SDU
UNIVERSITY OF
SOUTHERN DENMARK

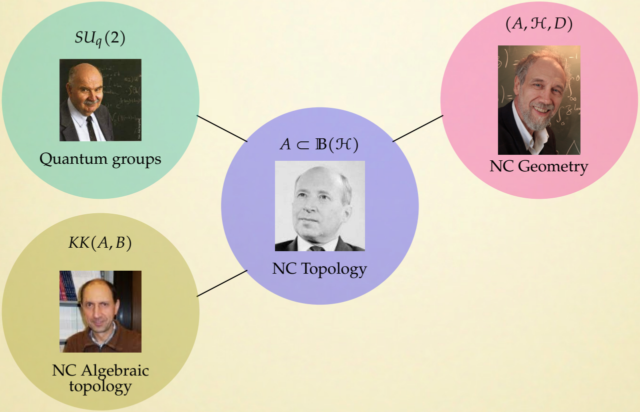
Current trends in non-commutative metric geometry

Operator Algebras That One Can See – Kick-off meeting

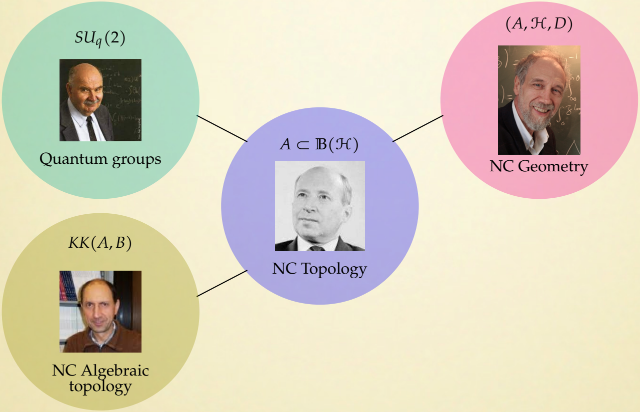
Januar 17, 2023

David Kyed

THE NON-COMMUTATIVE LANDSCAPE



The non-commutative landscape



Question (Rieffel, 1990's)

What is the non-commutative analogue of a compact metric space?

⇝ definition

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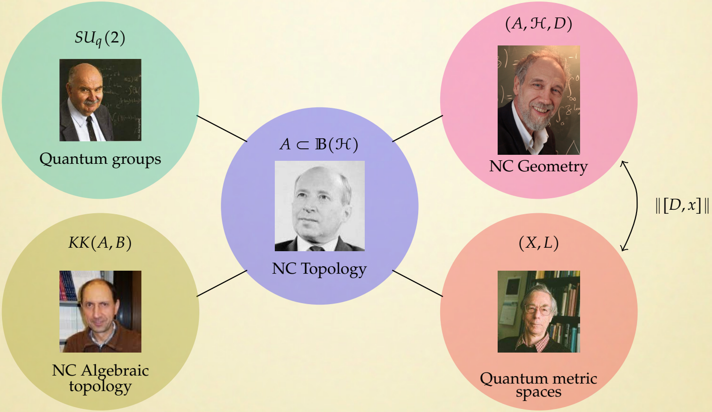

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COMPACT QUANTUM METRIC SPACES

Compact Quantum Metric Spaces

Definition (Rieffel)

Let A be a unital C-algebra equipped with a seminorm $L\colon A \to [0,\infty]$ satisfying that $L(x^*) = L(x)$ for all $x \in A$. Then (A, L) is called a compact quantum metric space if*

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(iii) $d_L(\mu, \nu) := \sup\{|\mu(a) - \nu(a)| : L(a) \leqslant 1\}$ *metrises the weak*-topology on* $\mathcal{S}(A)$.

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Example

If (X, d) is a compact metric space then $C(X)$ becomes a CQMS by setting $L_d(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x \neq y\right\}$.

⇝ NC examples

Examples

- If $G \overset{\alpha}{\curvearrowright} A$ is an ergodic, strongly continuous action of a compact metric group on a unital C^*-algebra A.

  Then one gets a Lip-norm [Rieffel, 1998] by setting

  $$L(a) := \sup\left\{ \frac{\|\alpha_g(a) - a\|}{d_G(g,e)} \;\middle|\; g \in G, g \neq e \right\}$$

  Examples include:

  (A) $SU(2) \curvearrowright \mathbb{M}_{n+1}(\mathbb{C})$ via conjugation with the $(n+1)$-dimensional irreducible representation.

  (B) $\mathbb{T}^2 \curvearrowright C(\mathbb{T}^2_\theta)$ by rescaling generators on the NC torus.

