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Summability and global property of transseries solution of Hamiltonian system

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Contents of Talk

Consider the Hamiltonian system with n degrees of freedom

$$\dot{q}_j = \frac{\partial}{\partial p_j} H, \quad \dot{p}_j = -\frac{\partial}{\partial q_j} H, \quad j = 1, 2, \ldots, n, \tag{1}$$

where H is a Hamiltonian function and $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$ are the variables in $\mathbb{R}^n$ or in $\mathbb{C}^n$ ($n \geq 2$). Let $H := H_0 + H_1$ with H_0 and H_1 given, respectively, by

$$H_0 = q_1^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j, \tag{2}$$

$$H_1 = -\sum_{j=2}^{n} q_j^2 B_j(q_1, q_1^{2\sigma} p_1, q), \tag{3}$$

where $B_j(q_1, s, q)$'s are holomorphic at the origin with respect to $(q_1, s, q) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-1}$. (1) appears in the geometric problem related to the geodesic flow. (cf. Taimanov).

Integrability

Eq. (1) is said to be C^ω-Liouville integrable in some domain Ω if there exist n first integrals $\phi_j \in C^\omega$ $(j = 1, \ldots, n)$ which are functionally independent on an open dense set in Ω and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$. If $\phi_j \in C^\infty$ $(j = 1, \ldots, n)$, then we say C^∞-Liouville integrable.

If, in addition, there exists $2n - 1$ funtionally independent first integrals, then we say that super integrable.

Assumption

Assume

$$B_\nu \equiv B_\nu(q_1, q_1^{2\sigma} p_1, q) = B_{\nu,0}(q_1, q) + q_1^{2\sigma} p_1 B_{\nu,1}(q_1, q), \quad \nu = 2, \dots, n, \tag{4}$$

where $B_{\nu,0}$ and $B_{\nu,1}$ are analytic at $(q_1, q) = (0, 0)$. Suppose that the Poincaré condition holds

$$Re\, \lambda_j > 0, \quad j = 2, 3, \dots, n. \tag{5}$$

Suppose that the nonresonance condition holds, i.e.,

$$\sum_{\nu=2}^{n} \lambda_\nu k_\nu - \lambda_j \neq 0, \quad \forall\ k_\nu \in \mathbf{Z}_+,\ \nu = 2, \dots, n,\ j = 2, \dots, n. \tag{6}$$

Definition (transseries)

Set $\lambda = (\lambda_2, \ldots, \lambda_n)$. Consider the formal series solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ whose component has the following form

$$\sum_{k \geq k_0,\ \ell \geq \ell_0} c_{k,\ell} t^{-\frac{\ell}{2\sigma - 1}} e^{\lambda k t}, \tag{7}$$

where $k = (k_2, \ldots, k_n)$, $\lambda k = \lambda_2 k_2 + \cdots + \lambda_n k_n$, and where $c_{k,\ell}$'s are complex constants and k_0 is a multiinteger and $\ell_0 \geq 0$ is an integer.

Remark. The series (7) is the special case of so-called transseries. In the general case one considers the series without the restriction $k \geq k_0$. Ecale considers more general form.

Motivation

In a series of papers including [1] Balser introduced the notion of the semi formal series and he showed that the general formal solution of the initial value problem of a nonlinear ODE without resonance is expressed as a semi formal series. Later, it was shown that the semi formal series is a special class of the transseries. Heuristically speaking, the semi formal series satisfies that the sume with respect to $\ell \geq \ell_0$ converges uniformly in k and the restriction $k \geq k_0$ is omitted.

Possible next step:

1. Construction of formal solution in a resonant case.
2. Geometrical or algrbraic characterization of formal series. (Ecale's group).
3. What is the true solution ? (Summability, connection)

Remark. By the results of a movable singular solution a general solution may have infite number of movable singular points accumulating in every direction. (no resurgence). In such a case the summability and the classical definition of the connection problem seem impossible.
Object of the study. We look for the subclass of semi formal solutions for the Hamiltonian equation for which the summability and the connection problem are formulated and are studied. We follow the complex analytic formulation being similar to the linear case. Indeed, we use the Borel sum of the semi formal series. Then the analytic continuation of the Borel sum is given and the connection problem is formulated.
The advantage is that the solution with dense movable singular points does not appear in these solutions. We study the Stokes function.

Formal transseries and summability

We first construct a formal transseries solution.

Theorem 1

Suppose that (4), (5) and (6). Then there exists a formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ *of (1) in the domain* $\{t \mid Re(\lambda_j t) < 0, j = 2, \ldots, n\}$.

Next we show the summability of the formal solutions.

