Fundamental theorem of projective geometry for W-power groups

Tengiz Bokelavadze

Akaki Tsereteli State University, Kutaisi, Georgia

LOOPS'23 Bedlewo, Poland
June 25 - July 2, 2023

Acknowledgement: This work is supported by the Shota Rustaveli National Science Foundation of Georgia Grant FR-21-4713.

Let V be a left vector space over a field k and V' be a left vector space over a field k' and assume that there exists an isomorphism $\mu : k \to k'$. Then we can generalize the notion of a homomorphism in an obvious fashion.

Definition

A map $\lambda : V \to V'$ is called semi-linear with respect to the isomorphism μ if

1. $\lambda(X + Y) = \lambda(X) + \lambda(Y)$
2. $\lambda(aX) = a^{\mu}\lambda(X)$

for all $X, Y \in V$ and $a \in k$.

The description of all semi-linear maps follows the same pattern as that of homomorphisms. If $A_1, A_2, \cdots, A_n$ is a basis for V and $B_1, B_2, \cdots, B_n$ are arbitrary vectors in V' (thought of as images of the basis vectors $A_1, A_2, \cdots, A_n$) and if $X = \sum_i x_i A_i$ is an arbitrary vector of V, then

$$\lambda(X) = \sum_i x_i^{\mu} B_i$$

is a semi-linear map of V into V' and every semi-linear map is of this form. This map is one-to-one if and only if $B_1, B_2, \cdots, B_n$ are independent, and one-to-one and onto V' if and only if $B_1, B_2, \cdots, B_n$ are a basis for V'.

With each left vector space V over k we form a new object, the corresponding projective space $\mathcal{P}(V)$. Its elements are no longer the single vectors of V, they are the subspaces U of V. To each subspace U of V we assign a projective dimension: $\dim_{\mathcal{P}} U = \dim U - 1$ just by 1 smaller that the ordinary dimension. We take over the terminology *points* and *lines* for subspaces with projective dimension 0, respectively 1. Thus the lines of V become the *points* of $\mathcal{P}(V)$ and the planes of V become the *lines* of $\mathcal{P}(V)$. The whole space V of ordinary dimension n gets (as element of $\mathcal{P}(V)$) the projective dimension $n-1$. The 0-subspace of V should be thought of as the empty element of $\mathcal{P}(V)$ and has projective dimension -1.

Definition

A map $\sigma : \mathcal{P}(V) \to \mathcal{P}(V)'$ of the element of a projective space $\mathcal{P}(V)$ onto the elements of a projective space $\mathcal{P}(V)'$ is called a collineation if

1. $\dim V = \dim V'$
2. σ is one-to-one and onto
3. $U_1 \subset U_2$ implies $\sigma(U_1) \subset \sigma(U_2)$

As an example of a collineation suppose that there exists a semi-linear map $\lambda : V \to V'$ which is one-to-one and onto. If we define $\sigma(U) \subset \lambda(U)$, then σ is obviously a collineation of $\mathcal{P}(V) \to \mathcal{P}(V)'$ and we say that σ is induced by λ.

Theorem

Let V and V' be left vector spaces of dimension $n \geq 3$ over the fields k respectively k' , $\mathcal{P}(V)$ and $\mathcal{P}(V)'$ the corresponding projective spaces. Let σ be one-to-one (onto) correspondence of the points of $\mathcal{P}(V)$ and the points of $\mathcal{P}(V)'$ which has the following property: Whenever three distinct points L_1, L_2, L_3 (they are lines of V) are collinear : $L_1 \subset L_2 + L_3$, then their images are collinear: $\sigma(L_1) \subset \sigma(L_2) + \sigma(L_3)$. Such a map can of course be extended in at most one way to a collineation but we contend more. There exists an isomorphism μ of k onto k' and a semi-linear map λ of V onto V' (with respect to μ) such that the collineation which λ induces on $\mathcal{P}(V)$ agrees with σ on the points of $\mathcal{P}(V)$. If λ_1 is another semi-linear map with respect to an isomorphism μ_1 of k onto k' which also induces this collineation, then $\lambda_1(X) = \lambda(\alpha X)$ for some fixed $\alpha \neq 0$ of k and the isomorphism μ_1 is given by $x^{\mu_1} = (\alpha x \alpha^{-1})^{\mu}$. For any $\alpha \neq 0$ the map $\lambda(\alpha X)$ will be semi-linear and induces the same collineation as λ. The isomorphism μ is therefore determined by σ up to inner automorphism of k.

My aim is in this talk to formulate the fundamental theorem of projective geometry for special types of groups. Ph. Hall has introduced one class of groups and called them W-power groups, or simply W-groups which are the generalization of the notion of W-modulus for the case of an arbitrary nilpotent groups. The meaning of W-groups in the general theory of abstract groups is defined by the fact that any finitely generated nilpotent torsion-free group is embedded in some W-group [1].

Definition

Let W be an integral domain of zero characteristic such that if $\lambda \in W$, then

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} \in W \quad \text{for all} \quad n = 1, 2, \cdots$$

We call such a ring binomial ring.
Let

$$(1) \qquad 1 = A_0 < A_1 < A_2 < \cdots < A_i < \cdots < A_n = G$$

be a central series of a group G, i.e. for $i = 0, 1, 2, \cdots, n-1$, A_{i+1}/A_i lies in the center of G/A_i in other words, if the commutator group $[A_{i+1}, G]$ lies in A_i.
A group G having a central series is called nilpotent. It is clear that every abelian group is nilpotent. We say that the nilpotent group G has nilpotency class c if the series (1) is stopped after c step. We call a group G locally nilpotent if every finitely generated subgroup of G is nilpotent.

