# Graphs and Groupoids

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Graph Algebras Kick Off Meeting

16 January 2023

# Motivation for the Work Package

- Many C*-algebras are (stably) isomorphic to (twisted) graph groupoid C*-algebras.

- They form an important class of combinatorially definable C*-algebras [Kumjian–Pask–Raeburn–Renault]

- From the very beginning of Renault's idea of modeling C*-algebras by groupoids, the question of relating

  - the combinatorial or topological structure of the model
  - the invariants of the C*-algebra

  has been regarded as a fundamental problem.

- Time has ripened for links between graph properties and C*-algebra properties to be established via graph groupoids.

# Groupoids

- Essentially Zakrzewski's idea

Groupoid = Group object in the symmetric monoidal category
of sets as objects and relations as morphisms

- The Zakrzewski functor

$$(\text{Groupoids, Z-relations}) \longrightarrow (\text{C*-algebras, *-homomorphisms})^{\text{op}}$$

# Zakrzewski relations between graphs

- The Zakrzewski functor restricted to the groupoids of one-sided infinite paths of directed graphs defines Z-relations between directed graphs and leads to a functor

$$(\text{Graphs, Z-relations}) \longrightarrow (\text{C*-algebras, *-homomorphisms})^{\text{op}}$$

categorifying the graph (and k-graph) C*-algebra construction.

- This relates the Zakrzewski category of groupoids to the *New morphisms of graphs* Work Package.

# Bicolimits in the bicategory of C*-correspondences

[Albandik–Meyer, Meyer–Fabre Sehnem]

**Quantized relations = C*-correspondences**

- Crossed products,
- Cuntz–Pimsner algebras,
- (topological) graph C*-algebras,
- groupoid C*-algebras,
- actions of Ore monoids on spaces by topological correspondences.

- Properties of the underlying groupoid are related to
  - the ideal structure and the pure infiniteness of the C*-algebra.

# Morita equivalence of graph C*-algebras

[Abrams–Tomforde, Eilers–Restorff–Ruiz–Sørensen]

- This translates into graph moves, and descends to an analogous concept at the level of the corresponding groupoids.

- However, in many cases, it is unclear how (*K*-theoretical) invariants and C*-algebraic properties translate to groupoid invariants.

  - Examples and counterexamples.

# Invariants of combinatorially defined C*-algebras

[Temperley–Lieb, Jones, Baxter, Martin, Martin–Saleur]

## The Temperley-Lieb algebra and the Jones polynomial

- statistical mechanics,
- integrable models,
- graph theory,
- knot theory,
- the braid group actions,
- quantum groups
- subfactors of von Neumann algebras

# Objectives of the Work Package

- Establishing links between graph properties and C*-algebra properties via graph groupoids.

- Finding topological and geometrical properties of groupoids allowing us to study, adapt and reinterpret their C*-algebras from the point of view of the Baum–Connes conjecture and related conjectures (e.g. the Matui HK conjecture).

- Extending the results about actions of Ore monoids on étale groupoids.

- Extending the Jones construction to a wider class of combinatorial structures like graphs, diagrams, categories (e.g., groupoids), etc.

# Proposed research methods

- In the first two objectives, case-by-case studies

- Third objective aims to integrate methods of groupoid models with methods worked out in search of C*-algebraic properties of the C*-algebras of

  - self-similar groups [Nekrashevych],
  - self-similar graphs [Exel–Pardo],
  - self-similar *k*-graph [Hui Li–Dilian Yang],
  - irreversible algebraic dynamical systems [Stammeier].

- For the fourth objective, we plan to fuse the methods of quantum groups, integrable systems, and topology with the fact of explicit computability of the K-theory of graph C*-algebras [Raeburn–Szymański, Drinen–Tomforde].