Theorem 2

Assume (4), (5) and (6). Then the formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ *of (1) is* $(2\sigma - 1)$*- Borel summable in every direction in* $\{t \mid Re(\lambda_j t) > 0, j = 2, \ldots, n\}$. *There exists a neighborhood* Ω_1 *of* $q_1 = 0$ *such that the Borel sum is the analytic transseries solution of (1) in the set* $\Omega_1 \cap \{t \mid Re(\lambda_j t) < 0, j = 2, \ldots, n\}$.

The proof of these theorems are given, after some preparations of first integrals.

We study the analytic continuation of the Borel sum in Theroem 2. First we extend Theorem 1

Theorem 3

Let $\xi_0 > 1$. Suppose that (4),(5) and (6) are satisfied. Then there exists a formal transseries solution $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ of (1) in the domain $\{t| \, |e^{(\lambda_j t}| < \xi_0, j = 2, \ldots, n\}$.

We consider the analytic continuation of the Borel sum in Theorem 2. Let E_0 be the closed convex proper cone with vertex at the origin containing $S_0(0)$

$$S_0(0) := \left\{z \in \mathbb{C} \,|\, \kappa z^{\kappa} + \lambda \cdot k = 0, \forall k \in \mathbb{Z}_+^{n-1} \setminus \{0\}\right\}, \tag{8}$$

where $\lambda = (\lambda_2, \ldots, \lambda_n)$. Define

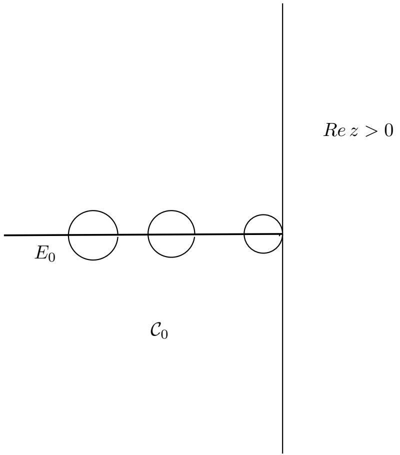
$$\mathcal{C}_0 := \left\{z \in \mathbb{C} |\, z = -\frac{t^{1-2\sigma}}{2\sigma - 1}, |e^{\lambda_j t}| < \xi_0, \quad j = 2, \ldots, n\right\}. \tag{9}$$

Then we have

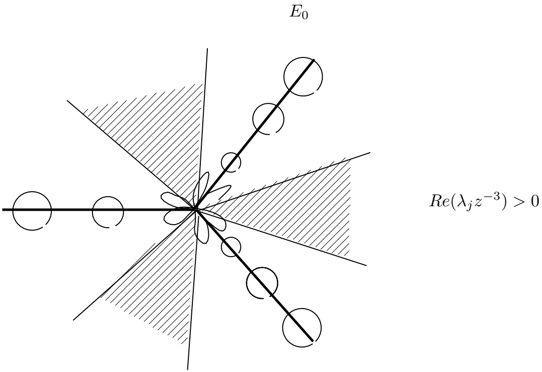
Theorem 4

The Borel sum in Theorem 2 is analytically continued to the set $\mathcal{C}_0 \cap E_0^c \cap \Omega_1$.

We omit the the proofs of Theorems 3 and 4.
We give some pictures of the set $\mathcal{C}_0 \cap E_0^c \cap \Omega_1$. Assume that $\lambda_j > 0$, $j = 2, 3, \ldots, n$. E_0 is the singular direction of the Hamiltonian system. We consider the cases $\sigma = 1$ and $\sigma = 2$.
We have $E_0 = \{z | z \leq 0\}$ if $\sigma = 1$.


$Re\, z > 0$
E_0
$\mathcal{C}_0$

$\sigma = 2$, z



Summability of transseries

Consider the formal series

$$\psi(q_1, q) = q_1^{\kappa} \sum_{n=0}^{\infty} v_n(q) q_1^n, \tag{10}$$

where $\kappa \geq 1$ is an integer and $v_n(q)$'s are holomorphic in $q \in V_0$ for some open sets V_0 independent of n. The formal κ- Borel transform $\hat{\mathcal{B}}_\kappa$ is defined by

$$\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q) := \sum_{n=0}^{\infty} v_n(q) \frac{\zeta^n}{\Gamma(\frac{n+\kappa}{\kappa})}, \tag{11}$$

where ζ is the dual variable of q_1 and $\Gamma(z)$ is the gamma function. For ϕ in (10) we have

$$\hat{\mathcal{B}}_\kappa(q_1^{\kappa+1} \frac{d}{dq_1} \psi)(\zeta, q) = \kappa \zeta^\kappa \hat{\mathcal{B}}_\kappa(\psi)(\zeta, q). \tag{12}$$

