Abstracts for contributed talks

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Isomorphisms of vector-matrix algebras

Mitchell Ashburn

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We begin by considering anticommutative *formed algebras*: anticommutative algebras with a distinguished bilinear form. Typically, we will consider bilinear forms that are symmetric, invariant, and possibly non-degenerate. The best examples of such formed algebras are Lie algebras along with their Killing form. From these formed algebras, we then construct a family of non-associative algebras called *vector-matrix algebras*, inspired by Zorn's vector-matrix construction of the split octonions [2], [3, p.49], [4].

Isomorphisms of these vector-matrix algebras can be reduced to constructing specific types of isotopies between their underlying formed algebras [1]. Furthermore, given such an isotopy of formed algebras, we construct the corresponding isomorphism between their vector-matrix algebras.

We explore the properties of these isotopies, looking for conditions under which two formed algebras will produce isomorphic vector-matrix algebras. Specifically, we examine anticommutative algebras along with their Killing forms, before focusing on examples of specific Lie algebras with bilinear forms that exhibit such isotopies.

References

[1] R.B. Brown, N.C. Hopkins, "Noncommutative matrix Jordan algebras", *Trans. Amer. Math. Soc.* **333** (1992), 137–55.

[2] Tae-il Suh, "Algebras formed by the Zorn vector matrix", *Pacific J. Math.* **30** (1969), 255–258.

[3] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, *Rings that are Nearly Associative* (tr. H.F. Smith), Academic Press, New York, NY, 1982.

[4] M. Zorn, "Alternativkörper und quadratische Systeme", *Abh. Math. Sem. Hamburg* **9** (1933), 395–402.

On the construction of manifolds from alternating n-quasigroups

Charlotte Aten

University of Denver (United States)

(Joint work with Semin Yoo)

In my recent work with Semin Yoo, we produced a generalization of a construction of Herman and Pakianathan which assigns to each finite noncommutative group a closed surface in a functorial manner. While Herman and Pakianathan built 2-manifolds from groups, we build n-manifolds from n-quasigroups, the n-ary analogue of quasigroups. We give a pair of functors whose domain is a subcategory of a variety of n-quasigroups. The first of these functors assigns to each such n-quasigroup a smooth, flat Riemannian manifold while the second assigns to each n-quasigroup a topological manifold which is a subspace of the metric completion of the aforementioned Riemannian manifold. I will give examples of these constructions, show some pictures, and prove that all homeomorphism classes of smooth orientable manifolds arise from this construction. I will then discuss a connection with the Evans Conjecture on partial Latin squares, give its implication for orientable surfaces, and state a related problem applicable to our construction for compact n-manifolds.

Fundamental theorem of projective geometry for W-power groups

Tengiz Bokelavadze

Akaki Tsereteli State University, Kutaisi, (Georgia)

Our aim of this talk is to formulate the fundamental theorem of projective geometry for special types of groups.

P.Hall has introduced one class of groups and called them W-power groups, or simply W-groups which are the generalization of the notion of W-modules for the case of an arbitrary nilpotent groups. The meaning of W-groups in the general theory of abstract groups is defined by the fact that any finitely generated nilpotent torsion-free group is embedded in some W-group [1], [2].

We study connections between lattice isomorphisms and semilinear isomorphisms of nilpotent W-power torsion free groups.

The problem on the induction of lattice isomorphisms of locally nilpotent torsion free groups by isomorphisms has been solved by A.Sadovskii [3]. An analogous problem for nilpotent Lie algebras over the field is solved negatively by A.Lashkhi [4].

However, if the basic ring possesses an infinite distributive lattice of ideals, i.e. if Lie algebra defined over the principal ideal domain different from the fields, then the problem is solved positively, viz, for the nilpotent Lie algebras over such rings the fundamental theorem of the projective geometry is valid [4].

Acknowledgement: This talk is supported by the Shota Rustaveli National Science Foundation of Georgia Grant FR-21-471-3.

References

[1] Hall, Philip. The Edmonton notes on nilpotent groups. Queen Mary College Mathematics Notes. *Queen Mary College, Mathematics Department, London*, 1969.

[2] Bokelavadze, T. On some properties of W-power groups. *Bull. Georgian Acad. Sci.* 172 (2005), no. 2, 202–204.

[3] Sadovskiĭ, L. E. Projectivities and isomorphisms of nilpotent groups. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 29 1965 171–208

[4] Lashkhi, A. A. The fundamental theorem of projective geometry in Lie algebras. (Russian) *Soobshch. Akad. Nauk Gruzii* 149 (1994), no. 2, 185–188 (1995).

