Reproducing kernel Hilbert spaces generated by some elliptic operators

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Connections between reproducing kernels of considered Hilbert spaces and Green's functions of their corresponding elliptic operators will be described.

STEP 1: The classical Bergman space

What is the (regular) Bergman space ?

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For a given domain $\Omega \subset \mathbb{C}^N$ consider the space :

$$L^2_H(\Omega) = \{f \in \mathcal{O}(\Omega); ||f||^2_\Omega = \int_\Omega |f|^2 dV < \infty\}$$

with the scalar product $\langle f, g\rangle = \int_\Omega f\overline{g} dV$.

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with the scalar product $\langle f, g\rangle = \displaystyle\int_\Omega f\overline{g}dV$. This is a Hilbert space, called the **Bergman space**.

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REMEMBER, $w \in \Omega$ IS ALREADY FIXED.

Calculation of the Bergman kernel

If $\{\varphi_k\}_{k=0}^{\infty}$ is an orthonormal complete system on $\Omega \subset \mathbb{C}^N$, then

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$$K_{D(0,1)}(z,w) = \sum_{k=0}^{\infty} \lambda_k z^k \lambda_k \overline{w}^k = \frac{1}{\pi}\sum_{k=0}^{\infty}(k+1)(z\overline{w})^k$$

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$$\begin{aligned} K_{D(0,1)}(z,w) &= \sum_{k=0}^{\infty} \lambda_k z^k \lambda_k \overline{w}^k = \frac{1}{\pi}\sum_{k=0}^{\infty}(k+1)(z\overline{w})^k \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty}(k+1)q^k = \frac{1}{\pi}\sum_{k=0}^{\infty}(q^{k+1})' \end{aligned}$$

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Weighted Bergman space

We can define the weighted Bergman space on a similar way.

Weighted Bergman space

Let $\Omega \subset \mathbb{C}^N$ be a domain, and let $W(\Omega)$ be the set of weights on Ω, i.e., $W(\Omega)$ is the set of all Lebesque measurable real - valued positive functions on Ω

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If $\mu \in W(\Omega)$, we denote by $L^2(\Omega, \mu)$ the space of all Lebesque measurable complex-valued μ-square integrable functions on Ω, equipped with the norm $|| \cdot ||_\mu$ given by the scalar product

$$\langle f, g \rangle_\mu := \int_\Omega f(z)\overline{g(z)}\mu(z)dV, \quad f, g \in L^2(\Omega, \mu).$$

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$$\langle f, g\rangle_\mu := \int_\Omega f(z)\overline{g(z)}\mu(z)dV, \quad f, g \in L^2(\Omega, \mu).$$

Define $L^2_H(\Omega, \mu) := \mathcal{O}(\Omega) \cap L^2(\Omega, \mu)$.

Admissible weights

We would like to know that $L^2_H(\Omega, \mu)$ is a Hilbert space for a given weight μ. Thus we will be working with special class of weights, called „admissible".

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Definition (Admissible weight)

A weight $\mu \in W(\Omega)$ is called an admissible weight, an a-weight for short, if $L^2_H(\Omega, \mu)$ is a closed subspace of $L^2(\Omega, \mu)$ and for any $z \in \Omega$ the evaluation functional $E_z f = f(z)$ is continuous on $L^2_H(\Omega, \mu)$.

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Remark

See that the definition of admissible weights provides that $L^2_H(\Omega, \mu)$ is a Hilbert space and the reproducing kernel exists uniquely by the Riesz theorem.

Admissible weights

Theorem (([Win, Corollary 3.1]))

Let $\mu \in W(\Omega)$. If the function μ^{-a} is locally integrable on Ω for some $a > 0$ then $\mu \in AW(\Omega)$.

Now, the space $L^2_H(\Omega, \mu) := \mathcal{O}(\Omega) \cap L^2(\Omega, \mu)$ is called the **weighted Bergman space**.

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Weighted Bergman kernel function

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$$K_{\Omega,\,\mu}(z, w) = \frac{\phi_\mu(z, w)}{||\phi_\mu||_\mu^2}.$$

We can expand the weighted Bergman kernel by means of orthonormal complete system, namely :

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$$K_{\Omega,\,\mu}(z, w) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(w)}$$

The weighted Bergman kernel and the Green's function for a domain $\Omega \subset \mathbb{C}$

It is well known that a Green's function for the Laplace operator takes the form

$$G_\Omega(z, w) = h_\Omega(z, w) - \ln|z - w|,$$

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where h_Ω is harmonic w.r.t $z \in \Omega$. Thus

$$\frac{\partial^2 G_\Omega}{\partial z \partial \overline{w}} = \frac{\partial}{\partial z}\left(\frac{\partial h_W}{\partial \overline{w}} - \frac{1}{2}\frac{\partial}{\partial \overline{w}}\left[\ln(z - w) + \ln(\overline{z} - \overline{w})\right]\right) = \frac{\partial^2 h_\Omega}{\partial z \partial \overline{w}}$$

It can be shown that any domain $\Omega \subset \mathbb{C}$ (or in $\mathbb{C}^n$) may be written as

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \quad \Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \dots ,$$

where $\partial\Omega_j$ consists of a finite number of smooth Jordan curves (we do not assume any regularity of $\partial\Omega$), for any $j \in \mathbb{N}$.