For the bisecting direction $d \in \mathbb{R}$ and the opening $\eta > 0$, define $S(d, \eta) := \{z \in \mathbb{C}; |\arg z - d| < \eta/2\}$. For the neighborhood $\Omega_0 \subset \mathbb{C}$ of the origin, define

$$\Sigma_0 := \Omega_0 \cup S(d, \eta). \tag{13}$$

We say that the formal power series $\psi(q_1, q)$ is κ-summable with respect to q_1 in the direction d if there exist $\theta > 0$ and a neighborhood Ω_1 of $\zeta = 0$ such that $\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q)$ converges when $(\zeta, q) \in \Omega_1 \times V_0$ and $\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q)$ can be analytically continued to $(\zeta, q) \in S(d, \eta) \times V_0$ and is of exponential type of order κ in $\zeta \in S(d, \eta)$. Namely, there exist $K_0 > 0$ and $K_2 > 0$ such that

$$|\hat{\mathcal{B}}_\kappa(\psi)(\zeta, q)| \le K_0 e^{K_2|\zeta|^\kappa}, \quad \zeta \in S(d, \eta),\ q \in V_0.$$

For the sake of simplicity we denotes the analytic continuation with the same notation. Then the κ- sum of the formal series $\psi(q_1, q)$, $\Psi(q_1, q)$ is defined by the Laplace transform

$$\Psi(q_1, q) := \int_0^{\infty e^{id}} e^{-(\zeta/q_1)^\kappa} \hat{\mathcal{B}}_\kappa(\psi)(\zeta, q) d\zeta^\kappa. \tag{14}$$

Summability of transseries.
Consider the transseries u given by (7). We write

$$u = \sum_{k \geq k_0, \ell \geq 0} c_{k,\ell} t^{-\ell/(2\sigma-1)} e^{\lambda k t} = \sum_{k \geq k_0} e^{\lambda k t} u_k(t), \tag{15}$$

where

$$u_k(t) = \sum_{j=0}^{2\sigma-2} t^{-j/(2\sigma-1)} u_{k,j}(t), \quad u_{k,j}(t) = \sum_{m=0}^{\infty} c_{k,m(2\sigma-1)+j} t^{-m}. \tag{16}$$

We say that u is κ- Borel summable in the direction d if there exist Σ_0 in (13) and the constant K_0 such that, for every j, $j = 0, \dots, 2\sigma - 1$ and every integer $k \geq 0$ the formal κ- Borel transform of $f_{k,j}(t) := e^{\lambda k t} u_{k,j}(t)$, $\mathcal{B}_\kappa(f_{k,j})(\tau)$ is extended to the holomorphic function on Σ_0 of order 1 uniformly in k, namely there exist $\exists C_k > 0$ satisfying $\sum_k C_k < \infty$ such that

$$|\mathcal{B}(f_{k,j})(\tau)| \leq C_k e^{K_0 |\tau|^\kappa}, \quad \forall \tau \in \Sigma_0, \tag{17}$$

where τ is the dual variable of t.

Connection of first integral

Consider the Borel sum of formal first integrals of the Hamiltonian H constructed in Theorem 14. Suppose that $\theta_0 \in E_0$ is not an accumulation point of E_0. Let Σ_1 and Σ_2 be the sectors in q_1-plane such that

$$\theta_0 \in \Sigma_1 \cap \Sigma_2, \quad \Sigma_1 \cap E_0 = \Sigma_2 \cap E_0 = \{\theta_0\}. \tag{18}$$

Assume that the formal first integrals $\phi := (\phi_1, \phi_2, \ldots, \phi_\nu)$ and $\psi := (\psi_1, \psi_2, \ldots, \psi_\nu)$ are Borel summable in Σ_1 and Σ_2, respectively, $(\nu \geq 1)$. By definition we see that ϕ_j's (or ψ_j's)) are functionally independent and are polynomials in p_1, p. Consider the connection relation in the sector $\Sigma_1 \cap \Sigma_2$

$$\phi(q_1, p_1, q, p) = \psi(q_1, p_1, q, p) + m(q_1, p_1, q, p). \tag{19}$$

Recall that every component $m_j(q_1, p_1, q, p)$ of $m(q_1, p_1, q, p)$ ($j = 1, \ldots, n$) is the first integral of (1). Then we have

Theorem 5

Suppose that the eqution

$$q_1^{2\sigma} \frac{dv}{dq_1} - 2\lambda_k v = B_k(q_1, 0, 0) \tag{20}$$

has no analytic solution v at the origin for $k = 2, 3, \ldots, n$. Assume that $m(q_1, p_1, q, p)$ is analytic in some neighborhood of the origin. Then, for every $j = 1, 2, \ldots, n$ there exists an analytic function of one variable ϕ_j at the origin such that $m_j(q_1, p_1, q, p) = \phi_j(H)$ in some neighborhood of the origin.

