ON ORTHOGONALITY OF PARASTROPHES OF TERNARY QUASIGROUPS

IRYNA FRYZ

(joint work with Fedir Sokhatsky)

Vasyl' Stus Donetsk National University,
Vinnytsia, Ukraine

LOOPS'23

Będlewo, Poland, June 26 - July 2, 2023

Table of Contents

Orthogonality concept

Definition 1.

A triplet of ternary quasigroup operations f_1, f_2, f_3 on a set Q is called *orthogonal*, if for all $a_1, a_2, a_3 \in Q$ the system of equations

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2, \\ f_3(x_1, x_2, x_3) = a_3 \end{cases} \tag{1}$$

has a unique solution.

Latin cube $\Longleftrightarrow$ ternary quasigroup

Definition 2.

A triplet of Latin cubes on Q of order m is called *orthogonal* if under their superimposition each of the m^3 ordered triplets appears exactly once.

Orthogonality concept

Definition 1.

A triplet of ternary quasigroup operations f_1, f_2, f_3 on a set Q is called *orthogonal*, if for all $a_1, a_2, a_3 \in Q$ the system of equations

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2, \\ f_3(x_1, x_2, x_3) = a_3 \end{cases} \qquad (1)$$

has a unique solution.

Latin cube $\iff$ ternary quasigroup

Definition 2.

A triplet of Latin cubes on Q of order m is called *orthogonal* if under their superimposition each of the m^3 ordered triplets appears exactly once.

Orthogonality concept

Let $\delta \subseteq \{1, 2, 3\}$.
A set of ternary operations is called *δ-retractly orthogonal*, if all tuples of similar δ-retracts of these operations are orthogonal.

Theorem 1. *

An orthogonal set of ternary quasigroups $f_1, f_2, \ldots, f_t$ defined on a set Q, where $t \geqslant 1$, is *strongly orthogonal* if and only if it is $\{i, j\}$-retractly orthogonal for each $i, j \in \{1, 2, 3\}$, where $i \neq j$.

* G. Belyavskaya, G.L. Mullen, *Strongly orthogonal and uniformly orthogonal many-placed operations*, Algebra Discrete Math., Vol. 5, №1, 2006, pp.1-17.

Orthogonal hypercubes and codes

Theorem 2. (E.T. Ethier, G.L. Mullen, 2012)*

A set of ℓ orthogonal d-cubes of order n is equivalent to an n-ary $(\ell, n^d, \ell - d + 1)$-code.

Theorem 3. (E.T. Ethier, G.L. Mullen, 2012)*

If $\ell > d$, a set of $(\ell - d)$ mutually strong orthogonal d-cubes of order n is equivalent to an n-ary $(\ell, n^d, \ell - d + 1)$-code.

* Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061. DOI: https://doi.org/10.1016/j.disc.2012.03.008

Theorem 2. (E.T. Ethier, G.L. Mullen, 2012)*

A set of ℓ orthogonal d-cubes of order n is equivalent to an n-ary $(\ell, n^d, \ell - d + 1)$-code.

Theorem 3. (E.T. Ethier, G.L. Mullen, 2012)*

If $\ell > d$, a set of $(\ell - d)$ mutually strong orthogonal d-cubes of order n is equivalent to an n-ary $(\ell, n^d, \ell - d + 1)$-code.

* Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061. DOI: https://doi.org/10.1016/j.disc.2012.03.008

For every permutation $\sigma \in S_4$, a *σ-parastrophe* ${}^{\sigma}\!f$ of an invertible ternary operation f is defined by

$$ {}^{\sigma}\!f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4, \qquad (2) $$

i.e.

$$ {}^{\sigma}\!f(x_1, x_2, x_3) = x_4 :\Longleftrightarrow f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}. \qquad (3) $$

A σ-parastrophe is called

- an *i-th division* if $\sigma = (i4)$ for $i = 1, 2, 3$;
- a *principal parastrophe* if $4\sigma = 4$.

For every permutation $\sigma \in S_4$, a *σ-parastrophe* ${}^{\sigma}f$ of an invertible ternary operation f is defined by

$$ {}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4, \qquad (2)$$

i.e.

$$ {}^{\sigma}f(x_1, x_2, x_3) = x_4 :\Longleftrightarrow f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}. \qquad (3)$$

A σ-parastrophe is called

- an *i-th division* if $\sigma = (i4)$ for $i = 1, 2, 3$;
- a *principal parastrophe* if $4\sigma = 4$.

