Quantum Symmetries

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Quantum Symmetries

Objectives:

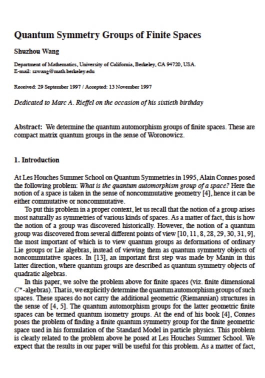
Our aim is to study, in full generality, the notion of quantum symmetry and isometry groups of graph C*-algebras, including these associated with higher-rank graphs.

Why:

Graph C^*-algebras comprise a vast range of noncommutative algebras, which are well known to *have symmetries*.

Examples: $M_n(\mathbb{C})$, Toeplitz algebra, quantum sphere(s), $SU_q(2)$ ...

Quantum symmetries



Quantum Symmetry Groups of Finite Spaces

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Dedicated to Marc A. Rieffel on the occasion of his sixtieth birthday

Abstract: We determine the quantum automorphism groups of finite spaces. These are compact matrix quantum groups in the sense of Woronowicz.

1. Introduction

At Les Houches Summer School on Quantum Symmetries in 1995, Alain Connes posed the following problem: *What is the quantum automorphism group of a space?* Here the notion of a space is taken in the sense of noncommutative geometry [4], hence it can be either commutative or noncommutative.

To put this problem in a proper context, let us recall that the notion of a group arises most naturally as symmetries of various kinds of spaces. As a matter of fact, this is how the notion of a group was discovered historically. However, the notion of a quantum group was discovered from several different points of view [10, 11, 8, 28, 29, 30, 31, 9], the most important of which is to view quantum groups as deformations of ordinary Lie groups or Lie algebras, instead of viewing them as quantum symmetry objects of noncommutative spaces. In [13], an important first step was made by Manin in this latter direction, where quantum groups are described as quantum symmetry objects of quadratic algebras.

In this paper, we solve the problem above for finite spaces (viz. finite dimensional C^*-algebras). That is, we explicitly determine the quantum automorphism groups of such spaces. These spaces do not carry the additional geometric (Riemannian) structures in the sense of [4, 5]. The quantum automorphism groups for the latter geometric finite spaces can be termed quantum isometry groups. At the end of his book [4], Connes poses the problem of finding a finite quantum symmetry group for the finite geometric space used in his formulation of the Standard Model in particle physics. This problem is clearly related to the problem above he posed at Les Houches Summer School. We expect that the results in our paper will be useful for this problem. As a matter of fact,

Definition
A left action of a compact quantum group A on a C^*-algebra B is a unital *-homomorphism α from B to $B \otimes A$ such that:

$$(\mathrm{id}_B \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id}_A)\alpha$$

$$(\mathrm{id}_B \otimes \epsilon)\alpha = \mathrm{id}_B$$

and there exists a dense subalgebra $\mathscr{B}$ such that α restricts to the right coaction of the canonical dense Hopf-* subalgebra $\mathscr{A}$ of A on $\mathscr{B}$.

Quantum symmetries (2)

Definition

The quantum automorphism group of B in the category of quantum transformation groups of B is a universal final object in this category (if it exists).

Example

For a finite space B other than $C(X_n)$ the quantum automorphism group does not exist for the category of all quantum transformation groups

Quantum symmetries (3)

States

Let ϕ be a continuous functional on the algebra B. We define quantum automorphism group of the pair $(B; \phi)$ to be the universal object in the category of quantum transformation groups of the pair $(B; \phi)$. [Wang]

Isometry

Quantum isometry groups - spectral triples $(\mathscr{B}, H, D)$ and CQG having a unitary representation on H that commutes with $D \otimes \mathrm{id}_A$. [Goswami, Bhomwick]

Filtration

Quantum symmetry group of a unital C*-algebra B equipped with an orthogonal filtration. [Banica, Skalski]

Quantum symmetries of graph C^*-algebras.

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Quantum Symmetries of Graph C^*-algebras

Simon Schmidt and Moritz Weber

Abstract. The study of graph C^*-algebras has a long history in operator algebras. Surprisingly, their quantum symmetries have not yet been computed. We close this gap by proving that the quantum automorphism group of a finite, directed graph without multiple edges acts maximally on the corresponding graph C^*-algebra. This shows that the quantum symmetry of a graph coincides with the quantum symmetry of the graph C^*-algebra. In our result, we use the definition of quantum automorphism groups of graphs as given by Banica in 2005. Note that Bichon gave a different definition in 2003; our action is inspired from his work. We review and compare these two definitions and we give a complete table of quantum automorphism groups (with respect to either of the two definitions) for undirected graphs on four vertices.