The condition (16) of Theorem 9 holds for general B_k's. Theorem 9 follows from the well known result: Under the condition of Theorem 9 every analytic first integral of the Hamiltonian system of H is expressed as $f(H)$ for some analytic function of one variable, $f(z)$.

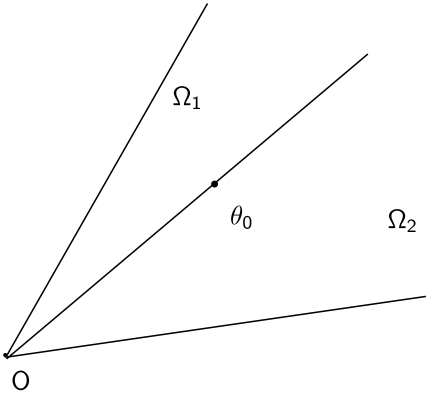


Figure: Choice of sectors

We consider the vanishing of the connection function m in (19). Let Ω_1 and Ω_2 be the adjacent sectors in the Borel plane whose boundaries has the common singular direction θ_0. (cf. Fig. 1). Then we have

Theorem 6

Assume that (5) and the Poincaré condition are satisfied. Suppose the condition

$$B_j(q_1, t, q) = \tilde{B}_j(t, q), \quad j = 2, \ldots, n, \tag{21}$$

are satisfied, where $\tilde{B}_j$ is a polynomial in t and analytic at $q = 0$. Then $m(q_1, p_1, q, p)$ in (19) vanishes as a formal power series.

Define the convex positive cone generated by λ_j $(j = 2, 3, \ldots, n)$, $\Omega(\lambda_2, \ldots, \lambda_n) \equiv \Omega(\lambda)$ by

$$\Omega(\lambda) = \{z = \sum_{j=2}^{n} t_j \lambda_j | \ t_j \geq 0, j = 2, 3, \ldots, n, \sum_{j=2}^{n} t_j > 0\}. \tag{22}$$

Then we have

Theorem 7

Suppose (5) and that λ_j's $(j = 2, \ldots, n)$ satisfy the Poincaré condition. Assume (4) with $B_{\nu,0}(q_1, q)$ $(\nu = 2, 3, \ldots, n)$ being polynomials in q with coefficients analytic at $q_1 = 0$. Then there exists a system of super integrable first integrals which are independent of p_1. For such a system of first integrals the connecting function m in (19) exists and it is holomorphic in q_1, q and p when $q_1 \neq 0$. There exists a neighborhood of the origin U such that m is a single-valued function of q_1 in $\{q_1 \in \mathbb{C} \cap U; q_1 \neq 0\}$. Moreover, m is not analytic at $q_1 = 0$ provided m does not vanish identically and the equation (20) has no analytic solution v at the origin for $k = 2, 3, \ldots, n$.

Formal first integral in the class of formal transseries

For $c \in \mathbb{C}$ and $\alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^{n-1}$, define

$$E_c \equiv E_c(q_1) = \exp\left(\frac{cq_1^{-2\sigma+1}}{2\sigma - 1}\right), \;\; E^\alpha = E_{\lambda_2}^{\alpha_2} \cdots E_{\lambda_n}^{\alpha_n}. \tag{23}$$

We denote by e_j the j-th unit vector, $e_j = (0, \cdots, 1, \cdots 0)$, $j = 2, 3, \ldots, n$.
We construct the first integral v of χ_H given by

$$v = \phi^{(\alpha)}(q_1, p_1, q, p)E^\alpha, \tag{24}$$

where $\phi^{(\alpha)}(q_1, p_1, q, p)$ is the formal power series of q_1, q, p_1 and p of the following form

i) If $\alpha = 0$, then

$$\phi^{(0)} \equiv \phi_j^{(0)} = p_j q_j + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \ldots, n, \tag{25}$$

where

$$U_{0,j} = U_{0,j}(q_1, q) = \sum_{\nu=0}^{\infty} U_{0,j,\nu}(q) q_1^{\nu}, \tag{26}$$

$$U_{1,j} = U_{1,j}(q_1, q) = \sum_{\nu=0}^{\infty} U_{1,j,\nu}(q) q_1^{\nu}, \tag{27}$$

are the formal power series of q_1 with coefficients analytic in q.
ii) If $\alpha = e_j$, $(2 \leq j \leq n)$, then

$$\phi^{(e_j)} = p_j q_j^2 (1 + U_{2,j}) + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \ldots, n, \tag{28}$$

where $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ are the formal power series of q_1 with coefficients analytic in q.
iii) If $\alpha = -e_j$, $(2 \leq j \leq n)$, then

$$\phi^{(-e_j)} = p_j (1 + U_{2,j}) + U_{0,j} + q_1^{2\sigma} p_1 U_{1,j}, \quad j = 2, \ldots, n, \tag{29}$$

where $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ are the formal power series of q_1 with coefficients analytic in q.