Parastrophes

For every permutation $\sigma \in S_4$, a *σ-parastrophe* ${}^{\sigma}\!f$ of an invertible ternary operation f is defined by

$$ {}^{\sigma}\!f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4, \tag{2} $$

i.e.

$$ {}^{\sigma}\!f(x_1, x_2, x_3) = x_4 :\Longleftrightarrow f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}. \tag{3} $$

A σ-parastrophe is called

- an *i-th division* if $\sigma = (i4)$ for $i = 1, 2, 3$;
- a *principal parastrophe* if $4\sigma = 4$.

A ternary quasigroup operation is called

- *asymmetric* if all its parastrophes are different;
- *parastrophic orthogonal* if it has a triplet of orthogonal parastrophes;
- *self-orthogonal* if it has a triplet of orthogonal principal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top-quasigroup*) if the set of its different parastrophes is triple-wise orthogonal.

A ternary quasigroup operation is called

- *asymmetric* if all its parastrophes are different;
- *parastrophic orthogonal* if it has a triplet of orthogonal parastrophes;
- *self-orthogonal* if it has a triplet of orthogonal principal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top-quasigroup*) if the set of its different parastrophes is triple-wise orthogonal.

A ternary quasigroup operation is called

- *asymmetric* if all its parastrophes are different;
- *parastrophic orthogonal* if it has a triplet of orthogonal parastrophes;
- *self-orthogonal* if it has a triplet of orthogonal principal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top-quasigroup*) if the set of its different parastrophes is triple-wise orthogonal.

A ternary quasigroup operation is called

- *asymmetric* if all its parastrophes are different;
- *parastrophic orthogonal* if it has a triplet of orthogonal parastrophes;
- *self-orthogonal* if it has a triplet of orthogonal principal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top-quasigroup*) if the set of its different parastrophes is triple-wise orthogonal.

A ternary quasigroup operation is called

- *asymmetric* if all its parastrophes are different;
- *parastrophic orthogonal* if it has a triplet of orthogonal parastrophes;
- *self-orthogonal* if it has a triplet of orthogonal principal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top-quasigroup*) if the set of its different parastrophes is triple-wise orthogonal.

Constructing orthogonal quasigroups

1. Evans T. *The construction of orthogonal k-skeins and latin k-cubes*, Aequationes Math, Vol. 13, Iss. 3 (1976), 485-491.
2. Trenkler M. *On orthogonal latin p-dimensional cubes*, Czechoslovak Mathematical Journal, 55 (130) (2005), 725-728.
3. Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061.

Constructing orthogonal quasigroups

1. Evans T. *The construction of orthogonal k-skeins and latin k-cubes*, Aequationes Math, Vol. 13, Iss. 3 (1976), 485-491.
2. Trenkler M. *On orthogonal latin p-dimensional cubes*, Czechoslovak Mathematical Journal, 55 (130) (2005), 725-728.
3. Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061.

1. Evans T. *The construction of orthogonal k-skeins and latin k-cubes*, Aequationes Math, Vol. 13, Iss. 3 (1976), 485-491.
2. Trenkler M. *On orthogonal latin p-dimensional cubes*, Czechoslovak Mathematical Journal, 55 (130) (2005), 725-728.
3. Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061.

1. Evans T. *The construction of orthogonal k-skeins and latin k-cubes*, Aequationes Math, Vol. 13, Iss. 3 (1976), 485-491.
2. Trenkler M. *On orthogonal latin p-dimensional cubes*, Czechoslovak Mathematical Journal, 55 (130) (2005), 725-728.
3. Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061.

Table of Contents

Problem

TO RESEARCH METHODS FOR CONSTRUCTION OF LINEAR TOP-QUASIGROUPS.

Syrbu P. Orthogonal and Self-orthogonal n-Operations (Ph.D. thesis), Academy of Science of Moldova SSR, 1990 (in Russian).

Belyavskaya G.B., Popovich T.V. *Totally conjugate orthogonal quasigroups and complete graphs*, J. Math. Sci., **185** (2012), No. 2, 184-191. DOI: https://doi.org/10.1007/s10958-012-0907-z

Table of Contents

Existence of linear n-ary top-quasigroups

If n-ary quasigroup $(Q; f)$ is *linear* over a group $(Q; +)$, then it has decomposition

$$f(x_1, x_2, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a, \qquad (4)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are automorphisms of $(Q; +)$ and $a \in Q$.

If $(Q; +)$ is abelian, then $(Q; f)$ is called a *central* or *T-quasigroup*.

Theorem 4 (I. Fryz, F. Sokhatsky).

Linear n-ary self-orthogonal quasigroups do not exist if $n > 3$.

