Principal equivariant spectral triples

Operator algebras that one can see

Graph algebras kick-off meeting

Bram Mesland

Mathematisch Instituut
Universiteit Leiden, Netherlands

January 17th, 2023

Noncommutative differential geometry connects geometry to C^*-algebras via *spectral triples*.

Principal bundles are a central notion in geometry and physics:

- G compact Lie group;
- P a compact G-space on which G acts freely and properly with quotient $B := P/G$;
- C^*-algebras $C(P)$ and $C(B) \simeq C(P)^G$;
- $\pi : P \to B$ smooth Riemannian submersion;
- $\pi_! : K^*(P) \to K^*(B)$ associated *wrong way map* in K-theory;
- $\pi_!$ is induced by a class $[\pi_!] \in KK_*(C(P), C(B))$.

In recent joint work with B. Ćaćić we studied these structures on an abstract G-C^*-algebra A.

Definition (Ellwood)

A G-C^*-algebra is *principal* if the map

$$\Phi : A \otimes A \to C(G, A), \quad \Phi(a_1 \otimes a_2)(g) := \alpha_g(a_1)a_2,$$

has dense range.

Fact: G acts freely on $P \Leftrightarrow C(P)$ is a principal G-algebra. Thus principal G-algebras give a notion of noncommutative topological principal bundles. We have extended this notion to the context of *G-spectral triples*.

Definition

A *G-spectral triple* $(\mathcal{A}, H, D)$ consists of

- a Hilbert space $H = H_+ \oplus H_-$ such that A and G are covariantly represented on $H_\pm$;
- $D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$ for a closed G-invariant operator $D_+ : H_+ \to H_-$, $D_- := D_+^*$ such that $a(1 + D^2)^{-1}$ is compact a operator,
- a dense subalgebra $\mathcal{A} \subset A$ such that for $a \in \mathcal{A}$ the commutators $[D_+, a]$ are bounded.

In particular $(\mathcal{A}, H, D)$ defines a class in K-homology $K_G^*(A)$.

Given $(\mathcal{A}, H, D)$ we can form (A^G, H^G, D^G) which is a spectral triple as well. The inclusion $A^G \to A$ can be viewed as the analogue of $\pi : P \to B$.

For a pair of C^*-algebras Kasparov constructed graded abelian group $KK_*(A, B)$ such that

- $KK_*(\mathbb{C}, A) \simeq K_*(A)$, the K-theory of A;
- $KK_*(A, \mathbb{C}) \simeq K^*(A)$, the K-homology of A;
- associative, bilinear product
  $KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C)$;
- recovers index pairing via
  $KK_0(\mathbb{C}, A) \times KK_0(A, \mathbb{C}) \to KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$;
- $x \in KK_*(A, B)$ defines a map $K^*(A) \to K^*(B)$;
- Question: can we think of $x \in KK_*(A, B)$ as being induced by a "morphism" of spectral triples?

Let M and N be manifolds. A *geometric correspondence* is a diagram

$$M \xleftarrow{f} (Z, E) \xrightarrow{g} N,$$

where Z is a manifold, E a vector bundle, f a smooth proper map and g a smooth K-oriented map.

Theorem (Connes-Skandalis)

The group $KK_*(C_0(M), C_0(N))$ is exhausted by equivalence classes of geometric correspondences and the Kasparov product

$$KK_*(C_0(M), C_0(L)) \times KK_0(C_0(L), C_0(N)) \to KK_0(C_0(M), C_0(N)),$$

is,up to transversality, given by the fiber product of correspondences:

$$[M \xleftarrow{f_1} (Z_1, E_1) \xrightarrow{g_1} L] \otimes [L \xleftarrow{f_2} (Z_2, E_2) \xrightarrow{g_2} N]$$
$$= [M \xleftarrow{f_1} (Z_1 \times_L Z_2, E_1 \times_L E_2) \xrightarrow{g_2} N]$$

KK-theory is given by homotopy classes of generalised smooth maps.