Formal first integral

Let χ_H be the Hamiltonian vector field of H.

Definition 8

We say that v in (24) is the formal first integral of χ_H if the following conditions are satisfied.
(i) $\chi_H v = 0$ as a formal power series.
(ii) If $\alpha = 0$, then $\phi^{(0)} \equiv \phi_j^{(0)}$, $(j = 2, \ldots, n)$ satisfies (25), (26) and (27) with $U_{0,j,\nu}(q)$'s and $U_{1,j,\nu}(q)$'s analytic in some neighborhood of the origin $q = 0$ independent of ν and j. If $\alpha = e_j$ (resp. $\alpha = -e_j$), $(j = 2, \ldots, n)$, then $\phi^{(\alpha)}$ has the form (28) (resp. (29)), with $U_{0,j}$'s, $U_{1,j}$'s and $U_{2,j}$'s satisfying the same conditions as the case $\alpha = 0$.

Gevrey order

Definition 9

We say that the formal series $U_{0,j}$ in (26) is Gevrey of order s (in short, s- Gevrey), for some $s \geq 0$, if there exist a neighborhood of the origin $q = 0$, Ω_0 and constants $C > 0$, $K > 0$ for which

$$\sup_{q \in \Omega_0} |U_{0,j,\nu}(q)| \leq C K^{\nu} \Gamma(1 + s\nu),$$

hold for all $\nu \geq 0$, where Γ denotes the Gamma function. If both $U_{0,j}$ and $U_{1,j}$ are s-Gevrey, then we say that $\phi_j^{(0)}$ is s-Gevrey. We say that $\phi^{(e_j)}$ (resp. $\phi^{(-e_j)}$) is s-Gevrey if $U_{0,j}$, $U_{1,j}$ and $U_{2,j}$ are s-Gevrey.

The following theorem shows the superintegrability in a formal transseries.

Theorem 10

Assume (5) and (6). Then χ_H has the formal first integrals, $\phi_j^{(0)}$, $\phi^{(e_j)}E^{e_j}$ and $\phi^{(-e_j)}E^{-e_j}$, $(j = 2, \ldots, n)$, which are $(2\sigma - 1)$- Gevrey.

For the proof we prepare a lemma.
Let $R_j > 0$ $(j = 2, \ldots, n)$ be given. Set $V_0 := \prod_{j=2}^{n}\{z_j \mid |z_j| < R_j\}$. Let $\mathcal{O}(V_0)$ be the set of holomorphic functions in V_0 continuous up to the boundary. Set $M_0(q) := \prod_{j=2}^{n}(R_j - |q_j|)$. For $f \in \mathcal{O}(V_0)$ we define the norm $\|f\|$ and the weighted norm $|||f|||$ by

$$\|f\| := \sup_{q\in V_0} |f(q)|, \quad |||f||| := \sup_{q\in V_0} |f(q)M_0(q)|.$$

$\mathcal{O}(V_0)$ is the Banach space with the norm $||| \cdot |||$.

Let $\lambda := (\lambda_2, \ldots, \lambda_n)$ and $\alpha = (\alpha_2, \ldots, \alpha_n)$. Consider the equation

$$Lu \equiv \left(\sum_{\nu=2}^{n} \lambda_\nu q_\nu \frac{\partial u}{\partial q_\nu} - \lambda \cdot \alpha \right) u = f \in \mathcal{O}(V_0), \quad f = O(|q|). \tag{30}$$

Then we have

Lemma 11

Let $\alpha = 0, \pm e_j, j = 2, \ldots, n$. Assume (5) and (6). Then there exists a constant $K > 0$ such that, for every $f \in \mathcal{O}(V_0)$ with $f = O(|q|)$ there exist a unique holomorphic solution u of (30) in $\mathcal{O}(V_0)$ such that $|||u||| \leq K\|f\|$.

The lemma is easily proved by Cauchy's integral formula in a polydisk. By Lemma 15 we prove Theorem 14 by estimating the coefficients of formal series by the recurrence relation.