Algebraically Describing Color Trades on Complete Bipartite Graphs

John Carr

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Two proper edge-colorings of a graph G are mate-colorings if and only if every vertex of G is incident to the same set of colors under each edge-coloring while each edge receives a different color under each edge-coloring. The color-trade-spectrum of a graph G is the set of all t for which there exist two mate-colorings of G using t colors. We fully determine the color-trade-spectrum of several families of graphs, and introduce some preliminary findings on algebraically describing color trades on complete bipartite graphs. In particular, we show the following: If Q_1 and Q_2 are isotopic quasigroups with $\alpha = \beta = \text{id}$ and where γ is any derangement, then there exist edge-colorings associated with these quasigroups which form a color trade.

On Doro's conjecture

Piroska Csörgő

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In 1978 S. Doro in his paper published the following conjecture:

If the nucleus of a Moufang loop is trivial, then the commutant is a normal subloop.

The following problem had been open for a while, and officially raised by A. Rajah in 2003:

Is the commutant of a Moufang loop normal in the loop?

First Gagola stated that the answer to this question is affirmative. Grishkov and Zavarnitsine showed that the answer is in fact generally negative by constructing two infinite series of Moufang loops of exponent 3, whose commutant is not a normal subloop. Their results reopened Doro's conjecture.

By using transversals belonging to the commutant, we characterize Moufang loops whose commutant is a normal subloop, i.e. we give necessary and sufficient conditions in the multiplication group for this purpose [1].

Applying these characterizations, by working in the multiplication group of the loop we prove that in case of finite Moufang loops with trivial nucleus the commutant is normal if and only if it is trivial [2].

References

[1] Piroska Csörgő, Characterization of Moufang loops whose commutant is a normal subloop, *Journal of Group Theory* **25**/4 (2022), 781–797.

[2] Piroska Csörgő, On Doro's conjecture for finite Moufang loops, *Intern. Journal of Algebra and Computation* **33**/4 (2023), 1–13.

Beams and scaffolds – the art of building modular Garside groups

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By results of Chouraqui and Rump, the structure groups of involutive non-degenerate set-theoretic solutions to the Yang-Baxter equation are exactly the Garside groups with a distributive lattice structure. It can be shown that all these groups G decompose canonically as a lattice-theoretic product $G \cong \prod_{i=1}^{k} \mathbb{Z}$.

The situation in case of a Garside group G with a modular lattice structure turns out to be quite similar – each such group contains a canonical distributive subgroup $\mathcal{D}(G) \leqslant G$ – the *distributive scaffold* – whose lattice-decomposition $\mathcal{D}(G) \cong \prod_{i=1}^{k} \mathbb{Z}$ is induced by a lattice decomposition $G \cong \prod_{i=1}^{k} \beth_i$ into primary lattices $\beth_i$ which are called the *beams* of G. In this sense, each modular Garside group contains the structure group of an involutive solution whose lattice-structure also controls the decomposition into beams.

In this talk, I give an outline of the architecture of modular Garside groups, starting with the decomposition of a modular Garside group into beams and ending with a characterization of the beams of *dimension* $\geqslant 4$.

Relative multiplication groups and Moufang p-loops

Aleš Drápal

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(Joint work with Petr Vojtěchovský)

For a subloop S of a loop Q denote by $\mathrm{Mlt}_Q(S)$ the subgroup of $\mathrm{Mlt}(Q)$ generated by all L_s and R_s, $s \in S$. It turns out that if Q is finite Moufang and S is a p-loop, then $\mathrm{Mlt}_Q(S)$ is a p-group. This result seems to have many consequences and offers itself for generalizations. Because of that the core of the talk will consist of the proof of this theorem.

The result has been already used in (1) a characterization of congruence soluble finite Moufang loops [2] and (2) a new and relatively short proof that finite Moufang p-loops are centrally nilpotent [1].

If time allows, I will also discuss the structure of finite Moufang loops Q that are of order coprime to three and possess a normal subloop S such that S is an abelian group and Q/S is cyclic [3].

References

[1] A. Drápal: *A short proof for the central nilpotency of Moufang loops of prime power order*, J. Alg. (a revised version is under review).

[2] A. Drápal and P. Vojtěchovský: *Congruence solvability in finite Moufang loops of order coprime to three* (submitted November 2022).

[3] A. Drápal and P. Vojtěchovský: *On abelian-by-cyclic Moufang loops* (submitted December 2022).

The Hadamard quasigroup product of orthogonal Latin squares

Raúl M. Falcón

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(This talk is based on joint work with V. Álvarez, J.A. Armario, M.D. Frau, F. Gudiel, M.B. Güemes and L. Mella.)