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where $\partial\Omega_j$ consists of a finite number of smooth Jordan curves (we do not assume any regularity of $\partial\Omega$), for any $j \in \mathbb{N}$. By Harnack's theorem we have

$$\lim_{j\to\infty} \frac{\partial^2 G_{\Omega_j}}{\partial z \partial \overline{w}} = \lim_{j\to\infty} \frac{\partial^2 h_{\Omega_j}}{\partial z \partial \overline{w}} = \frac{\partial^2 h_{\Omega}}{\partial z \partial \overline{w}} = \frac{\partial^2 G_{\Omega}}{\partial z \partial \overline{w}} .$$

It was shown in [Kra-Woj] that assuming $\Omega \subset \mathbb{C}$ is a domain with $L^2(\Omega) \neq 0$ and $\varrho(z) = |\mu(z)|^2$, where $\log \varrho$ is harmonic on a neighbourhood of $\overline{\Omega}$ (and μ has no zeros on $\overline{\Omega}$) the following connection between the weighted Bergman kernel of the weighted Bergman space $L^2_H(\Omega, \varrho)$ and the Green's function of the operator $\frac{\partial}{\partial \overline{z}} \left(\frac{1}{\varrho(z)} \frac{\partial}{\partial z} \right)$ holds:

$$K_{\Omega,\,\rho}(z,w) = -\frac{2}{\pi \rho(z)\rho(w)} \frac{\partial^2 G_{\Omega,\,\rho}(z,w)}{\partial z \partial \overline{w}}\,.$$

That is the improvement of the Garabedian's result in [G].

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That is the improvement of the Garabedian's result in [G]. We get the classical Bermgan-Schiffer identity (see [B-S]) taking $\varrho(z) = 1$.

STEP 2: A general Bergman space

A general Bergman space

Consider a differential operator $\mathcal{L} = \sum_{|\alpha| \leq m} C_\alpha \partial^\alpha$ of order $m \geq 1$ with constant coefficients. As usual, we denote by $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \ldots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}$.

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This time we consider (see [Mal]) the space

$$P_{\mathcal{L}}(\Omega) = \{u \in L^2(\Omega);\ \mathcal{L}u = 0\},$$

where $\Omega \in \mathbb{R}^n$ is a bounded, open set. It is called a (general) Bergman space.

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So for $\mathcal{L} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$ we get the classical Bergman space.
Taking $\mathcal{L} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ we get the space of harmonic L_2-functions.

A general Bergman space

The Bergman space $P_{\mathcal{L}(\Omega)}$ is a closed subspace of the space $L_2(\Omega)$. Hence, the projector

$$\mathcal{B}_{P_{\mathcal{L}(\Omega)}} : L_2(\Omega) \to P_{\mathcal{L}(\Omega)}$$

is properly defined. So there is a natural question on the representation of the projector, in particular, under which conditions one has

$$\mathcal{B}_{P_{\mathcal{L}(\Omega)}} u(x) = \int_\Omega K(x, y)u(y)dy?$$

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$$\mathcal{B}_{P_{\mathcal{L}(\Omega)}} u(x) = \int_\Omega K(x, y) u(y) dy?$$

The kernel $K(x, y)$ of the above integral operator is called the Bergman kernel. If the Bergman kernel exists, then

$$u(x) = \int_\Omega K(x, y) u(y) dy$$

for any function $u \in P_{\mathcal{L}(\Omega)}$.

Existence of the kernel

Theorem ([Mal])

Assume that an embedding

$$P_{\mathcal{L}(\Omega)} \hookrightarrow C(\Omega)$$

exists. Then there exists the Bergman kernel $K(x,y)$ *and*

$$K(x,y) = f_1(x)\overline{f_1(y)} + f_2(x)\overline{f_2(y)} + \dots$$

where $f_1, f_2, \dots$ *is an orthonormal basis in the Bergman space* $P_{\mathcal{L}(\Omega)}$.

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where $f_1, f_2, \dots$ *is an orthonormal basis in the Bergman space* $P_{\mathcal{L}(\Omega)}$.