Preparatory lemma. (Theorem 1)

Define

$$\mathcal{C} := \left\{ z \in \mathbb{C} \,|\, Re\,(\overline{\lambda_j} z^{2\sigma-1}) > 0 \quad j = 2, \ldots, n \right\}. \tag{31}$$

Let $\phi_j^{(0)}$ and $\phi^{(-e_j)} E^{-e_j}$ ($j = 2, \ldots, n$) be the formal first integrals given by the preceeding theorem. Let C_j, $\tilde{C}_j$ and C_0 be constants. For $z \in \mathcal{C}$, we solve the system of equations for q, p, p_1

$$\phi_j^{(0)} = C_j,\ \phi^{(-e_j)} E^{-e_j} = \tilde{C}_j,\ H = C_0, \quad j = 2, \ldots, n, \tag{32}$$

where $H = H_0 + H_1$ is given by

$$H = z^{2\sigma} p_1 + \sum_{j=2}^{n} \lambda_j q_j p_j + \sum_{j=2}^{n} q_j^2 B_j(z, z^{2\sigma} p_1, q). \tag{33}$$

Here the unknown quantities are

$$q = q(z, T), \ p = p(z, T), \ p_1 = p_1(z, T), \tag{34}$$

where

$$q = \sum_{n=0}^{\infty} c_n z^n, c_n = c_n(T^{-1}), \quad T = (T_j)_j, \quad T_j = \tilde{C}_j E^{e_j}, \tag{35}$$

is a formal series of z with $c_n(T^{-1})$ convergent in T in some neighborhood of $T = \infty$. The Tayler series of p has the same form as q. As for p_1 we have

$$p_1 z^{2\sigma} = \sum_{n=0}^{\infty} \rho_n z^n, \rho_n = \rho_n(T^{-1}), \tag{36}$$

with $\rho_n(T^{-1})$ convergent in T in some neighborhood of $T = \infty$.

By (25) and (29) we have

$$p_j q_j + \tilde{A}_j(z, z^{2\sigma} p_1, q) = C_j, \quad j = 2, \ldots, n, \tag{37}$$

$$p_j(1 + D_j(z, q)) + \tilde{D}_j(z, z^{2\sigma} p_1, q) = T_j, \quad j = 2, \ldots, n, \tag{38}$$

$$H = C_0. \tag{39}$$

Then we have

Lemma 12

Assume (5). Then (37)-(39) has the formal solution (q, p, p_1) *for* $z \in \mathcal{C}$ *given by (34), (35) and (36).*

The proof is the calculation of the recurrence formula.

Construction of formal transseries solution

Let z satisfy $\dot{z} = z^{2\sigma}$. Namely

$$t = -\frac{z^{1-2\sigma}}{2\sigma - 1}. \tag{40}$$

Let $q \equiv q(z)$, $p_1 \equiv p_1(z)$ and $p \equiv p(z)$ be the formal series given by Lemma 12. By (40) they are the transseries of t. The exponential part is given by $e^{\lambda k t}$ for $k \geq -1$. For the sake of simplicity we write the transseries with the same letter $q \equiv q(t)$, $p_1 \equiv p_1(t)$ and $p \equiv p(t)$. Then we have

Lemma 13

Suppose that (5) and (6) are satisfied. Then there exists a formal transseries solution $q_1(t)$ of $\dot{q}_1 = H_{p_1}$ in $\{t|Re(\lambda_j t) < 0,\ j = 2, \ldots, n\}$ such that $(q_1(t), q(t), p_1(t), p(t))$ is the formal transseries solution of (1) in $\{t|Re(\lambda_j t) < 0,\ j = 2, \ldots, n\}$.

Theorem 1 follows from Lemma 13.

Summability of first integrals

Set $\kappa = 2\sigma - 1$ and $\lambda := (\lambda_2, \ldots, \lambda_n)$. Let $\alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^{n-1}$ and $k = (k_2, \ldots, k_n) \in \mathbb{Z}_+^{n-1}$. Define

$$S_0(\alpha) := \left\{ z \in \mathbb{C} \mid \kappa z^\kappa + \lambda \cdot (k - \alpha) = 0, \forall k \in \mathbb{Z}_+^{n-1} \setminus \{0\} \right\}. \qquad (41)$$

Let $B_{\nu,0}$ and $B_{\nu,1}$ be given by (4). Assume

$$B_{\nu,0}(q_1, q) = O(q_1^\kappa), \quad B_{\nu,1}(q_1, q) = O(q_1^\kappa), \quad \nu = 2, \ldots, n. \qquad (42)$$

Then we have

Theorem 14

Assume (4), (5), (6) and (42). Let $v = E^\alpha \phi^{(\alpha)}$ $(\alpha = 0, \pm e_j, j = 2, \ldots, n)$ be the formal first integrals given by Theorem 10. Then $\phi^{(\alpha)}$ is κ-summable with respect to q_1 in every direction d such that $d \notin S_0(\alpha)$.

