Isomorphisms of vector-matrix algebras

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Outline

Formed algebras

Definition

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We will consider formed algebras with the following properties:

- (A, β) is *anticommutative* if $x^2 = 0$ for all $x \in A$,
- (A, β) has *symmetric* bilinear form if $\beta(x, y) = \beta(y, x)$ for all $x, y \in A$,
- (A, β) has *invariant* bilinear form if $\beta(x, yz) = \beta(xy, z)$ for all $x, y, z \in A$.

Vector-matrix algebras

Out of any anticommutative formed algebra (A, β) with symmetric invariant bilinear form, we construct the vector-matrix algebra $Z(A, \beta)$ composed of vector-matrices

$$\begin{bmatrix} t & x \\ y & s \end{bmatrix}$$

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Addition and scaling are defined componentwise, and multiplication in this algebra is defined by

$$\begin{bmatrix} t_1 & x_1 \\ y_1 & s_1 \end{bmatrix} \begin{bmatrix} t_2 & x_2 \\ y_2 & s_2 \end{bmatrix} = \begin{bmatrix} t_1 t_2 + \beta(x_1, y_2) & t_1 x_2 + s_2 x_1 - y_1 y_2 \\ s_1 y_2 + t_2 y_1 + x_1 x_2 & s_1 s_2 + \beta(y_1, x_2) \end{bmatrix}.$$

Zorn's vector-matrices

The motivation for the construction of vector-matrices is the algebra of Zorn's vector-matrices. Take $\mathbb{R}^3$ with the cross product for the anticommutative product, and bilinear form β to be the dot product. Then Zorn's vector-matrices are exactly $Z(\mathbb{R}^3, \beta)$ under our construction.

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These form a simple quadratic alternative algebra, providing an alternate construction of the split-octonions.

Properties

For an anticommutative formed algebra (A, β) with symmetric invariant bilinear form we have the following properties:

- $Z(A, \beta)$ is flexible: $x(yx) = (xy)x$ for all $x, y \in Z(A, \beta)$.

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- $Z(A, \beta)$ is power-associative.

Killing form

For any anticommutative algebra A we may define a bilinear form κ called the Killing form by

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(y)\mathrm{ad}(x))$$

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This bilinear form is always symmetric. For Lie algebras it is always invariant. Over $\mathbb{R}$, a Lie algebra has non-degenerate Killing form if and only if it is semisimple.

Autotopies

Fix a formed algebra (A, β), take $S \in GL(A)$ such that

$$\beta(xS, y) = \beta(x, yS), \quad \text{and} \quad (xS)(yS) = (xy)S^{-1}. \tag{1}$$

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Then we can define a new product $\circ_S$ and a new bilinear form β_S on the algebra A by

$$x \circ_S y = (xy)S, \quad \text{and} \quad \beta_S(x, y) = \beta(xS^{-1}, y).$$

We denote the resulting formed algebra (A_S, β_S).

Isomorphism theorem

Theorem (Brown, Hopkins 1992)

Suppose that (A, β) *and* (A', β') *are anticommutative formed algebras with non-degenerate invariant symmetric bilinear forms.* $Z(A, \beta) \cong Z(A', \beta')$ *if and only if there is an* $S \in GL(A)$ *satisfying conditions (1) and an isomorphism of algebras with bilinear form* $R : (A_S, \beta_S) \to (A', \beta')$.

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In such a case, we construct the isomorphism $Z(A, \beta) \to Z(A', \beta')$ by

$$\begin{bmatrix} t & x \\ y & s \end{bmatrix} \mapsto \begin{bmatrix} t & xSR \\ yR & s \end{bmatrix}.$$

The $\mathfrak{sl}(2)$ example

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We consider $\mathbb{R}^3$ as a Lie algebra with the cross product as Lie bracket. Note that $\frac{-1}{2}\kappa(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, so $Z(\mathbb{R}^3, \frac{-1}{2}\kappa)$ is the usual Zorn vector-matrix construction of the split-octonions.

The $\mathfrak{sl}(2)$ example

Theorem

$Z(\mathfrak{sl}(2), c\kappa) \cong Z(\mathbb{R}^3, c\kappa)$ for any $c \in \mathbb{R}$. Moreover, $(\mathfrak{sl}(2), c\kappa)$ and $(\mathbb{R}^3, c\kappa)$ are the only two formed algebras (up to isomorphism) that result in this vector-matrix algebra.

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The proof involves showing that for any S satisfying conditions (1), $(\mathfrak{sl}(2)_S, c\kappa_S)$ will again be a Lie algebra. Furthermore, $c\kappa_S$ will be c times the Killing form of $\mathfrak{sl}(2)_S$. The bilinear form will remain non-degenerate, so the Lie algebra $\mathfrak{sl}(2)_S$ is semisimple. Meaning it can only be either $(\mathfrak{sl}(2), c\kappa)$ or $(\mathbb{R}^3, c\kappa)$, then S needs to be chosen to get $(\mathbb{R}^3, c\kappa)$.

The $\mathfrak{so}(4)$ example

$\mathfrak{sl}(2) \cong \mathfrak{so}(2,1)$ and $\mathbb{R}^3 \cong \mathfrak{so}(3)$, so does anything similar happen in general for $\mathfrak{so}(n)$ and $\mathfrak{so}(p,q)$?

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Then for any $S_1, S_2 \in GL(\mathfrak{so}(3))$ satisfying conditions (1) for $(\mathfrak{so}(3), c\kappa)$, $S_1 \oplus S_2 \in GL(\mathfrak{so}(4))$ satisfies conditions (1) for $(\mathfrak{so}(4), c\kappa)$ too.

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Furthermore we find that

$$(\mathfrak{so}(3) \oplus \mathfrak{so}(3))_{S_1 \oplus S_2} = \mathfrak{so}(3)_{S_1} \oplus \mathfrak{so}(3)_{S_2}.$$

The $\mathfrak{so}(4)$ example

Then by the $\mathfrak{sl}(2)$ example before, we know that there are appropriate choices of $S_1, S_2 \in GL(\mathfrak{so}(3))$ such that $\mathfrak{so}(3)_{S_1} \oplus \mathfrak{so}(3)_{S_2}$ becomes each of the following Lie algebras

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With a bit more work, we can show that these are all Lie algebras obtainable as $(\mathfrak{so}(4)_S, c\kappa_S)$. In particular, $(\mathfrak{so}(3,1), c\kappa)$ cannot be written in this form.

Sources

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