Let $\mathcal{A}(n)$ and $\mathcal{L}(n)$ denote, respectively, the set of $n \times n$ arrays, and the set of Latin squares of order n, all of them with entries in the set $[n] := \{1, \ldots, n\}$. Let $A, B \in \mathcal{A}(n)$ and $L \in \mathcal{L}(n)$. As a natural generalization of the classical Hadamard product, the *Hadamard quasigroup product* $A \odot_L B \in \mathcal{A}(n)$ has recently been introduced [1] so that

$$(A \odot_L B)[i,j] := L[A[i,j], B[i,j]], \text{ for all } i,j \in [n]. \tag{1}$$

Let $\mathcal{OL}(n)$ denote the set of pairs of orthogonal Latin squares in $\mathcal{L}(n)$. In this talk, we are interested in studying under which conditions $L_1 \odot_{L_3} L_2 \in \mathcal{L}(n)$, for $(L_1, L_2) \in \mathcal{OL}(n)$ and $L_3 \in \mathcal{L}(n)$. It requires the existence of localized Latin transversals within L_3, which give rise to the involution

$$\begin{aligned} \varphi : \mathcal{OL}(n) &\to \mathcal{OL}(n) \\ (L_1, L_2) &\to (\varphi^{\rho}_{L_2}(L_1), \varphi^{\ell}_{L_1}(L_2)) \end{aligned}$$

where $\begin{cases} \varphi^{\rho}_{L_2}(L_1)\,[L_1[i,j],\, L_2[i,j]] := i, \\ \varphi^{\ell}_{L_1}(L_2)\,[L_1[i,j],\, L_2[i,j]] := j. \end{cases}$, for all $i, j \in [n]$.

Theorem 1. *The following statements hold.*

a) $L_1 \odot_{L_3} L_2 \in \mathcal{L}(n)$ *if and only if* $\varphi^{\rho}_{L_2}(L_1)$, $\varphi^{\ell}_{L_1}(L_2)$ *and* L_3 *are MOLS.*

b) L_1, L_2 *and* L_3 *are MOLS if and only if* $\varphi^{\rho}_{L_2}(L_1)$, $\varphi^{\ell}_{L_1}(L_2)$ *and* $\varphi^{\rho}_{L_2}(L_1) \odot_{L_3} \varphi^{\ell}_{L_1}(L_2)$ *are MOLS.*

Based on this theorem, we describe illustrative examples showing how the involution φ establishes a new way to connect distinct species of sets of three MOLS.

References

[1] R.M. Falcón, V. Álvarez, J.A. Armario, M.D. Frau, F. Gudiel, M.B. Güemes, *A computational approach to analyze the Hadamard quasigroup product.* Electronic Research Archive **31** (2023), 3245–3263.

On orthogonality of parastrophes of ternary quasigroups

Iryna Fryz

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(Joint work with Fedir Sokhatsky)

A triplet of ternary quasigroups defined on the same set is called *orthogonal* if each possible triplet of the elements of the carrier set occurs exactly once when the corresponding hypercubes are superimposed; *strongly orthogonal* if all their corresponding subhypercubes are orthogonal. A set of ternary quasigroups are orthogonal if each triplet in this set is orthogonal.

Parastrophic orthogonal ternary medial quasigroups are under consideration. It is known that each medial quasigroup is linear over a commutative group with commuting decomposition coefficients (see [1, 2]).

Criteria for a medial ternary quasigroup to be 1) self-orthogonal (all principal parastrophes are different and orthogonal) [3]; 2) totally-parastrophic orthogonal or, more briefly, a top-quasigroup (all parastrophes are different and orthogonal); 3) strongly self-orthogonal [3] have been found. Hence, the algorithms for constructing these ternary quasigroups are obtained.

Earlier, the criterion for a central binary quasigroup to be totally-parastrophic orthogonal was stated by G. Belyavskaya and T. Popovich in [4].

We prove that there are no central top-quasigroups with the condition to be strongly orthogonal for arity $n > 2$, and strongly self-orthogonal linear quasigroups (and so linear top-quasigroups with the condition to be strongly orthogonal) for $n > 3$ [3].

References

[1] V. D. Belousov, n-ary quasigroups, Chisinau: Stiintsa, 1972 (in Russian).

[2] F. Sokhatsky, Factorization of operations of medial and abelian algebras, Visnyk DonNY. Ser. A: Pryrodnychi nauky, 2017, No. 1-2, 84–96.

[3] I. Fryz, F. Sokhatsky, Construction of medial ternary self-orthogonal quasigroups, Bul. Acad. Ştiinţe Repub. Mold. Mat. (in print).

[4] G. B. Belyavskaya, T. V. Popovich, Totally conjugate orthogonal quasigroups and complete graphs, J. Math. Sci., 2012, Vol. 185, Iss. 2, 184-191.