Sketch of proof of Theorem 2

We prove the theorem by five steps.
Step 1. Consider (37)-(39). If we show the summability of q we have the summability of p and p_1 as well.
Set $\kappa = 2\sigma - 1$. Let t and z satisfy $t = -\kappa^{-1} z^{-\kappa}$. Let z_0 be such that $Re(\lambda_j z_0^{-\kappa}) > 0$. Define Σ_0 by (13) with $d = \arg z_0$. We show that there exist constants $C_0 > 0$, $C_1 > 0$ and $\eta > 0$ such that, for Σ_0 given by (13) with $d = \arg z_0$ we have

$$|\kappa z^\kappa + \lambda_j k| \geq C_0 |z|^\kappa, \quad \forall z,\ |z| > C_1, z \in \Sigma_0, \tag{43}$$

$$|\kappa z^\kappa + \lambda_j k| > C_0, \quad \forall z,\ |z| \leq C_1,\ z \in \Sigma_0, \tag{44}$$

for $j = 2, \ldots, n$ and $k = 1, 2, \ldots$.

Step 2. By deleting the unknown functions p, p_1, q_1 from (37)-(39) we obtain the equation of q. Let ζ be the dual variable of z. Let

$$q_0 = \sum_{n=0}^{\infty} c_n(\xi) z^n \tag{45}$$

be the formal series solution. Define $\tilde{q}$ by

$$q = \tilde{q} + \rho, \quad \rho = \sum_{n=0}^{\kappa-1} c_n(\xi) z^n. \tag{46}$$

Clearly we have $\tilde{q} = O(z^\kappa)$. Rewriting $\tilde{q}$ as q we are reduced to solving the equation of q

$$q = G(z, q, \xi). \tag{47}$$

Step 3. $G(z, q, \xi)$ is the formal power series of z with coefficients being holomorphic in ξ and q in some neighborhood of the origin $\xi = 0$, $q = 0$ which is uniform among the coefficients. By expanding the coefficients in the power series of ξ and q and rearranging them we obtain the series of ξ and q whose coefficients are the formal series of z. We show that the coefficients of the series of G with respect to ξ and q are summable in z which are uniform among the coefficients. We denote the uniform summability property by (P).
By the definition of $G = (G_j)$ it is sufficient to show that (P) holds for the first integrals constructed in Theorem 10. Let $C(z, q, \xi)$ be any formal first integral constructed in Theorem 10. For every pair of multiintegers $m \geq 0, n \geq 0$ we consider the coefficient of $q^m \xi^n$ of the Tayler series of $C(z, q, \xi)$

$$C_{m,n}(z) = \frac{1}{(2\pi i)^2} \iint_{|w_j|=\epsilon_1, |s_\nu|=\epsilon_2} \frac{C(z,w,s)}{w^{m+1}s^{n+1}} dwds, \tag{48}$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are small constants. Let $\hat{C}(\zeta, w, \xi)$ be the formal Borel transform of $C(z, w, \xi)$ with respect to z, where ζ is the dual variable of z. By the formal Borel transform of (48) we have

$$\hat{C}_{m,n}(\zeta) = \frac{1}{(2\pi i)^2} \iint_{|w_j|=\epsilon_1, |s_\nu|=\epsilon_2} \frac{\hat{C}(\zeta,w,s)}{w^{m+1}s^{n+1}} dwds. \tag{49}$$

Since $C(z, w, s)$ is Borel summable, there exist Σ_0 in (13) and the neighborhoods V_0 and V_1 of $q = 0$ and $\xi = 0$, respectively, such that $\hat{C}(\zeta, w, s)$ is holomorphic in $(\zeta, w, s) \in \Sigma_0 \times V_0 \times V_1$. Moreover, $\hat{C}(\zeta, w, s)$ is of exponential order of one in $\zeta \in \Sigma_0$ for every $(w, s) \in V_0 \times V_1$.

By the scale change of the variables $q \mapsto \epsilon q$ and $\xi \mapsto \epsilon \xi$ we may assume that V_0 and V_1 contain a disk with sufficiently large radius. Therefore, by (49) we have the summability of $C_{m,n}(z)$ uniformly in m and n. In the following we assume the condition.

Step 4. We prove the summability of q as the transseries. It is sufficient to show the summability with respect to the variable z instead of t. Expand $c_n(\xi)$ in (45) in the power series of ξ and consider

$$q(z) = \sum_{j \geq 0} \xi^j q_j(z). \tag{50}$$

By (46) it is sufficient to show the summability of q in (47). Note that, by the definition of the summability of the transseries it is sufficient to show the uniform summability of q_j's and the convergence of the sum (50) with q_j replaced by its Borel sum.

