dfrac{x_2}{x_1^{2/3}}.$

8. Connection formula

Using Theorem 7.1, we can take analytic continuation with respect to y of $\psi_{\ell,B}$ if $|x_2|$ is small ($\ell = 1, 2, 3$).

Recall: The singularity u_ℓ of $\psi_{\ell,B}$ is given by

$$u_\ell = -\frac{1}{4}(3x_1 S_{-1}^{(1),\ell} + 2x_2 S_{-1}^{(2),\ell})$$

and the branch of $S_{-1}^{(1),\ell}$ is specified by $S_{-1}^{(1),\ell}\Big|_{x_2=0} = -\dfrac{x_1^{1/3}}{4^{1/3}} e^{2\pi i \ell/3}$.
Since $h = x_1 g$ is holomorphic at $t = 0$, we can specify the branches g_j ($j = 1, 2, 3, 4$) of g at $x_2 = 0$ and $g_j^{(\ell)}$ at u_ℓ ($j = 1, 2, 3, 4; \ell = 1, 2, 3$) by setting

$$x_1 g_j\big|_{t=0} = h_j(s, 0), \quad x_1 g_j^{(\ell)}\big|_{t=0} = h_j^{(\ell)}(s, 0),$$

respectively ($s = y/x_1^{4/3},\ t = x_2/x_1^{2/3}$). Relations (6.1) yield the following relations for small $|x_2|$:

$$\text{(8.1)} \qquad \begin{cases} g_1 = g_4^{(3)} = g_2^{(1)} = g_3^{(2)}, \\ g_2 = g_3^{(3)} = g_4^{(1)} = g_2^{(2)}, \\ g_3 = g_1^{(3)} = g_3^{(1)} = g_4^{(2)}, \\ g_4 = g_2^{(3)} = g_1^{(1)} = g_1^{(2)}. \end{cases}$$

Notation:
Let f be an analytic function germ (possibly 2-valued) at $y = u_\ell$ (x_1, x_2 are fixed).

- $c^*_{\ell k} f$: the analytic continuation of f along the segment $u_\ell u_k$.
- $c^*_\ell f$: another branch of f if f has a square-root type singularity at $y = u_\ell$.
- $c^*_\ell f = f$ if f is holomorphic at $y = u_\ell$.

Using (8.1), we can take analytic continuation of $g_j^{(\ell)}$ to the possible singularity u_k:

$$\begin{cases} c^*_{12} g_1^{(1)} = g_1^{(2)}, \quad c^*_{13} g_1^{(1)} = g_2^{(3)}, \quad c^*_{23} g_1^{(2)} = g_2^{(3)}, \\ c^*_{12} g_2^{(1)} = g_3^{(2)}, \quad c^*_{13} g_2^{(1)} = g_4^{(3)}, \quad c^*_{23} g_2^{(2)} = g_3^{(3)}, \\ c^*_{12} g_3^{(1)} = g_4^{(2)}, \quad c^*_{13} g_3^{(1)} = g_1^{(3)}, \quad c^*_{23} g_3^{(2)} = g_4^{(3)}, \\ c^*_{12} g_4^{(1)} = g_2^{(2)}, \quad c^*_{13} g_4^{(1)} = g_3^{(3)}, \quad c^*_{23} g_4^{(2)} = g_1^{(3)}, \end{cases}$$

$$\begin{cases} c^*_1 g_1^{(1)} = g_2^{(1)}, \quad c^*_1 g_2^{(1)} = g_1^{(1)}, \quad c^*_1 g_3^{(1)} = g_3^{(1)}, \quad c^*_1 g_4^{(1)} = g_4^{(1)}, \\ c^*_2 g_1^{(2)} = g_2^{(2)}, \quad c^*_2 g_2^{(2)} = g_1^{(2)}, \quad c^*_2 g_3^{(2)} = g_3^{(2)}, \quad c^*_2 g_4^{(2)} = g_4^{(2)}, \\ c^*_3 g_1^{(3)} = g_2^{(3)}, \quad c^*_3 g_2^{(3)} = g_1^{(3)}, \quad c^*_3 g_3^{(3)} = g_3^{(3)}, \quad c^*_3 g_4^{(3)} = g_4^{(3)}. \end{cases}$$

Consider the case $x_1^{4/3} > 0$. If $|x_2|$ is sufiiciently small, then

$$u_\ell \sim p_\ell x_1^{4/3} = 3/4^{4/3} e^{2\pi i \ell/3} x_1^{4/3}.$$

Take the half lines in the y-plane starting at u_k ($k = 1, 2, 3$) with the positive real direction as new branch cuts of $\psi_{\ell,B}$.

