Classification and structure

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Contents

Outline

Observation

Most classification results by K-theoretic invariants apply to classes of C^*-algebras $\mathfrak{A}$ enjoying at least one of the following properties:

- $\mathfrak{A}$ is simple
- $\mathfrak{A}$ is stably finite with real rank zero
- $\mathfrak{A}$ is purely infinite

Takeaway

The classification theory for graph C^*-algebras applies in some cases satisfying none of these properties.

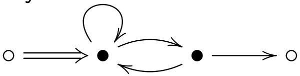
Definition

A graph is a tuple (E^0, E^1, r, s) with

$$r, s : E^1 \to E^0$$

and E^0 and E^1 countable sets.

We think of $e \in E^1$ as an edge from $s(e)$ to $r(e)$ and often represent graphs visually



or by an adjacency matrix

$$\mathsf{A}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Singular and regular vertices

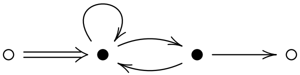
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular* if v is a sink or an infinite emitter. v is *regular* if it is not singular.



Graph algebras

Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Theorem

Gauge invariant ideals are induced by **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \implies r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \implies$ *[$v \in V$ or v is singular]*

and when there are no **breaking vertices**, *all such ideals arise this way.*

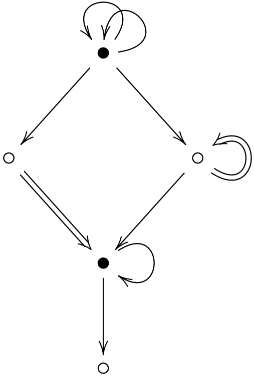
The gauge simple case

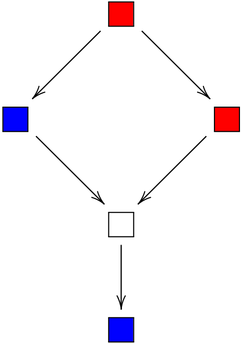
Theorem

If a graph C^*-algebra has no non-trivial gauge invariant ideals, it is either

- *a simple AF algebra;*
- *a Kirchberg algebra; or*
- $C(\mathbb{T}) \otimes \mathbb{K}(H)$ *for some Hilbert space H.*

It is easy to tell from the graph which case occurs: The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.





Filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with only finitely many gauge invariant ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with gauge invariant $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $FK^\gamma(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $FK^{\gamma,+}(\mathfrak{A})$.

$FK^{\gamma,+}(C^*(E))$ is readily computable when $|E^0| < \infty$.

The unital case

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

1. $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$
2. *There is a finite sequence of moves of type*

   (S),**(R)**,**(O)**,**(I)**,**(C)**,**(P)**

   and their inverses, leading from E to F.
3. $FK^{\gamma,+}(C^*(E)) \simeq FK^{\gamma,+}(C^*(F))$

The classification result is strong, and isomorphism is decidable.

The unital case

Theorem (E-Restorff-Ruiz-Sørensen, Arklint-E-Ruiz)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras. Then the following are equivalent

1. $C^*(E) \simeq C^*(F)$
2. *There is a finite sequence of moves of type*

   (R+),**(O)**,**(I+)**,**(C+)**,**(P+)**

   and their inverses, leading from E to F.
3. $(FK^{\gamma,+}(C^*(E)), [1_{C^*(E)}]) \simeq (FK^{\gamma,+}(C^*(F)), [1_{C^*(F)}])$

The classification result is strong, and isomorphism is decidable.

Outline

Weaknesses of [ERRS]

- Unitality required
- Finitely many gauge invariant ideals
- Internal classification only

The gauge simple case

Theorem

The invariant

$$K_*(-) = [K_0(-), K_0(-)_+, K_1(-)]$$

is complete for the class of gauge simple graph algebras up to stable isomorphism.

Theorem

If a C^*-algebra $\mathfrak{A}$ is either

- *an AF algebra;*
- *a Kirchberg algebra with UCT; or*
- *$C(\mathbb{T}) \otimes \mathbb{K}(H)$ for some Hilbert space H.*

and if for some graph E we have $K_*(\mathfrak{A}) \simeq K_*(C^*(E))$, then $\mathfrak{A} \otimes \mathbb{K} \simeq C^*(E) \otimes \mathbb{K}$.