Definition

Let G be an arbitrary locally nilpotent group and W be a binomial ring. The group G is called W -power group, or simply W -group if on the product $G \times W$ there is a mapping with the value in $G : (x, \lambda) \to x^{\lambda}$, $x \in G$, $\lambda \in W$ such that the following axioms are fulfilled:

1. $x^1 = x, x^{\lambda+\mu} = x^{\lambda} \cdot x^{\mu}$, $x^{\lambda\mu} = (x^{\lambda})^{\mu}$
2. $y^{-1}x^{\lambda}y = (y^{-1}xy)^{\lambda}$
3. $x_1^{\lambda}x_2^{\lambda} \cdots x_n^{\lambda} =$
$$t_1^{\lambda}(x_1, x_2, \cdots, x_n) t_2^{\binom{\lambda}{2}}(x_1, x_2, \cdots, x_n) \cdots t_c^{\binom{\lambda}{c}}(x_1, x_2, \cdots, x_n)$$

where c is a class of nilpotency of the group $\{x_1, x_2, \cdots, x_n\}$ and $t_i(x_1, x_2, \cdots, x_n)$ is the product of commutator weights, not less than i, $i = 1, 2, \cdots, c$ and $t_1(x_1, x_2, \cdots, x_n) = x_1 \cdot x_2 \cdots x_n$.

From these axioms it follows that $1^\lambda = 1, x^0 = 1, x^{-\lambda} = (x^\lambda)^{-1}$. We can easily see that if G is abelian, then these axioms turn into an ordinary definition of W-modulus in which usual additive notions are replaced by multiplicative ones. It is evident that the W-group is multi-operator group. The notion of W-subgroup, W-factor group and W-homomorphism is defined customarily. We shall say that the subgroup H is W-admissible in G if it is W-subgroup. Examples of W-groups:
If $W = Z$, where Z is a ring of integers, then an arbitrary locally nilpotent group is W-group (Z-group).
An arbitrary W-modulus is the abelian W-group.
Let $W = R$, where R is the ring of rational numbers. Then the complete locally nilpotent torsion free group G is the R-group.

Definition

Let X and Y be W-power groups defined over the rings W_1 and W_2 respectively. We say that the mapping $f : X \to Y$ is the semi-linear W-isomorphism with respect to the isomorphism $\sigma : W_1 \to W_2$ (is the semi-linear isomorphism), if for all $\lambda_1, \lambda_2 \in W_1$ and $x_1, x_2 \in X$ the equality

$$f(x_1^{\lambda_1}, x_2^{\lambda_2}) = f(x_1)^{\sigma(\lambda_1)} \cdot f(x_2)^{\sigma(\lambda_2)}$$

is valid.

For the W-power group X we denote by $L(X)$ the lattice of all subgroups. If $A \subseteq X$ is an arbitrary set, then $\langle A \rangle$ denotes the W-subgroups of X generated by A. We say that for the W-power group X the fundamental theorem of the projective geometry is valid, if the lattice isomorphism is induced by the semi-linear isomorphism.

Theorem

Let X and Y be the nilpotent W-power proper groups defined over the principal ideal domains W_1 and W_2, respectively. Let $\phi : L(X) \to L(Y)$ be the lattice isomorphism. If $\dim(Z(X)) > 2$, then there exists the isomorphism $\lambda : W_1 \to W_2$ and the σ-semilinear isomorphism $f : X \to Y$ such that for any subgroup $A \in L(X)$ the equality $f(A) = \phi(A)$ is valid.

Proving this fundamental theorem we use the generalizations of the fundamental theorem of the projective geometry for modules over the rings [5,6,7]. This theorem allows one to construct both the ring isomorphism $\sigma : W_1 \to W_2$ and the σ-semilinear isomorphism on the Abelian groups.

First the connection between lattice isomorphism and semi-linear isomorphisms of nilpotent W-power torsion free groups was studied. The problem on the introduction of lattice isomorphism of locally nilpotent torsion free groups by isomorphism has been solved by Sadovskii [2]. An analogous problem for nilpotent Lie algebras over fields is solved negatively by A. Lashkhi. [3] If Lie algebra defined over the principial ideal domain different from the fields, then the problem is solved positively. For the nilpotent Lie algebras over such rings the fundamental theorem of the projective geometry is valid. [4]

References

1. Artin, E. Geometric algebra. Reprint of the 1957 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1988.
2. Sadovskii, L. E. Projectivities and isomorphisms of nilpotent groups. Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), 171-208.
3. Lashkhi, A. A. Projection of Magnus Lie rings and algebras. V. I. Lenin. Sakharth. Politekh. Inst. Łrom. 1971, no. 8 (148), 7–11.
4. Lashkhi, A. A. The fundamental theorem of projective geometry in Lie algebras. Soobshch. Akad. Nauk Gruzii 149 (1994), no. 2, 185–188 (1995).

5. Camillo, V. P. Inducing lattice maps by semilinear isomorphisms. Rocky Mountain J. Math. 14 (1984), no. 2, 475–486.
6. Lashkhi, A. A. Harmonic mappings and collineations of modules. Soobshch. Akad. Nauk Gruzin. SSR 133 (1989), no. 3, 497–500.
7. Ojanguren, M.; Sridharan, R. A note on the fundamental theorem of projective geometry. Comment. Math. Helv. 44 (1969), 310–315.
8. Bokelavadze, T. Z. Lashkhi, A. A.; A subgroup lattice and the geometry of Hall's W-power groups. Dokl. Akad. Nauk 429 (2009), no. 6, 731–734; translation in Dokl. Math. 80 (2009), no. 3, 891-894.

Thanks for your attention!