Given a spectral triple $(\mathcal{B}, H, D)$ there is a canonical module of 1-forms

$$\Omega^1_D := \overline{\left\{\sum_i b_i^0[D, b_i^1] : b_i^k \in \mathcal{B}\right\}}.$$

If $\mathcal{X}$ is a right inner product $\mathcal{B}$-module with completion X, a *connection* is a map $\nabla : \mathcal{X} \to X \otimes_B^h \Omega^1_D$ satifying $\nabla(xb) = \nabla(x)b + x \otimes [D, b]$.

Definition

Let $(\mathcal{A}, H_0, D_0)$ and $(\mathcal{B}, H_1, D_1)$ be spectral triples. A *differential correspondence between* $(\mathcal{A}, H_0, D_0)$ and $(\mathcal{B}, H_1, D_1)$ is a triple $(\mathcal{X}, S, \nabla)$ such that

$$\begin{aligned}(\mathcal{X}, S, \nabla) \otimes_{\mathcal{B}} (\mathcal{B}, H_1, D_1) &:= (\mathcal{A}, X \otimes_B H_1, S \otimes 1 + 1 \otimes_\nabla D_1) \\ &\simeq (\mathcal{A}, H_0, D_0),\end{aligned}$$

where $\simeq$ is a combination of unitary equivalence and "admissible" (possibly unbounded) perturbations.

Goal: For a principal G-spectral triple $(\mathcal{A}, H, D)$, construct a differential correspondence between $(\mathcal{A}, H, D)$ and $(\mathcal{A}^G, H^G, D^G)$.

The conditional expectation

$$A \to A^G, \quad a \mapsto \int_G \alpha_g(a)\mathrm{d}g,$$

can be used to construct a (A, A^G) bimodule $L^2_v(A)$.

Using Clifford algebra techniques, this module carries a natural vertical operator

$$S_{\mathfrak{g},\rho} = \varepsilon^i \cdot \varepsilon_i - \frac{1}{6}\langle \varepsilon_i, (\rho^{-1})^t[\varepsilon_j, \varepsilon_k]\rangle \varepsilon^i \varepsilon^j \varepsilon^k.$$

Here

- $\langle \cdot, \cdot \rangle$ is a fixed Ad-invariant inner product on $\mathfrak{g}$;
- $\varepsilon_i, \varepsilon^i$ are a dual pair of bases for $\mathfrak{g}, \mathfrak{g}^*$;
- ρ is a *vertical Riemannian metric*.

Proposition

If $(\mathcal{A}, H, D)$ is a G-spectral triple with A a principal G-C^*-algebra, then $(L^2_v(A), S_{\mathfrak{g},\rho})$ is a KK-cycle for (A, A^G).

Next step: find a connection on $L^2_v(A)$. Note that $H = L^2_v(A) \otimes_{A^G} H^G$ and write $D_v := S_{\mathfrak{g},\rho} \otimes 1$.

Definition

Let $(\mathcal{A}, H, D)$ be a G-spectral triple. A *strong remainder* is an odd G-invariant operator Z on H such that

$$[D - D_v - Z, A] \subset A \cdot [D - D_v - Z, A^G].$$

Theorem (Ćaćić-Mesland)

Given a strong remainder Z, there is an induced module connection

$$\nabla^Z : \mathcal{A} \to L^2_v(A) \otimes^h_{A^G} \Omega^1_{D^G},$$

such that $(L^2_v(A), S_{\mathfrak{g},\rho}, \nabla^Z)$ is a noncommutative correspondence between $(\mathcal{A}, H, D)$ and $(\mathcal{A}^G, H^G, D^G)$.

So far our result applies to

- all classical principal bundles;
- θ-deformations of classical principal bundles such as $S^7_\theta \to S^4_\theta$ with fiber $G = S^3$;
- crossed products by $\mathbb{Z}^N$.

One important research effort will be to construct more examples, such a Cuntz-Krieger and graph C^*-algebras.

Further work will be done in the following directions:

- replace the G by a *compact quantum group*;
- noncompact structure groups;
- fully develop noncommutative gauge theory;
- construct more examples.