Working conjecture [E-Restorff-Ruiz 2010]

$FK^{\gamma,+}(-)$ is a complete invariant, up to stable isomorphism, for graph C^*-algebras of real rank zero (*i.e.*, with no $\square$ subquotients) and finitely many ideals.

No counterexamples are known, not even allowing for $\square$ subquotients, but then we would have to say:

Conjecture

$FK^{\gamma,+}(-)$ is a complete invariant, up to stable isomorphism, for graph C^*-algebras with finitely many gauge invariant ideals.

Unmixed graph C^*-algebras: General classification

Theorem

When a graph algebra has only finitely many gauge invariant ideals, and they are exclusively of one type ■,■, or □, then $FK^{\gamma,+}(-)$ is a complete invariant up to stable isomorphism.

- The ■ case was done by Elliott with no ideal restrictions in 1978.
- The ■ case was solved by Bentmann and Meyer in 2014, but using a different invariant. Unpublished work by Restorff and Ruiz shows that the $FK^{\gamma,+}(-)$ invariant is complete too.
- The □ case can be shown from [ERRS] by passing to a (unital) full corner

Unmixed graph C^*-algebras: External classification

Theorem

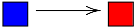
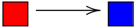
If a C^*-algebra $\mathfrak{A}$ is either

- ■ *an AF algebra;*
- ■ *a purely infinite C^*-algebra with finitely many ideals and UCT;*

and if for some graph E we have $FK^{\gamma,+}(\mathfrak{A}) \simeq FK^{\gamma,+}(C^*(E))$, where E must be finite with no sinks in the latter case, then $\mathfrak{A} \otimes \mathbb{K} \simeq C^*(E) \otimes \mathbb{K}$.

- The ■ case was done by Elliott 1978.
- The ■ case was done by Bentmann, proving that there are no "phantom Cuntz-Krieger" algebras.

Mixed graph C^*-algebras

Case	Classified?	Strongly?
	√	(f.g.)
	√	(f.g.)
	(f.g)	÷
	(f.g)	÷
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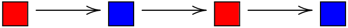
	√	(f.g)
	÷	÷

Avenues for progress?

- Ordering the Bentmann-Meyer invariant
- Elliott intertwining with select morphisms and unital building blocks
- Fullness
- Semiprojectivity
- Algebraic methods, cf. the Abrams-Tomforde conjectures

Outline

Preserving γ

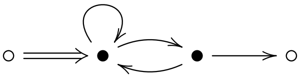
$$\begin{array}{ccc} C^*(E) & \xrightarrow{\varphi} & C^*(F) \\ \gamma\downarrow & & \downarrow\gamma \\ C^*(E) & \xrightarrow[\varphi]{} & C^*(F) \end{array}$$

$$\begin{array}{ccc} C^*(E)\otimes\mathbb{K} & \xrightarrow{\varphi} & C^*(F)\otimes\mathbb{K} \\ \gamma\otimes\mathrm{id}_{\mathbb{K}}\downarrow & & \downarrow\gamma\otimes\mathrm{id}_{\mathbb{K}} \\ C^*(E)\otimes\mathbb{K} & \xrightarrow[\varphi]{} & C^*(F)\otimes\mathbb{K} \end{array}$$

Note that such φ must preserve

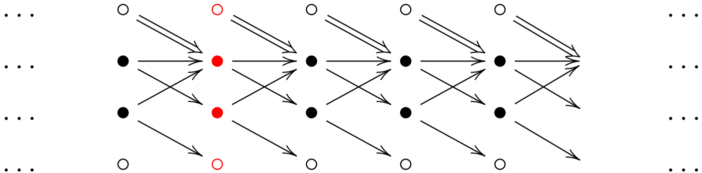
$$C^*(E)^\gamma = \{x \in C^*(E) \mid \forall t \in \mathbb{T} : \gamma_t(x) = x\}$$

Fixed point algebra



Theorem

$C^*(E)^\gamma$ is itself a corner of a graph C^*-algebra which is AF. It is best described as $1^0 C^*(E \times_1 \mathbb{Z}) 1^0$ with 1^0 and $E \times_1 \mathbb{Z}$ as indicated below.



Conjecture

Definition

For E a graph with finitely many vertices we define the **dimension triple** by

$$\mathcal{DT}(E) := (K_0(C^*(E \times_1 \mathbb{Z})), K_0(C^*(E \times_1 \mathbb{Z}))_+, \sigma_*)$$

Here, σ is the natural right shift on $E \times_1 \mathbb{Z}$.