If $j = 0$, then the summability of $q_0 \equiv 0$ is trivial. Suppose that the uniform summability of q_j for $j = 0, \ldots, k-1$ holds. Namely, the formal Borel transform of q_j, $\hat{q}_j$ is holomorphic in Σ_0 and has the same exponential order for $j = 0, \ldots, k-1$. Consider q_k. Substitute (50) into (47).

Since G is analytic at $q = 0$ we consider the term

$$C_\ell(\xi, z)(\sum_{j,|j|>0} q_j \xi^j)^\ell, \tag{51}$$

where $\ell \geq 0$ is a multiinteger and $C_\ell(\xi, z)$ is analytic in ξ and a formal power series of z. Expand $C_\ell(\xi, z)$ in the power series of ξ, $C_\ell(\xi, z) = \sum_{|\nu|\geq 1} K_{\ell,\nu}(z)\xi^\nu$. We introduce the weight ϵ_0^j in front of q_j by the scale change $\xi \mapsto \epsilon_0 \xi$ (cf. step 4), where $\epsilon_0 > 0$ is a sufficiently small number. Then the coefficient of ξ^k appearing from $G(z, q, \xi)$ is given by

$$\sum \frac{K_{\ell,\nu}(z)\ell!\epsilon_0^{|k|}}{m_1! \cdots m_\mu!} q_{j_1}^{m_1} q_{j_2}^{m_2} \cdots q_{j_\mu}^{m_\mu}, \tag{52}$$

where the summation is taken over the pair of multiintegers, $m_1, \ldots, m_\mu$ satisfying

$$m_1 + \cdots + m_\mu = \ell, \quad j_1|m_1| + j_2|m_2| + \cdots + j_\mu|m_\mu| = k - \nu, \tag{53}$$

where μ is an integer and $j_1, \ldots, j_\mu \geq 0$ are multiintegers.

By the result of Step 3 $K_{\ell,\nu}(z)$ is uniformly summable in ℓ and ν and $\sum_{\ell,\nu} \|K_{\ell,\nu}\| < \infty$. By (52) and (47) we see that the formal Borel transform of $q_k(z)$, $\hat{q}_k(\zeta)$ is holomorphic in Σ_0 and has the same exponential order as $\hat{q}_j$'s.
It remains to estimate $\|q_k\|$, where $\|q_k\|$ is a certain maximal norm. Suppose that

$$\|q_j\| \leq K_1 \epsilon_2^{|j|}, \quad |j| < |k|, \tag{54}$$

for some positive constants K_1 and ϵ_2, where ϵ_2 is chosen sufficiently small. Take $\epsilon_0 \leq 1$ and $2\epsilon_0 < \epsilon_2$. We have

$$\begin{aligned}
&\sum \frac{\ell!}{m_1! \cdots m_\mu!} (\|q_{j_1}\|)^{m_1} (\|q_{j_2}\|)^{m_2} \cdots (\|q_{j_\mu}\|)^{m_\mu} \\
&\leq \sum \frac{\ell!}{m_1! \cdots m_\mu!} (K_1 \epsilon_2^{|j_1|})^{|m_1|} (K_1 \epsilon_2^{|j_2|})^{|m_2|} \cdots (K_1 \epsilon_2^{|j_\mu|})^{|m_\mu|} \\
&\leq (K_1 \sum_{j,|j|>0} \epsilon_2^{|j|})^{|\ell|} \leq (C K_1 \epsilon_2)^{|\ell|},
\end{aligned} \tag{55}$$

where the summation is taken over all combinations satisfying (53) and where C satisfies $\sum_{j,|j|>0} \epsilon_2^{|j|} \leq C\epsilon_2$.

Then the term (52) is estimated by

$$\epsilon_2^k \sum_{\ell,\nu} (CK_1\epsilon_2)^{|\ell|} \|K_{\ell,\nu}\|. \tag{56}$$

By taking ϵ_2 sufficiently small we have

$$\sum_{\ell\neq 0,\nu} (CK_1\epsilon_2)^{|\ell|} \|K_{\ell,\nu}\| \leq \frac{K_1}{2}. \tag{57}$$

On the other hand we may assume $\|K_{0,\nu}\| \leq K_1/2$ since $\nu \geq 1$. Hence (56) is estimated by $K_1\epsilon_2^n$, which proves the convergence of the sum. Step 5. We prove the summability of q_1. If we prove the summability of q_1 and q, then we have the summability of p_1 and p as well.

Thank you very much for your attention !!

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