Corollary 1 (F. Sokhatsky, Ie. Pirus, 2014).*

Linear n-ary top-quasigroups do not exist if $n > 3$.

* Sokhatsky F., Pirus Ie. *About top-quasigroups*, Proceedings of the Third Conference of Mathematical Society of Moldova IMCS-50, August 19-23, 2014, Chisinau, Republic of Moldova, 162-165.

Existence of linear n-ary top-quasigroups

If n-ary quasigroup $(Q; f)$ is *linear* over a group $(Q; +)$, then it has decomposition

$$f(x_1, x_2, \dots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a, \tag{4}$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are automorphisms of $(Q; +)$ and $a \in Q$.

If $(Q; +)$ is abelian, then $(Q; f)$ is called a *central* or *T-quasigroup*.

Theorem 4 (I. Fryz, F. Sokhatsky).

Linear n-ary self-orthogonal quasigroups do not exist if $n > 3$.

Corollary 1 (F. Sokhatsky, Ie. Pirus, 2014).*

Linear n-ary top-quasigroups do not exist if $n > 3$.

* Sokhatsky F., Pirus Ie. *About top-quasigroups*, Proceedings of the Third Conference of Mathematical Society of Moldova IMCS-50, August 19-23, 2014, Chisinau, Republic of Moldova, 162-165.

Existence of linear n-ary top-quasigroups

Theorem 5 (I. Fryz, F. Sokhatsky, 2022).*

Linear n-ary strongly self-orthogonal quasigroups do not exist if $n > 3$.

Theorem 6 (I. Fryz).

Central n-ary strongly top-quasigroups do not exist if $n > 2$.

* Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

Existence of linear n-ary top-quasigroups

Theorem 5 (I. Fryz, F. Sokhatsky, 2022).*

Linear n-ary strongly self-orthogonal quasigroups do not exist if $n > 3$.

Theorem 6 (I. Fryz).

Central n-ary strongly top-quasigroups do not exist if $n > 2$.

* Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

Theorem 7 (I. Fryz).

A linear quasigroup $(Q; f)$ over a group $(Q; +)$ with canonical decomposition

$$f(x, y) = \alpha x + \beta y + a \tag{5}$$

is a top-quasigroup if and only if the mappings

$$-I_t\alpha^{-1}\beta + \beta^{-1}\alpha, \quad I_{Ia}\alpha + \iota, \quad \iota + \beta, \quad I_{\beta a}\beta^2 + \alpha, \quad \beta + \alpha^2,$$
$$\beta\alpha - I_a, \quad I_t I + \alpha^2, \quad I_t I_a^{-1} I\beta + \beta^{-1} I_a, \quad I_t I_a^{-1} I\beta + \alpha \tag{6}$$

are permutations for any $t \in Q$, where $I_a(x) := a + x - a$, $I(x) := -x$.

Sokhatsky F.M., Fryz I.V. *Invertibility criterion of composition of two multiary quasigroups*, Comment. Math. Univ. Carolin., Vol. 53 (2012), No. 3, 429-445.

Shcherbacov V.A. *Orthogonality of linear (alinear) quasigroups and their parastrophes*, arXiv:1212.1804v1 [math.GR] 8 Dec 2012, 1-23.

Corollary 2 (Belyavskaya G., Popovich T., 2010*).

A central quasigroup $(Q; f)$ over an abelian group $(Q; +)$ with canonical decomposition (5) is a top-quasigroup if and only if all mappings

$$\alpha+\iota, \quad \alpha-\iota, \quad \beta+\iota, \quad \beta-\iota, \quad \alpha^2+\beta, \quad \beta^2+\alpha, \quad \alpha-\beta, \quad \alpha+\beta, \quad \beta\alpha-\iota \tag{7}$$

are permutations of Q.

*Belyavskaya G.B., Popovich T.V. *Totally conjugate orthogonal quasigroups and complete graphs*, J. Math. Sci., **185** (2012), No. 2, 184-191. DOI: https://doi.org/10.1007/s10958-012-0907-z

Binary case ($n = 2$)

Top-quasigroups according to symmetry groups

$S_3 := \{\iota, s, \ell, r, s\ell, sr\}$ and $s := (12)$, $\ell := (13)$, $r := (23)$.

A quasigroup is called *strictly commutative*, if $\mathrm{Sym}(f) = \{\iota, s\}$, i.e.

$$f = {}^{s}f, \qquad {}^{\ell}f = {}^{sr}f, \qquad {}^{r}f = {}^{s\ell}f.$$

Corollary 3.