Conjecture (Hazrat, E-Ruiz)

- $(C^*(E), \gamma) \simeq (C^*(F), \gamma) \iff (\mathcal{DT}(E), [1^0]) \simeq (\mathcal{DT}(F), [1^0])$
- $(C^*(E) \otimes \mathbb{K}, \gamma \otimes \mathrm{id}_{\mathbb{K}}) \simeq (C^*(F), \gamma \otimes \mathrm{id}_{\mathbb{K}}) \iff (\mathcal{DT}(E), I([1^0])) \simeq (\mathcal{DT}(F), I([1^0]))$

The conjecture is known to hold when $C^*(E)$ is AF, or a simple Cuntz-Krieger algebra, and in a few other cases.

Avenues for progress?

- Cuntz-Pimsner algebras
- Equivariant KK-theory
- Shift equivalence
- Algebraic methods, cf. the Hazrat conjectures

Outline

Definition

$\mathfrak{A}$ is **semiprojective** when

$$\begin{array}{ccccccccc} & & & & & & & & \mathfrak{A} \\ & & & & & \swarrow & & & \downarrow \\ \mathfrak{B}/\mathfrak{I}_1 & \longrightarrow & \mathfrak{B}/\mathfrak{I}_2 & \longrightarrow \dots \longrightarrow & \mathfrak{B}/\mathfrak{I}_{n_0} & \longrightarrow & \dots & \longrightarrow & \mathfrak{B}/\overline{\bigcup \mathfrak{I}_n} \end{array}$$

This notion due to Blackadar is a key concept in all C^*-algebra theory, including classification.

Definition

$\mathfrak{A}$ is **weakly semiprojective** when

$$\begin{array}{ccc} & & \mathfrak{A} \\ & \swarrow & \downarrow \\ \prod_{n=1}^{\infty} \mathfrak{B}_n & \longrightarrow & \prod_{n=1}^{\infty} \mathfrak{B}_n / \sum_{n=1}^{\infty} \mathfrak{B}_n \end{array}$$

Theorem

Let $C^*(E)$ be a gauge simple graph C^*-algebra.

- When $C^*(E)$ is AF, it is semiprojective precisely when it is unital.
- When $C^*(E)$ is a Kirchberg algebra, it is semiprojective precisely when it has finitely generated K_*. (Spielberg)
- When $C^*(E) \simeq C(\mathbb{T}) \otimes \mathbb{K}(H)$, it is semiprojective precisely when it is unital (i.e. $\dim H < \infty$).

Corollary (Szymański)

Any simple and unital graph C^*-algebra is semiprojective.

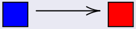
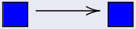
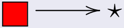
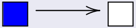
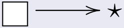
Theorem

Let $C^*(E)$ be a gauge simple graph C^*-algebra.

- When $C^*(E)$ is AF, it is weakly semiprojective precisely when it is unital.
- When $C^*(E)$ is a Kirchberg algebra, it is weakly semiprojective precisely when K_* is a direct sum of cyclic groups. (Spielberg, Lin)
- When $C^*(E) \simeq C(\mathbb{T}) \otimes \mathbb{K}(H)$, it is weakly semiprojective precisely when it is unital (i.e. $\dim H < \infty$).

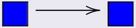
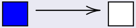
Lemma (E-Katsura)

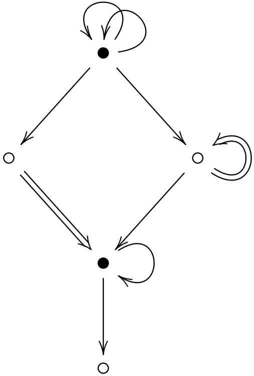
Among the unital graph algebrs with one non-trivial ideals, we have

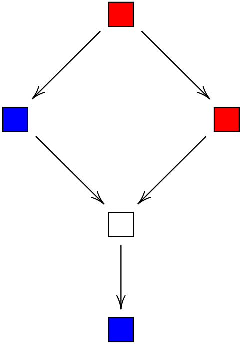
Semiprojective	$\neg$ Semiprojective
	
	
	

Theorem (E-Katsura)

Let $C^*(E)$ be a unital graph algebra. The following are equivalent

- $C^*(E)$ *is semiprojective*
- $C^*(E)$ *is weakly semiprojective*
- *There are no*  *or*  *subquotients*





Avenues for progress?

- The  case
- Weak semiprojectivity of AF algebras
- F.g. of ordered groups