Moufang loops and non-commuting graphs

Mark Greer

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(Joint work with J. Carr & A. Johnson)

Geometric group theory has been well studied and there has been progress to expand these ideas into loop theory. We will discuss some results and difficulties. The main focus will be on the connection between Moufang loops and their non-commuting graphs.

Minimal non-solvable Bieberbach groups. Math databases matter.

Rafał Lutowski

University of Gdańsk (Poland)

In 2022 Jonathan Hillman asked a question: What is a minimal Hirsch length of a torsion-free virtually solvable group, which is not solvable itself? Together with Andrzej Szczepański we were able to give the exact answer in the "virtually abelian" case. We obtained the result with heavy usage of computer algebra system GAP. In my talk I will focus on this aspect of our research. To be even more precise, I will put particular attention on the usage of databases of various algebraic objects available in GAP, which resulted in significant reduction of computational time and resources.

Varieties of quasigroups with inverse properties

Alla Lutsenko

Vasyl' Stus Donetsk National University (Ukraine)

(Joint work with F.M. Sokhatsky)

We study quasigroups in which the sets of translations of the same type coincide. Each of these quasigroups has an inverse properties. The classification is carried out using the concepts and results of parastrophic symmetry [1].

The main results of our research is the following: quasigroup classes with inverse properties in which the sets of translations of different directions coincide have been found [2, 3]; the distribution of the corresponding classes of quasigroups into parastrophic orbits (trusses) according to parastrophic symmetry has been described [3]; it is proved that these classes of quasigroups with inverse properties are varieties and it is found the corresponding identities [3]; invertibility functions for each variety of quasigroups with inverse properties has been found [3]; the classification of group isotopes with inverse properties has been found and constructed the bunch of varieties with inverse properties [4]; matrix IP quasigroups and CIP quasigroups are described [5].

References

[1] F. M. Sokhatsky, Parastrophic symmetry in quasigroup theory. Visnyk Donetsk national university, Ser. A: natural sciences. 2016. No. 1–2. P. 70–83.

[2] F. M. Sokhatsky, A. V. Lutsenko, Classification of quasigroups according to directions of translations I. Commentationes Mathematicae Universitatis Carolinae. 2020. Vol. 61, No. 4. P. 567–579.

[3] F. M. Sokhatsky, A. V. Lutsenko, Classification of quasigroups according to directions of translations II. Commentationes Mathematicae Universitatis Carolinae. 2021. Vol. 62, No. 3. P. 309–323.

[4] A. V. Lutsenko, Classification of group isotopes according to their inverse properties. Applied problems of mechanics and mathematics. 2020. Vol. 13, 48–62.

[5] F. M. Sokhatsky, A. V. Lutsenko, I. V. Fryz, Constructing quasigroups with invertibility property. Math. Methods and Physic. Fields. 2021. 64, No.4 (in Ukrainian).

Deformed solutions of the Yang-Baxter equation coming from skew braces

Marzia Mazzotta

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In 2007, W. Rump traced a research line for determining set-theoretic solutions of the Yang-Baxter equation by introducing *braces*. These algebraic structures with interesting generalizations have been intensively studied over the years by many authors, among these *skew braces*. It is well-known that any skew brace $(B,+,\circ)$ determines a bijective non-degenerate solution given by $r(a,b) = \big(-a + a\circ b, (-a + a\circ b)^{-} \circ a \circ b\big)$, for all $a, b \in B$. Recently, Doikou and Rybołowicz [2] have shown that a bigger family of solutions can be obtained from any skew brace B by "deforming" the map r by certain parameters $z \in B$. What we mean is that the following map

$$r_z(a,b) = \big(-a\circ z + a\circ b\circ z, (-a\circ z + a\circ b\circ z)^{-} \circ a\circ b\big)$$

gives rise to a new solution on B under some assumptions on z. In particular, if z is the identity of B, we obtain the usual solution r.

In this talk, we present this new family of solutions coming from a skew brace B and show the parameters that fit well are only those belonging to the set

$$\mathcal{D}_r(B) = \{z \in B \mid \forall\, a,b \in B \quad (a+b)\circ z = a\circ z - z + b\circ z\}$$

which we call the *right distributor* of B and is a subgroup of $(B,\circ)$. We will discuss some natural issues concerning such a set and kind of solutions. Moreover, we will show that all the results can be extended to the more general structure of *dual weak brace* [1].
This talk is based on joint work with B. Rybołowicz and P. Stefanelli [3].

References

[1] F. Catino, M. Mazzotta, M. M. Miccoli, P. Stefanelli, Set-theoretic solutions of the Yang-Baxter equation associated to weak braces, Semigroup Forum 104 (2) (2022) 228–255.