Choose the branch of $\psi_{\ell,B}$ near $y = u_\ell$ as $0 \leq \arg(y/x_1^{4/3} - u_\ell) < 2\pi$ ($\ell = 1, 2, 3$).

Discontinuity: For a function f defined near $y = u_\ell$ analytic outside $\{u_\ell\}$, we set

$$\Delta_{u_\ell} f(y) = f(y) - f(u_\ell + (y - u_\ell)e^{2\pi i})$$

for $y \in \{\, u_\ell + s \,|\, s \geq 0 \,\}$. Its analytic continuation is also denoted by $\Delta_{u_\ell} f$.

Abbreviation:

$$\begin{aligned} \Delta_{u_k} c^*_{\ell k} \psi_{\ell,B} &= \Delta_{u_k} \psi_{\ell,B} \quad (\ell \neq k), \\ \Delta_{u_j} c^*_{kj} c^*_k c^*_{\ell k} \psi_{\ell,B} &= \tilde{\Delta}_{u_j} \psi_{\ell,B} \quad (j, k, \ell : \text{distinct}). \end{aligned}$$

Theorem 8.1

Fix a point $q = (q_1, 0)\,(q_1 > 0)$. For a point $x = (x_1, x_2)$ and distinct ℓ, j, k, we assume that the point $u_\ell\,(t_1, t_2)$ does not cross the segment $u_j\,(t_1, t_2)\,u_k\,(t_1, t_2)$ when a point (t_1, t_2) moves along the segment qx. Then we have

$$\Delta_{u_k}\psi_{\ell,B} = (-1)^\ell\psi_{k,B}, \tag{8.2}$$

$$\tilde{\Delta}_{u_k}\psi_{\ell,B} = 0 \tag{8.3}$$

for $k, \ell = 1, 2, 3; k \neq \ell$.

- We can track the possible singularities u_k's of $\psi_{\ell,B}$ when x_1, x_2 vary.
- If the paths of analytic continuation $c^*_{\ell k}$ and $c^*_{kj}c^*_k c^*_{\ell k}$ of $\psi_{\ell,B}$'s are deformed suitably, the discontinuity formulas (8.2), (8.3) keep hold.
- Thus we can deduce, in principle, connection formulas of WKB solutions ψ_ℓ across the Stokes set in a neighborhood of arbitrary generic point on the Stokes set.

Proof We prove (8.2), (8.3) for $\ell = 1, k = 3$. Theorem 7.1 and (8.1) yield

$$\psi_{1,B} = \frac{i}{\sqrt{\pi}}(g_1 - g_4) = \frac{i}{\sqrt{\pi}}(g_2^{(1)} - g_1^{(1)})$$

and

$$c_{13}^* \psi_{1,B} = \frac{i}{\sqrt{\pi}}(g_4^{(3)} - g_2^{(3)}).$$

Since $\Delta_{u_3} g_4^{(3)} = 0$ and $\frac{i}{\sqrt{\pi}}\Delta_{u_3} g_2^{(3)} = \frac{i}{\sqrt{\pi}}\Delta_{u_3} g_4 = \psi_{3,B}$, we have (8.2).
Since $c_{12}^* g_2^{(1)} = g_3^{(2)}$, $c_{12}^* g_1^{(1)} = g_1^{(2)}$, $c_2^* g_3^{(2)} = g_3^{(2)}$ and $c_2^* g_1^{(2)} = g_2^{(2)}$, we have

$$c_2^* c_{12}^* \psi_{1,B} = \frac{i}{\sqrt{\pi}}(g_3^{(2)} - g_2^{(2)}).$$

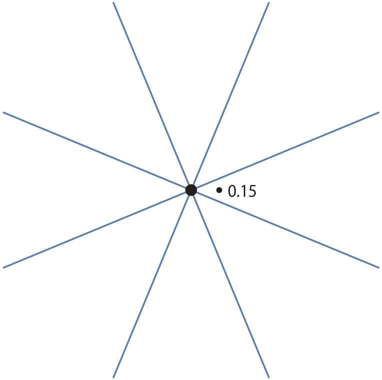
Moreover, $c_{23}^* g_3^{(2)} = g_4^{(3)}$ and $c_{23}^* g_2^{(2)} = g_3^{(3)}$ yield

$$c_{23}^* c_2^* c_{12}^* \psi_{1,B} = \frac{i}{\sqrt{\pi}}(g_4^{(3)} - g_3^{(3)}).$$