- If $(\mathcal{A}, H, D)$ is a spectral triple then the *Connes seminorm*

  $$L_D(a) := \|[D,a]\| \quad (a \in \mathcal{A})$$

  sometimes, but not always, gives rise to a CQMS (examples coming soon).

⇝ q-GH-dist

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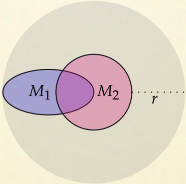
- A prominent feature is the analogue of the classical Gromov-Hausdorff distance between compact metric spaces



- Rieffel's original version [Rieffel, 2004] provides a distance function $\mathrm{dist}^{\mathrm{Q}}_{\mathrm{GH}}$ between compact quantum metric spaces.
- It is symmetric and satisfies the triangle inequality...
  ...but distance zero only means Lip-norm preserving isomorphism of the underlying order unit spaces, and not $*$-isomorphism!
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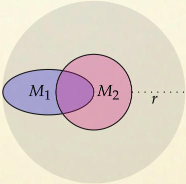

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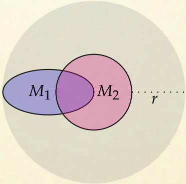

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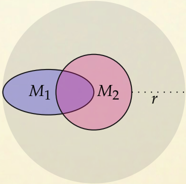

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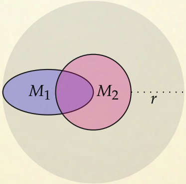

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Convergence and continuity results

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2. Non-commutative tori [Rieffel, 2004]
3. Spectral truncations [D'Andrea-Lizzi-Martinetti, 2014] [van Suijlekom, 2021]
4. Crossed products [Kaad-K, 2020]
5. Non-commutative solenoids [Latrémolière-Packer, 2017]
6. AF-algebras [Aguilar-Latrémolière, 2015]
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So what is left to do?

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METRIC- VERSUS DIFFERENTIAL NCG

- A lot of current research goes into the connections between Connes' NCG and compact quantum metric spaces.
- On the propinquity side, Latrémolière and co-authors are currently working on a *spectral propinquity*.

  This is applicable to spectral triples (A, H, D) for which the Connes seminorm $L_D(a) = \|[D, a]\|$ gives rise to a CQM structure.

  And is designed so that propinquity zero corresponds to unitary equivalence of the spectral triples.

  Convergence in spectral propinquity implies a "pointwise convergence" of the Dirac operators' spectra [Latrémolière, 2021].
- Also the original approximation of S^2 by fuzzy spheres is being upgraded in a more geometric direction [van Suijlekom, 2021], [Rieffel, 2022] and so is the theory for NC solenoids [Farsi-Landry-Larsen-Packer, 2022] $\rightsquigarrow$ *q*-deformations

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q-DEFORMED SPACES

- Also *q-deformations* is an active area of research
- The quantised 2-sphere S^2_q fits into a spectral triple [Dąbrowski-Sitarz, 2003]
- And the Connes seminorm gives a CQMS [Aguilar-Kaad, 2018]
- Moreover, $\lim_{q\to 1} S^2_q = S^2$ in dist^Q_{GH} [Aguilar-Kaad-K, 2021]
- Also Woronowicz' $SU_q(2)$ can be endowed with a Dirac operator (not a full spectral triple) and a seminorm L_q turning it into a CQMS [Kaad-K, 2022]
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- Ongoing work treats the quantum projective plane $\mathbb{C}P^2_q$, the quantum 5-sphere and higher dimensional analogues [Kaad-Mikkelsen]
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- Also Woronowicz' $SU_q(2)$ can be endowed with a Dirac operator (not a full spectral triple) and a seminorm L_q turning it into a CQMS [Kaad-K, 2022]
- And also here we have $\lim_{q \to 1} SU_q(2) = SU(2)$ [Kaad-K, 2022]
- Ongoing work treats the quantum projective plane $\mathbb{C}P_q^2$, the quantum 5-sphere and higher dimensional analogues [Kaad-Mikkelsen]

  Note the graph algebra connection via [Hong-Szymański, 2022]

⇝ groups

Group related research avenues

- Given a discrete group Γ and a proper length function $\ell : \Gamma \to \mathbb{R}_{\geqslant 0}$ it is a difficult problem to determine if the Dirac operator
$$D_\ell(\delta_\gamma) := \ell(\gamma)\delta_\gamma$$
gives rise to a CQMS.
No counter examples known for word lenghts!
Positive results for hyperbolic and nilpotent groups
[Ozawa-Rieffel, 2005], [Christ-Rieffel, 2015], [Long-Wu, 2021] .
- Much more work is needed in order to clarify this.

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THANK YOU!