[2] A. Doikou, B. Rybołowicz, Novel non-involutive solutions of the Yang-Baxter equation from (skew) braces, preprint arXiv:2204.11580

[3] M. Mazzotta, B. Rybołowicz, P. Stefanelli, Deformed solutions of the Yang-Baxter equation coming from dual weak braces and unital near-trusses, preprint arXiv:2304.05235

Quasi-projections and factorizations of monoids

Bachuki Mesablishvili*

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It was realized a long time ago that the problem of deciding whether a given mathematical object has a particular property can be solved by means of reducing it to the problem of representing the object as a "union" of two (usually simpler) subobjects with minimal intersection and then to solve the problem for these subobjects for which one has techniques that were not available to begin with. Such a representation is called a *factorization* of the object.

In this talk, I will explain some of my recent results on factorization of monoids. For this, the notions of left and right quasi- projections on a monoid are introduced. Then they are combined with the results of [1] and [2] to give a full classification of factorizations of monoids in terms of complementary pairs of quasi-projections. Connections with descent 1-cocycles and Rota-Baxter operators on groups are established.

References

[1] A. Z. Balogh and T. Mesablishvili, Some results on factorization of monoids, J. Algebra Appl. (DOI:10.1142/S0219498824501184), 2023.

[2] V. Bovdi and B. Mesablishvili, Descent cohomology and factorizations of groups, Alg. Represent. Theory (DOI:10.1007/s10468-022-10139-0), 2022.

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Dihedral solutions of the set-theoretic Yang-Baxter equation

Alex W. Nowak

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The isomorphic correspondence between quandles of the form $x \rhd y = 2x - y \in \mathbb{Z}_n$ and the conjugation quandle generated by reflections in D_n is well-known. In this talk, we draw attention to the fact that the associated set-theoretic Yang-Baxter solution $\sigma : \mathbb{Z}_n^2 \to \mathbb{Z}_n^2; (x, y) \mapsto (2x-y, x)$ furnishes $\mathbb{Z}_n^2$ with D_n-set structure, as $\sigma^n = \mathrm{id}$, and $(\tau\sigma)^2 = \mathrm{id}$, where $\tau : (x, y) \mapsto (y, x)$ is the trivial SYBE solution. When we restrict our attention to Latin SYBE solutions (ones for which the *derived*, or *structure rack* is a quasigroup), we obtain a partial converse: the structure rack of a Latin *dihedral* SYBE solution – one where $(\tau\sigma)^2 = \mathrm{id}$ – is an involutory quandle. We will demonstrate how to obtain nontrivial dihedral solutions from symmetric spaces. Furthermore, we will explore how the added condition of triality, $\sigma^3 = \mathrm{id}$, imposes a remarkable level of rigidity on the underlying set.

Characterization of Extra Polyloop-I

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(Joint work with T.G. Jaiyeola and K.G. Ilori)
(Obafemi Awolowo University, Osun State, Nigeria)

Keywords: polyquasigroup, polyloops, extra polyloop-I

This work is devoted to studying some algebraic properties of newly introduced algebraic hyperstructure christened, 'polyloops'. The purpose of this work is to introduce a class of polyloops (i.e. extra Polyloop-I) and characterize them. Thus, a study of a non-associative algebraic hyperstructure of this type, namely, extra polyloop-I is carried out. We also investigate the the notion of autotopism and pseudo-automorphism in this algebraic hyperstructure.

References

[1] B. Davvaz, Polygroup Theory, and Related System, World Scientific Publishing Company Pte. Ltd 5 Toh Tuck Link, Singapore 596224, 2013.

[2] F. Fenyves, *Extra loops I*, Publ. Math. Debrecen, 15 (1968), 235-238.

[3] F. Fenyves, *Extra loops II*, Publ. Math. Debrecen,16 (1969).

[4] O. O. Oyebola and T. G. Jaiyeola, *Non-associative algebraic hyperstructures and their applications to biological inheritance*, Nonograffas Matematicas Garcia de Galdeano 42, (2019) 229-241.

[5] T. Vougiouklis *Hyperstructures and their Representations*, Hadronic Press Monographs in Mathematics, Palm Harbor Florida (1994).

When is the commutant of a Moufang loop normal?

J.D. Phillips

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The *commutant* of a loop, L, is the set of those elements that commute with every element in the loop: $\mathrm{C}(L) = \{x \in \mathrm{L}\colon \forall y \in L, xy = yx\}$. The communtant need not be a subloop [3], but in Moufang loops it is [4]. Unlike in groups, in Moufang loops the commutant need not be normal [2]. But often it is. We analyze conditions under which the commutant is normal. Our proofs, many of which have been found by automated deduction, are complicated (some of them are tens of thousands of lines long and require fairly advanced techniques to find). This research program has a long and colorful history (the early parts of which may found in [1]). We outline our results in the context of this lively story.