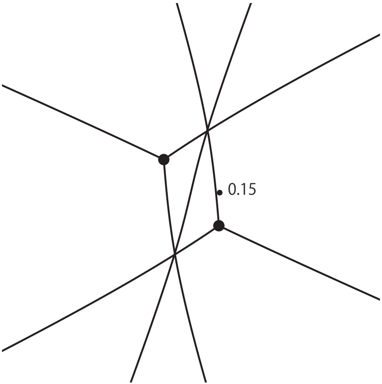
Using $\Delta_{u_3} g_3^{(3)} = \Delta_{u_3} g_4^{(3)} = 0$, we obtain (8.3). Other cases can be proved similarly. $\square$

Example Analytic continuation from $x^{(1)} = (0.15, 0)$ to $x^{(13)} = (0.45 + 0.69i, 0.5 + 0.5i)$:

Sections of the Stokes set
For $x_2 = 0$ and for $x_2 = 0.5 + 0.25i$:

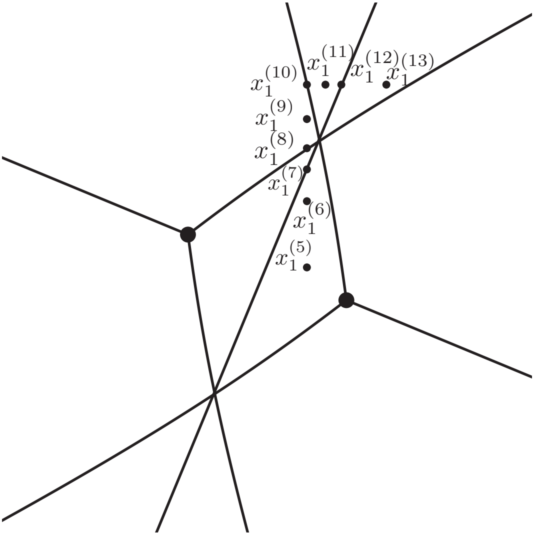


$x = x^{(1)} = (0.15, 0)$

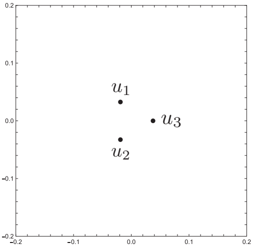


$x = x^{(4)} = (0.15, 0.5 + 0.25i)$

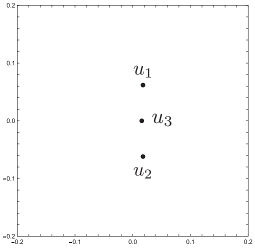
For $x_2 = (1+i)/2$:



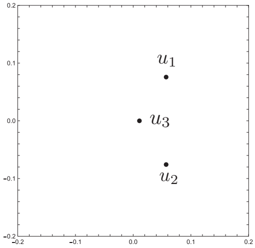
$x = (x_1^{(j)}, 0.5 + 0.5i)$



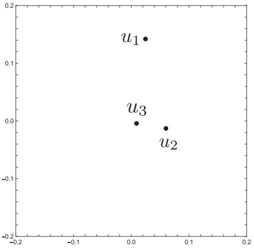
$$x^{(1)} = (0.15, 0)$$



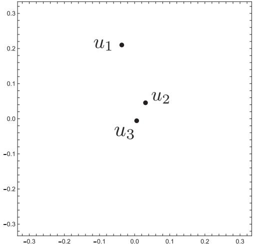
$$x^{(2)} = (0.15, 0.32)$$



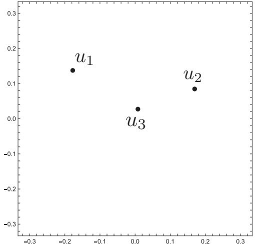
$$x^{(3)} = (0.15, 1/2)$$



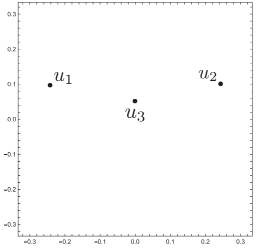
$$x^{(4)} = (0.15, 0.5 + 0.25i)$$



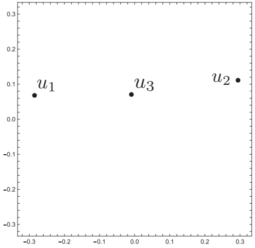
$$x^{(5)} = (0.15, (1 + i)/2)$$



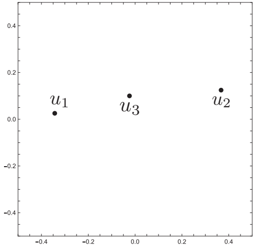
$$x^{(6)} = (0.15 + 0.25i, (1 + i)/2)$$



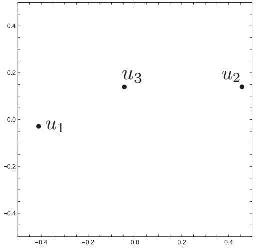
$x^{(7)} = (0.15 + 0.37i, \frac{1+i}{2})$



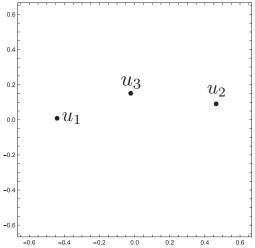
$x^{(8)} = (0.15 + 0.45i, \frac{1+i}{2})$



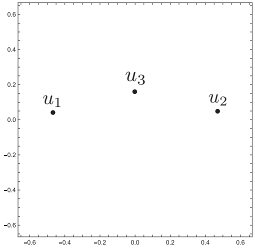
$x^{(9)} = (0.15 + 0.56i, \frac{1+i}{2})$



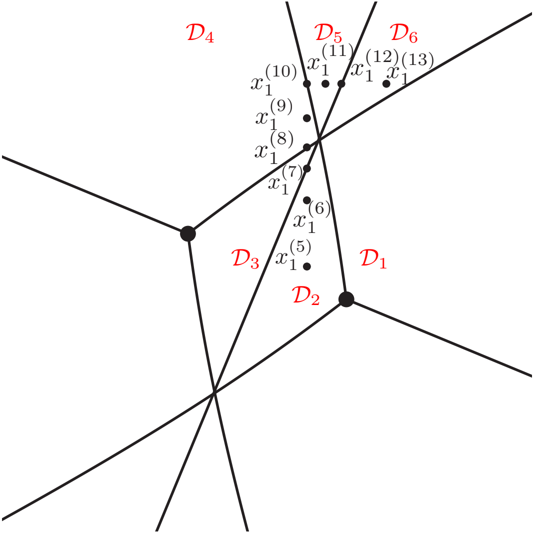
$x^{(10)} = (0.15 + 0.69i, \frac{1+i}{2})$



$x^{(11)} = (0.22 + 0.69i, \frac{1+i}{2})$



$x^{(12)} = (0.28 + 0.69i, \frac{1+i}{2})$



Section of Stokes regions $\mathcal{D}_k$ for $x_2 = 0.5 + 0.5i$.

Let Ψ_ℓ^k denote the Borel sum of ψ_ℓ for $x = (x_1, x_2) \in \mathcal{D}_k$.

During the analytic continuation of $\psi_{\ell,B}$ (in x-variable) from $x^{(1)}$ to $x^{(5)}$, u_1 never crosses the moving segment u_2u_3. Near $x = x^{(4)}$, $\mathrm{Im}(u_3 - u_2) \sim 0$ and ψ_3 is dominant. Hence there are no Stokes phenomena for ψ_1, ψ_2 between $\mathcal{D}_1$ and $\mathcal{D}_2$. Modifying the path of integration of the definition of Ψ_3^1 and using Theorem 8.1, we have

$$\begin{cases} \Psi_1^1 = \Psi_1^2, \\ \Psi_2^1 = \Psi_2^2, \\ \Psi_3^1 = \Psi_3^2 - \Psi_2^2. \end{cases}$$