A linear quasigroup $(Q; f)$ over a group $(Q; +)$ is strictly commutative if and only if $(Q; +)$ is abelian and $\alpha = \beta \neq \iota$, i.e., its canonical decomposition is

$$f(x, y) = \alpha x + \alpha y + a. \qquad (8)$$

A strictly commutative quasigroup $(Q; f)$ is a top-quasigroup if and only if the mappings

$$\alpha + \iota, \qquad \alpha - \iota$$

are permutations of Q.

Kirnasovsky O.U. *Linear isotopes of small orders groups*, Quasigroups Related Systems, **2** (1995), No. 1(2), 51–82.

Krainichuk H. *Classification of group isotopes according to their symmetry groups*, Folia Mathematica, Vol. 19 (2017), No. 1, 84-98.

A quasigroup is called *strictly left symmetric*, if $\mathrm{Sym}(f) = \{\iota, r\}$, i.e.

$$f = {}^r f, \qquad {}^\ell f = {}^{s\ell} f, \qquad {}^s f = {}^{sr} f.$$

Corollary 4.

A linear quasigroup $(Q; f)$ over a group $(Q; +)$ is strictly left symmetric if and only if $(Q; +)$ is abelian and $\beta = -\iota \neq \alpha$, i.e., its canonical decomposition is

$$f(x, y) = \alpha x + Iy + a \tag{9}$$

A strictly left symmetric quasigroup $(Q; f)$ is a top-quasigroup if and only if the mappings

$$\alpha + \iota, \qquad \alpha - \iota$$

are permutations of Q.

Kirnasovsky O.U. *Linear isotopes of small orders groups*, Quasigroups Related Systems, **2** (1995), No. 1(2), 51–82.
Krainichuk H. *Classification of group isotopes according to their symmetry groups*, Folia Mathematica, Vol. 19 (2017), No. 1, 84-98.

A quasigroup is called *strictly right symmetric*, if $\mathrm{Sym}(f) = \{\iota, \ell\}$, i.e.

$$f = {}^{\ell}f, \qquad {}^{r}f = {}^{sr}f, \qquad {}^{s}f = {}^{s\ell}f.$$

Corollary 5.

A linear quasigroup $(Q; f)$ over a group $(Q; +)$ is strictly right symmetric if and only if $(Q; +)$ is abelian and $\alpha = -\iota \neq \beta$, i.e., its canonical decomposition is

$$f(x, y) = Ix + \beta y + a. \tag{10}$$

A strictly right symmetric quasigroup $(Q; f)$ is a top-quasigroup if and only if the mappings

$$\beta + \iota, \qquad \beta - \iota$$

are permutations of Q.

Kirnasovsky O.U. *Linear isotopes of small orders groups*, Quasigroups Related Systems, **2** (1995), No. 1(2), 51–82.

Krainichuk H. *Classification of group isotopes according to their symmetry groups*, Folia Mathematica, Vol. 19 (2017), No. 1, 84-98.

Binary case ($n = 2$)

Top-quasigroups according to symmetry groups

A quasigroup is called *strictly semi-symmetric*, if $\mathrm{Sym}(f) = A_3$, i.e.

$$f = {}^{s\ell}f = {}^{sr}f, \qquad {}^{s}f = {}^{\ell}f = {}^{r}f.$$

Corollary 6.

A linear quasigroup $(Q; f)$ over a group $(Q; +)$ is strictly semi-symmetric if and only if α is an anti-automorphism of $(Q; +)$,

$$\beta = \alpha^{-1}, \qquad \alpha a = -a, \qquad \alpha^3 = -I_a,$$

$(Q; +)$ is non-abelian or $\alpha \neq -\iota$, so its canonical decomposition is

$$f(x, y) = \alpha x + \alpha^{-1} y + a. \tag{11}$$

A strictly semi-symmetric quasigroup $(Q; f)$ is a top-quasigroup if and only if the mapping

$$-I_t + \alpha^2$$

is a permutation of Q for any $t \in Q$.

Krainichuk H. *Classification of group isotopes according to their symmetry groups*, Folia Mathematica, Vol. 19 (2017), No. 1, 84-98.

Theorem 8 (V.D. Belousov, 1972).*

A ternary quasigroup $(Q; f)$ is medial if and only if there exists an abelian group $(Q; +)$ such that

$$f(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + a, \qquad (12)$$

where φ_1, φ_2, φ_3 are pairwise commuting automorphisms of $(Q; +)$ and $a \in Q$.

Lemma 1.