References

[1] S. Doro, Simple Moufang Loops, *Math. Proc. Camb. Phil. Soc.* **83** (1978), 377–392

[2] A. Grishkov and A. Zavarnitsine, Moufang loops with nonnormal commutative centre, *Math. Proc. Cambridge Philos. Soc.* **170** (2021), no. 3, 609–614.

[3] M. K. Kinyon, J. D. Phillips and P. Vojtěchovský, When is the commutant of a Bol loop a subloop? *Transactions of the American Mathematical Society*, **360** (2008), no.5, 2393–2408.

[4] H. O. Pflugfelder, *Quasigroups and Loops, Introduction*, Sigma Series 7 in Pure Mathematics, Helderman-Verlag, 1990.

A common divisor graph for skew braces

Silvia Properzi

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(Joint work with A. Van Antwerpen)

In the combinatorial study of solutions to the Yang–Baxter equation (YBE), recently introduced rings-theoretical objects play a fundamental role: skew braces. A skew brace is a set with two group operations $+$ and $\circ$ satisfying a compatibility condition. In a skew brace $(A, +, \circ)$, the group $(A, \circ)$ acts on $(A, +)$ by automorphism via the so-called λ-map, $\lambda\colon (A, \circ) \to \mathrm{Aut}(A, +)$. This action is involved in the (universal) construction of the set-theoretic solutions to the YBE provided by skew braces. Furthermore, the λ-action deeply influences the structure of a finite skew brace, as e.g. ideals are λ-invariant normal subgroups (for both group structures). Motivated by similar ideas in representation theory of finite groups (see [3]) and by the work of of Bertram, Herzog, and Mann [4], we study a *common divisor graph*: the simple undirected graph whose vertices are the non-trivial λ-orbits and two vertices are adjacent if their sizes are not coprime. We provide some examples and prove that it has at most two connected components and that, in the connected case, its diameter is at most four. The main result is a complete classification of finite skew braces with a one-vertex graph. In particular, we have the following enumeration.

Theorem 1. *There are only three non-isomorphic finite skew braces with an abelian additive group and one-vertex graph.*

Theorem 2. *The number of non-isomorphic skew braces of size $2^m d$ (with d odd) whose graph has only one vertex is $ma(d)$ if $m \leqslant 3$ and $2a(d)$ if $m \geqslant 4$, where $a(d)$ is the number of isomorphism classes of abelian groups of order d.*

References

[1] M.L. Lewis, *An overview of graphs associated with character degrees and conjugacy class sizes in finite groups*, The Rocky Mountain Journal of Mathematics 38 (175-211), 2008.

[2] E.A. Bertram, M. Herzog, Marcel, A. Mann, *On a graph related to conjugacy classes of groups*, The Bulletin of the London Mathematical Society 6 (569-575), 1990.

Barycentric algebras and barycentric coordinates

Anna B. Romanowska

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(Joint work with J.D.H. Smith, Iowa State University, Ames, Iowa, U.S.A.,
A. Zamojska-Dzienio, Warsaw University of Technology, Warsaw, Poland)

Real affine spaces are (abstract) algebras with non-associative binary operations indexed by real numbers. Convex subsets are subalgebras under the operations indexed by real numbers taken from the open unit interval. The algebras defining convex sets generate the variety of *barycentric algebras*. Convex polytopes considered as barycentric algebras are generated by their vertices. In particular, simplices are free barycentric algebras over their vertex sets. Each element of a simplex is presented as a convex combination of its vertices with barycentric coordinates defined in a unique way. Each general convex polytope is a homomorphic image of a simplex. Hence each of its elements can also be presented by convex combinations, but not necessarily in any unique way. Thus, the following problem is important in many applications of polytopes:

> Given the set of vertices of a convex polytope, determine algorithms for the barycentric coordinates of each point of the polytope.

There exist several methods of solving the problem for specific convex polytopes. We offer new methods based on decompositions of such polytopes into unions of simplices, with interesting combinatorial properties in the case of polygons (2-dimensional polytopes) that relate to the parsing trees of non-associative products and coproducts.

References

[1] M.S. Floater, "Generalized barycentric coordinates and applications", *Acta Numer.* **24** (2015), 161-–214.

[2] A.B. Romanowska, A. Zamojska-Dzienio and J.D.H. Smith, "Barycentric algebra and convex polygon coordinates", in preparation.

[3] A. Romanowska, "Convex sets and barycentric algebras", in: Nonassociative Mathematics and its Applications, 243-–259, *Contemp. Math.* **721**, Amer. Math. Soc., Providence, RI, 2019.

[4] A.B. Romanowska, J.D.H. Smith, "Modes", World Scientific, Singapore, 2002.

BL-algebras with the identities and diassociative loops.