Take a point $x \in \Omega$ and consider the linear functional

$$\Phi_x : P_{\mathcal{L}(\Omega)} \to \mathbb{C},$$

given by

$$\Phi_x(u) = u(x), \quad u \in P_{\mathcal{L}(\Omega)}.$$

The embedding $P_{\mathcal{L}(\Omega)} \hookrightarrow C(\Omega)$ implies the continuity of the operator Φ_x.

Existence of the kernel

Since the space $P_{\mathcal{L}(\Omega)} \hookrightarrow C(\Omega)$ is a closed subspace of the Hilbert space $L_2(\Omega)$, it is Hilbert space too. So by the Riesz representation theorem there exists exactly one function $\varphi_x \in P_{\mathcal{L}(\Omega)}$ with

$$\Phi_x(u) = \langle u, \varphi_x \rangle_{L_2(\Omega)}, \quad \forall u \in P_{\mathcal{L}(\Omega)}.$$

It turns out (see [Mal]) that the function

$$K(x,y) = \overline{\varphi_x(y)}, \quad x, y \in \Omega$$

is the Bergman kernel (reproducing kernel of the space $P_{\mathcal{L}(\Omega)}$).

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is the Bergman kernel (reproducing kernel of the space $P_{\mathcal{L}(\Omega)}$). As in the classical case, this Bergman kernel resolves the variational problem on the minimal norm in $P_{\mathcal{L}(\Omega)}$.

Existence of the kernel

Thus the problem is the existence of an embedding

$$P_{\mathcal{L}(\Omega)} \hookrightarrow C(\Omega).$$

It turns out that it exists for

(1) the Cauchy-Riemann operator $\mathcal{L} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$ (the classical Bergman space),

(2) $\mathcal{L} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ (the classical space of harmonic L_2 functions),

(3) $\mathcal{L}$ is the heat operator $\mathcal{L} = \frac{\partial u}{\partial t} - c\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$

(4) arbitrary elliptic operator $\mathcal{L}$

(5) arbitrary hypoelliptic operator $\mathcal{L}$

The Bergman kernel and the Green's function

Assume that $\mathcal{L}$ is a real elliptic operator of order $l > \dfrac{n}{2}$. Denote $G(x, y)$ the Green's function of the operator $\mathcal{L}\mathcal{L}'$, where $\mathcal{L} = \sum_{|\alpha| \leq l} C_\alpha \partial^\alpha$ and $\mathcal{L}' = \sum_{|\alpha| \leq l} (-1)^{|\alpha|} C_\alpha \partial^\alpha$ is the adjoint differential operator. Define a distribution $\Lambda \in D'(\Omega \times \Omega)$ by the formula

$$(\Lambda, \varphi) = \int_\Omega \varphi(x, x) dx, \quad \varphi \in C_0^\infty(\Omega \times \Omega)$$

(so it is vanishing outside the diagonal). Then

$$K(x, y) = \Lambda - \mathcal{L}'_x \mathcal{L}'_y G(x, y),$$

where $K(x, y)$ is the reproducing kernel of the space $P_{\mathcal{L}(\Omega)}$.

STEP 3: Linear operators with nonconstant coefficients

It was shown in [Żyn-Sad-Kra-Wój] that if Ω is a domain in $\mathbb{R}^2$ with the boundary of class C^1 and $\mathcal{L}$ is a strongly elliptic operator of 2nd order:

$$\mathcal{L}f = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}(x_1, x_2)\frac{\partial f}{\partial x_i}\right) + \sum_{i=1}^{2} b_i(x_1, x_2)\frac{\partial f}{\partial x_i} + c(x_1, x_2)f,$$

where $a_{ij} \in C^1(\Omega), b_i, c \in L^\infty(\Omega)$, then the space $P_{\mathcal{L}}(\Omega)$ has the reproducing kernel $K(x, y)$.

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where $a_{ij} \in C^1(\Omega), b_i, c \in L^\infty(\Omega)$, then the space $P_{\mathcal{L}}(\Omega)$ has the reproducing kernel $K(x, y)$.

On the other hand the space $P_{\mathcal{L}}(\Omega)$ for $\Omega = D(0, 1) \subset \mathbb{R}^2$ and $\mathcal{L} = \dfrac{\partial^2}{\partial x \partial y}$ does NOT have any reproducing kernel ([Żyn-Sad-Kra-Wój]).

STEP 4: Further research should involve other operators $\mathcal{L}$ and the relationship between the weighted Bergman kernel of weighted space $P_{\mathcal{L}}(\Omega)$ and the corresponding Green's function.

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Thank You for your attention!