Let $\mathcal{D}_3$ be the Stokes region containing $x = (0.15 + 0.4i, (1+i)/2)$. In the process of the analytic continuation from $x^{(1)}$ to $x^{(7)}$, u_3 crosses the (moving) segment u_1u_2 once. It follows from Theorem 8.1 that $\Delta_{u_2}\psi_{1,B} = 0$ for $x = x^{(7)}$. Hence we have

$$\begin{cases} \Psi_1^2 = \Psi_1^3, \\ \Psi_2^2 = \Psi_2^3, \\ \Psi_3^2 = \Psi_3^3. \end{cases}$$

This means that between there is no Stokes phenomenon for ψ_ℓ between $\mathcal{D}_2$ and $\mathcal{D}_3$.

On the other hand, u_2 never crosses the segment $u_1 u_3$ during the analytic continuation from $x^{(1)}$ to $x^{(8)}$. Therefore, if we denote by $\mathcal{D}_4$ the Stokes region containing $x^{(9)}$, we have

$$\begin{cases} \Psi_1^3 = \Psi_1^4 - \Psi_3^4, \\ \Psi_2^3 = \Psi_2^4, \\ \Psi_3^3 = \Psi_3^4. \end{cases}$$

Let $\mathcal{D}_5$ and $\mathcal{D}_6$ denote the Stokes region containing $x^{(11)}$ and $x^{(13)}$, respectively. Similar discussion as above shows

$$\begin{cases} \Psi_1^4 = \Psi_1^5, \\ \Psi_2^4 = \Psi_2^5, \\ \Psi_3^4 = \Psi_3^5 - \Psi_2^5 \end{cases}$$

and

$$\begin{cases} \Psi_1^5 = \Psi_1^6 - \Psi_2^6, \\ \Psi_2^5 = \Psi_2^6, \\ \Psi_3^5 = \Psi_3^6. \end{cases}$$

We note that u_2 never crosses the segment $u_1 u_3$ during the analytic continuation, while u_3 crosses once again when x moves from $x^{(8)}$ to $x^{(12)}$.

As is pointed out in [A] and [H], the restriction of the Pearcey system to $x_2 = c$ (c is a constant) yields the equation investigated by Berk-Nevins-Roberts [BNR]. The restriction of the Pearcey system to $x_2 = c$ is given by

$$R_i\psi = 0 \quad (i = 1, 2, 3), \tag{8.4}$$

where we set $x_1 = x$ and

$$\begin{aligned}
R_1 &= (8c^3\eta + 32c\eta^2)\partial_\eta^2 + (8c^3\eta x - 6cx + 27\eta x^3)\partial_x \\
&\qquad + (32c\eta - 36\eta^2 x^2)\partial_\eta + 4c^3\eta + 6c^2\eta^2 x^2 - 9\eta x^2 - 2c, \\
R_2 &= 8c\eta\partial_\eta\partial_x + (2c^3\eta - 4c + 9\eta x^2)\partial_x - 12\eta^2 x\partial_\eta + c^2\eta^2 x - 3\eta x, \\
R_3 &= 2c\partial_x^2 + 3x\eta\partial_x - 4\eta^2\partial_\eta - \eta.
\end{aligned}$$

We call (8.4) the BNR system. Restricting our discussions concerning the Pearcey system to $x_2 = c$, we obtain the counterparts for the BNR system. It can be seen from the discussion in [HKT, Theorem A.1.1], [H], [T] that the WKB solutions to the BNR equation

$$(4\partial_x^3 + 2c\eta^2\partial_x + x\eta^3)\psi = 0 \qquad (c \neq 0)$$

are Borel summable under the general assumption.

9. Summary and concluding remarks

- The Borel transform of the WKB solutions to the Pearcey system with a large parameter are algebraic.
- The Stokes set of the Pearcey system is semialgebraic.
- Explicit forms of the Borel transform of the WKB solutions can be given in terms of the algebraic function coming from the integral representation of the Pearcey function.
- Analytic continuation of the Borel transform can be obtained by using the algebraic function.
- Connection formula of the Borel transform can be given by using analytic continuation of the algebraic function.
- We expect that the Pearcey system gives a WKB theoretic canonical form of the 2-dimensional holonomic systems with a large parameter having a cusp turning point set.

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Thank you for your attention.