Liudmila Sabinina

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We will give a survey on our recent results on the theory of binary Lie algebras. We will discuss the properties of the Binary Lie algebras with the identities $J(x, y, zt) = 0$ and their corresponding diassociative loops.

References

[1] R. Carrillo-Catalan, M.Rasskazova, L.Sabinina *The Moufang theorem for all analytic loops from a non-Moufang variety of loops.* Comm. Algebra 48 (2020), no. 2,589-594

[2] O.Garza-Guajardo, M.Rasskazova, L. Sabinina *Levi and Malcev theorems for finite-dimensional algebras from the variety defined by the identities x2=J(x,y,zu)=0.* Algebra Colloq. 28 (2021), no. 1, 87–90.

Superquasigroups and their multiplication groups

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(Joint work with B. Im,
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Supersymmetry is an important concept in mathematical physics and the treatment of structures such as Clifford algebras [4]. When applied to linear spaces, it entails a direct sum decomposition with two *homogeneous* summands, respectively described as *even* and *odd*. In a combinatorial or set-theoretical context, it becomes simpler: just a disjoint union decomposition into two uniands, again described as *even* and *odd*. Sets, groups, quasigroups decomposed appropriately in this way are called *supersets, supergroups, superquasigroups*. We present a purely set-theoretical version of the superalgebra tensor product which will be applicable equally to groups, quasigroups and loops. Our work [1] is part of a project to make supersymmetry an effective tool for the study of combinatorial structures.

Starting from supergroup and superquasigroup structures on four-element supersets, our superproduct unifies the construction of the eight-element quaternion and dihedral groups with that of a loop structure, the *quatedral loop*, which hybridizes the two groups. All three of these loops share the same character table. Thus, the quatedral loop solves a long-standing problem from the combinatorial character theory of quasigroups [2, 3], showing that a group and a loop which is not associative may have the same character table. Supersymmetry plays a key role in helping humans identify the multiplication group of the quatedral loop.

References

[1] B. Im and J.D.H. Smith, "Combinatorial supersymmetry: Supergroups, superquasigroups, and their multiplication groups", submitted.

[2] K.W. Johnson and J.D.H. Smith, "Characters of finite quasigroups", *Europ. J. Combinatorics*, **5** (1984), 43–50.

[3] J.D.H. Smith, *An Introduction to Quasigroups and Their Representations*, Chapman and Hall/CRC, Boca Raton, FL, 2007.

[4] V.S. Varadarajan, Supersymmetry for Mathematicians: An Introduction, American Mathematical Society, Providence, RI, 2004.

Public key cryptographic algorithms on vector-valued functions

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Let Q be a set, $\mathcal{O}_2$ the set of binary operations on Q, $e_1(x,y)=x$, $e_2(x,y)=y$, and

$$(f\circ g)(x,y):=f(g(x,y),y),\qquad (f\bullet g)(x,y):=f(x,g(x,y)) \tag{1}$$

be the left and right multiplications of binary operations. $(Q;f)$ is a quasigroup iff f is invertible element in both of the monoids $(\mathcal{O}_2;\circ,e_1)$ and $(\mathcal{O}_2;\bullet,e_2)$ (The functional definition of a quasigroup).

A mapping $g:Q^n\to Q^k$ is called an *n-ary vector-valued operation of the rank k*. Let $\mathcal{O}_{n,\infty}$ ($\mathcal{O}_{n,k}$) be the set of all n-ary vector-valued operations of all ranks (resp. of the rank k). Each of the operations, say g, defines and is defined by a sequence of n-ary operations $g_1,\dots,$ g_k:

$$g(x_1,\dots,x_n)=(g_1(x_1,\dots,x_n),\dots,g_k(x_1,\dots,x_n)).$$

Consider a generalization of (1). Define *$\varkappa$-multiplication* $\underset{\varkappa}{\otimes}$ for each $\varkappa:=\{j_1,\dots,j_k\}\subseteq\{1,\dots,n\}$ on the set $\mathcal{O}_{n,\infty}$:

$$\left(f\underset{\varkappa}{\otimes}g\right)(\bar{x}):=f\left(x_1^{j_1-1},g_1(\bar{x}),x_{j_1+1}^{j_2-1},g_2(\bar{x}),\dots,x_{j_{k-1}+1}^{j_k-1},g_k(\bar{x}),x_{j_k+1}^{n}\right),$$

$$e_\varkappa(x_1,\dots,x_n):=(x_{j_1},\dots,x_{j_k}).$$

where $g=(g_1,\dots,g_k)$, $k:=\operatorname{rank}(g)\leqslant n$, $\bar{x}:=(x_1,\dots,x_n)$, x_i^j is the sequence x_i, $x_{i+1},\dots,x_j$. An n-ary vector-valued operation of the rank k is called *$\varkappa$-invertible* if it is invertible element in the monoid $(\mathcal{O}_{n,k};\underset{\varkappa}{\otimes},e_\varkappa)$.