Introduction

Symmetry constitutes one of the most important properties of a graph. It is captured by its automorphism group

$$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \mid \sigma\varepsilon = \varepsilon\sigma\} \subseteq S_n,$$

where $\Gamma = (V, E)$ is a finite graph with n vertices and no multiple edges, $\varepsilon \in M_n(\{0,1\})$ is its adjacency matrix, and S_n is the symmetric group. In modern mathematics, notably in operator algebras, symmetries are no longer described only by groups, but by quantum groups. In 2005, Banica [1] gave a definition of a quantum automorphism group of a finite graph within Woronowicz's theory of compact matrix quantum groups [20]. In our notation, $G^+_{aut}(\Gamma)$ is based on the C^*-algebra

$$\begin{aligned} &C(G^+_{aut}(\Gamma)) \\ &\quad := C(S_n^+)\langle u\varepsilon = \varepsilon u\rangle \\ &\quad = C^*\Big(u_{ij}, i, j = 1, \dots, n \;\Big|\; u_{ij} = u_{ij}^* = u_{ij}^2, \sum_l u_{il} = 1 = \sum_l u_{lj}, R_{Ban}\Big), \end{aligned}$$

where S_n^+ is Wang's quantum symmetric group [18] and R_{Ban} are the relations

$$\sum_k u_{ik}\varepsilon_{kj} = \sum_k \varepsilon_{ik}u_{kj}.$$

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AMS subject classification: 46LXX, 05CXX, 20B25.
Keywords: finite graph, graph automorphism, automorphism group, quantum automorphism, graph C^*-algebra, quantum group, quantum symmetry.

Theorem:

The quantum symmetry of a finite graph without multiple edges coincides with the quantum symmetry of the associated graph C*-algebra, $G^+_{aut}(\Gamma)$.

Quantum symmetry of a graph (Banica)

Definition

Let Γ be a directed graph with adjacency matrix $\epsilon \in M_n(\{0,1\})$, then:

$$C(G^+_{\text{aut}}(\Gamma)) = C^*\left(\begin{cases} u_{ij} = u_{ij}^* = u_{ij}^2, \\ \sum_k u_{ik} = 1 = \sum_k u_{kj}, \\ \sum_k u_{ik}\epsilon_{kj} = \sum_k \epsilon_{ik} u_{kj}, \end{cases}\right)$$

Remark

A result by Joardar & Mandal [Quantum symmetry of graph C*-algebras associated with connected graphs] shows that for finite, connected graphs with no multiple edges or loops the quantum automorphism group of the graph is a quantum subgroup of the quantum automorphism group they consider (linear faithful action).

Quantum Symmetries - objectives

Objective 1

Quantum symmetries of higher-rank graphs and quantum graphs: extend the main result of Schmidt & Weber to higher-rank graphs so as to arrive at a definition of the quantum symmetry group of a higher-rank graph C*-algebra.

Possible extension, links to noncommutative graphs that appear in the context of quantum channels?

Quantum Symmetries - objectives

Objective 2

KMS quantum symmetry groups: determine quantum symmetry groups that preserve a family of KMS-states over graph C*-algebras and provide conditions under which they are isomorphic to each other as quantum groups.

Joardar & Mandal: Quantum symmetry of graph C-algebras at critical inverse temperature. Studia Math. 256 (2021), no. 1, 1–20*
Invariance of KMS states on graph C-algebras under classical and quantum symmetry, Proceedings of the Edinburgh Mathematical Society , Volume 64 , Issue 4 , November 2021 , pp. 762 - 778*

Braided symmetries?

Braided quantum symmetries of graph C*-algebras, arXiv:2201.09885 [Bhattacharjee, Joardar, Roy]

Thm: Let E be a finite, directed graph without sinks such that the KMS state exists. Then there is a universal braided compact quantum group acting linearly, faithfully on $C^*(E)$ and preserving the KMS state.

Quantum Symmetries - objectives

Objective 3

Quantum isometry groups: construct quantum isometry groups of the graph C*-algebras and higher-rank graph C*-algebras. Moreover, develop a theory of quantum automorphisms groups preserving a given metric structure over graph C*-algebras.

Pask, Rennie, Sims: Noncommutative Manifolds from Graph and k-Graph C-Algebras, Commun. Math. Phys. 292, 607–636 (2009)*

Some practical things:

A list with all relevant papers (and preprints) ?

THANK YOU !