Banach Center
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NEW MORPHISMS OF GRAPHS

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Graph Algebras, 2023–2026

Staff Exchange network of 78 mathematicians from 26 places:

IMPAN (Poland), University of Warsaw (Poland), University of Wrocław (Poland), Jagiellonian University (Poland), University of Copenhagen (Denmark), University of Southern Denmark, Odense (Denmark), SISSA, Trieste (Italy), University of Naples Federico II (Italy), Leiden University (Netherlands), University of Göttingen (Germany), University of Haifa (Israel), University of Hawai'i at Mānoa (USA), University of California at Berkeley (USA), Pomona College (USA), Arizona State University (USA), University of Denver (USA), University of Colorado at Boulder (USA), University of Colorado at Colorado Springs (USA), University of Kansas at Lawrence (USA), Kansas State University (USA), Penn State University (USA), State University of New York at Buffalo (USA), Fields Institute (Canada), Western University (Canada) University of New Brunswick at Fredericton (Canada), University Michoacana de San Nicolás de Hidalgo (Mexico).

Morphisms of graphs

Definition

A homomorphism $f\colon E \to F$ of graphs is a pair of maps

$$(f^0 : E^0 \to F^0, f^1 : E^1 \to F^1)$$

satisfying the conditions:

$$s_F \circ f^1 = f^0 \circ s_E\,, \qquad t_F \circ f^1 = f^0 \circ t_E\,.$$

We call it proper iff both these maps are finite-to-one.

Definition

A path homomorphism of graphs is a map $f\colon FP(E) \to FP(F)$ satisfying:

1. $f(E^0) \subseteq F^0$,
2. $s_F \circ f = f \circ s_E\,, \quad t_F \circ f = f \circ t_E\,,$
3. $\forall\, p, q \in FP(E)$ such that $t(p) = s(q)$: $f(pq) = f(p)f(q)$.

We call it proper iff the above map is finite-to-one.

Contravariant conditions

1. POG is the category of graphs and proper graph homomorphisms.

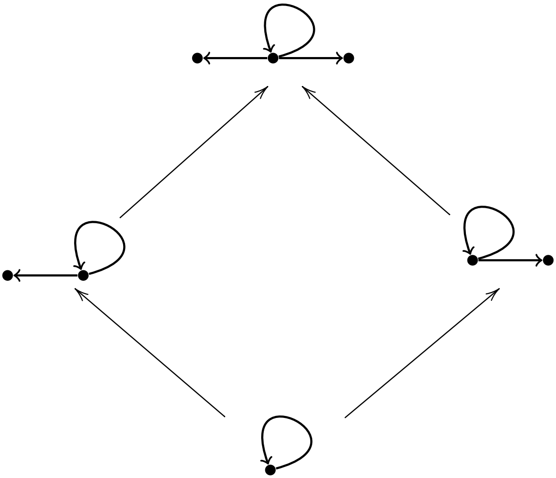
2. $TBPOG$ is the subcategory of POG whose morphisms $f\colon E \to F$ satisfy the target-bijectivity condition:

$$\forall\, x \in F^1\colon\ f^{-1}(x) \ni e \longmapsto t_E(e) \in f^{-1}(t_F(x)) \text{ is bijective.}$$

3. $CRTBPOG$ is the subcategory of $TBPOG$ whose morphisms $f\colon E \to F$ satisfy

$$f\big(E^0 \setminus \mathrm{reg}(E)\big) \subseteq F^0 \setminus \mathrm{reg}(F).$$

Equatorial Podleś quantum sphere



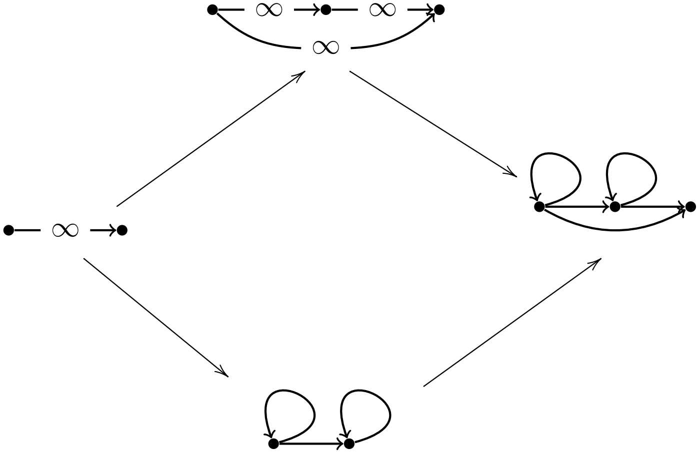
Theorem (P. M. H., M. Tobolski)

A pushout-to-pullback theorem for Leavitt path algebras and graph C-algebras.*

Covariant conditions

1. IPG is the category of graphs and path homomorphisms of graphs that are injective when restricted to vertices.
2. $MIPG$ is the subcategory of IPG whose morphisms satisfy
$$f(e) \preceq f(e') \quad \Rightarrow \quad e = e'$$
when restricted to the sets of edges.
3. $RMIPG$ is the subcategory of $MIPG$ whose morphisms satisfy the regularity conditions:
   (A) For any $v \in \mathrm{reg}(E)$, the vertex $f(v)$ emits $|s_E^{-1}(v)|$-many positive-length paths $p_1, \ldots, p_{n_v}$, $n_v := |s_E^{-1}(v)|$, whose all edges begin at regular vertices. Also, we require that the set $FP_{f(v)} := \{p_1, \ldots, p_{n_v}\}$ is constructed in the following way: we take $x \in s_F^{-1}(f(v))$ and either set it aside as a length-one element of $FP_{f(v)}$, or extend it by all edges emitted from $t_F(x)$. Any thus obtained path of length two, we either set aside as an element of $FP_{f(v)}$, or extend it by all edges emitted from its end. Then we iterate this procedure until we obtain the set $FP_{f(v)}$.
   (B) For any $v \in \mathrm{reg}(E)$, the map f when restricted to $s_E^{-1}(v)$ is a bijection onto $FP_{f(v)}$.

Quantum $\mathbb{C}\mathrm{P}^2$



Theorem (A. Chirvasitu, P. M. H., M. Tobolski)

A pullback theorem for graph C-algebras.*

Main research objectives

- Develop new and most general concepts of morphisms of graphs that make the assignment of graph C*-algebras to graphs either a contravariant functor turning pushouts of graphs into pullbacks of graph C*-algebras, or a covariant functor respecting limits.
- Unify the pushout-to-pullback theorems by proving them for higher-rank graphs allowing both sinks and infinite emmiters.

First steps

- Examples of non-injective pushouts with breaking vertices in $CRTBPOG$.
- Translate pushout conditions for relative graphs into pushout conditions for induced graphs.
- Extend the pushout-to-pullback theorem to higher-rank graphs and Cuntz–Pimsner algebras.
- Relax the covariant conditions by allowing shrinking single loops at sinks, study examples.
- Extend the CHT pullback theorem to higher-rank graphs starting with trimmable graphs.