Using these notions and obtained results, public key cryptographic algorithms are constructed.

Half-automorphism group of a class of Bol loops

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A *Bol loop* is a loop that satisfies the Bol identity $(xy \cdot z)y = x(yz \cdot y)$. If L is a loop and $f : L \to L$ is a bijection such that $f(xy) \in \{f(x)f(y), f(y)f(x)\}$, for every $x, y \in L$, then f is called a *half-automorphism* of L.

In this work, we describe the half-automorphism group of the Bol loop, L_M defined over $\mathbb{Z}_2 \times \mathbb{Z}_2 \times M$, where M is an abelian group, with the following operation

$$(l, s, x) * (u, v, y) = \begin{cases} (l, s, xy), & \text{if} \quad u = v = 0, \\ (l + u, s + v, x^{-1}y), & \text{otherwise.} \end{cases}$$

The main result is the following:

Theorem 1. *Let M be an abelian group of odd order and let $L_M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times M$ be the Bol loop defined above. Denote by $Aut(L_M)$ the automorphim group of L_M and $HAut(L_M)$ the half-automorphim group of L_M. Then*

$$Aut(L_M) \cong S_3 \times Aut(M) \quad and \quad HAut(L_M) \cong C_2 \times S_3 \times Aut(M).$$

This is a joint work with Giliard Souza dos Anjos.

References

[1] de Barros, D. A. S, dos Anjos, G. S. *Half-autmorphism group of a class of Bol loops*, Communications in Algebra, (2023) vol. 51, no. 3, 1229-1241.

On Conjugation Quandle Coloring of Torus Knots - A characterization of $\mathrm{GL}(2, q)$ colorability

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The classification of knots is a wide problem that has been addressed through various methods. Notably, the study of knot invariants supported by tools of quandle theory has proven to be highly advantageous in the research of a quandle coloring, that is, a way to distinguish one knot from another by assigning a mathematical object to each strand of its diagram. In this talk, we present a characterization for determining the colorability of torus knots using matrices in $\mathrm{GL}(2, q)$.

Supernilpotent loops

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Finite nilpotent loops, in general, do not admit a direct decomposition into p-primary components. This issue was addressed recently in a novel way in universal algebra (in far greater generality) [1], resulting in a stronger concept of nilpotence, called supernilpotence. The approach is based on another fundamental property: the limited essential arity of absorbing polynomial operations. I will discuss what the theory of supernilpotence may bring to the theory of loops. In particular, we compare the degree of nilpotence of Q, the degree of nilpotence of $\mathrm{Mlt}(Q)$ and the degree of supernilpotence of Q [2], and we address the problem of equational axiomatization of supernilpotent loops of degree $\leqslant k$ [3].

References

[1] E. Aichinger and N. Mudrinski, Some applications of higher commutators in Mal'cev algebras, Algebra Universalis 63 (2010), no. 4, 367–403.

[2] Ž. Semanišinová, D. Stanovský, Three concepts of nilpotence in loops, Results in Math. 78, No. 4, Paper No. 119, 15 p. (2023).

[3] D. Stanovský, P. Vojtěchovský, Supernilpotent groups and 3-supernilpotent loops, to appear in J. Algebra and its Applications.

Classical solvability and congruence solvability in Moufang loops

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There are two concepts of solvability in loops: classical solvability generalized from groups and congruence solvability specialized from universal algebra. These two concept coincide in groups. Congruence solvability is strictly stronger than classical solvability in loops. It is an open questions whether the two concepts coincide in Moufang loops.

To prove that the two concepts coincide in a variety V, it suffices to show that every abelian normal subalgebra of an algebra A in V induces an abelian congruence of A. This is the case in groups, but we will construct counterexamples to this sufficient condition in nilpotent Moufang loops. On the other hand, we will show that the two concepts of solvability coincide in 6-divisible Moufang loops and in Moufang loops of odd order. Thus every Moufang loop of odd order is congruence solvable.

Relations on nets and MOLS

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A k-net is a geometry equivalent to $(k-2)$ Mutually Orthogonal Latin Squares (MOLS). A *relation* is a linear dependence in the point-line incidence matrix of the net. In 2014 Dukes and Howard showed that any 6-net of order 10 satisfies at least two non-trivial relations. This opens up a possibile avenue towards showing the non-existence of 4 MOLS of order 10. We generated all 4-nets of order 10 that satisfy a non-trivial relation and also ruled out one type of relation on 5-nets. I will discuss these computations, as well as some of the theory of relations on nets/MOLS more generally.