Let $(Q; f)$ be a medial ternary quasigroup with (12) and τ_1, τ_2, $\tau_3 \in S_4$. The parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are orthogonal if and only if

$$\begin{vmatrix} \varphi_{1\tau_1} & \varphi_{2\tau_1} & \varphi_{3\tau_1} \\ \varphi_{1\tau_2} & \varphi_{2\tau_2} & \varphi_{3\tau_2} \\ \varphi_{1\tau_3} & \varphi_{2\tau_3} & \varphi_{3\tau_3} \end{vmatrix} \qquad (13)$$

is an automorphism of the group $(Q; +)$, where $\varphi_4 := I$.

* Belousov V.D. *n-ary quasigroups*, Chishinau: Stiintsa, 1972. (in Russian)

Theorem 8 (V.D. Belousov, 1972).*

A ternary quasigroup $(Q; f)$ is medial if and only if there exists an abelian group $(Q; +)$ such that

$$f(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + a, \tag{12}$$

where φ_1, φ_2, φ_3 are pairwise commuting automorphisms of $(Q; +)$ and $a \in Q$.

Lemma 1.

Let $(Q; f)$ be a medial ternary quasigroup with (12) and τ_1, τ_2, $\tau_3 \in S_4$. The parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are orthogonal if and only if

$$\begin{vmatrix} \varphi_{1\tau_1} & \varphi_{2\tau_1} & \varphi_{3\tau_1} \\ \varphi_{1\tau_2} & \varphi_{2\tau_2} & \varphi_{3\tau_2} \\ \varphi_{1\tau_3} & \varphi_{2\tau_3} & \varphi_{3\tau_3} \end{vmatrix} \tag{13}$$

is an automorphism of the group $(Q; +)$, where $\varphi_4 := I$.

* Belousov V.D. *n-ary quasigroups*, Chishinau: Stiintsa, 1972. (in Russian)

Let $\vec{\nu} := (\nu_1, \nu_2, \nu_3)$ be a triplet of injections of the set $\{1, 2, 3\}$ into the set $\{1, 2, 3, 4\}$. For each triplet $\vec{\nu}$ a polynomial $d_{\vec{\nu}}$ over a commutative ring K is defined as follows:

$$d_{\vec{\nu}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) := \begin{vmatrix} \gamma_{1\nu_1} & \gamma_{2\nu_1} & \gamma_{3\nu_1} \\ \gamma_{1\nu_2} & \gamma_{2\nu_2} & \gamma_{3\nu_2} \\ \gamma_{1\nu_3} & \gamma_{2\nu_3} & \gamma_{3\nu_3} \end{vmatrix}. \qquad (14)$$

Definition 3.

A polynomial p over a commutative ring K will be called *invertible-valued* over a subset $H \subseteq K$, if $p(a, b, c)$ is invertible in K whenever a, b, c are in H.

Lemma 2 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12) is self-orthogonal if and only if the polynomials

$$\gamma_1 - \gamma_2, \qquad \gamma_1 + \gamma_2 + \gamma_3, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \gamma_1\gamma_2 - \gamma_1\gamma_3 - \gamma_2\gamma_3 \tag{15}$$

are invertible-valued over the automorphisms $\varphi_1, \varphi_2, \varphi_3$ of the group $(Q, +)$.

Theorem 9 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12), is self-orthogonal if and only if the mappings

$$\varphi_1 - \varphi_2, \qquad \varphi_1 - \varphi_3, \qquad \varphi_2 - \varphi_3, \qquad \varphi_1 + \varphi_2 + \varphi_3, \qquad (\varphi_1 + \varphi_2 + \varphi_3)^2 - 3(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3) \tag{16}$$

are automorphisms of the group $(Q, +)$.

* Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

Lemma 2 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12) is self-orthogonal if and only if the polynomials

$$\gamma_1 - \gamma_2, \qquad \gamma_1 + \gamma_2 + \gamma_3, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \gamma_1\gamma_2 - \gamma_1\gamma_3 - \gamma_2\gamma_3 \tag{15}$$

are invertible-valued over the automorphisms $\varphi_1, \varphi_2, \varphi_3$ of the group $(Q, +)$.

Theorem 9 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12), is self-orthogonal if and only if the mappings

$$\varphi_1 - \varphi_2, \qquad \varphi_1 - \varphi_3, \qquad \varphi_2 - \varphi_3, \qquad \varphi_1 + \varphi_2 + \varphi_3, \qquad (\varphi_1 + \varphi_2 + \varphi_3)^2 - 3(\varphi_1\varphi_2 + \varphi_1\varphi_3 + \varphi_2\varphi_3) \tag{16}$$

are automorphisms of the group $(Q, +)$.

* Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

Lemma 3 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12) is strongly self-orthogonal if and only if the polynomials (15) and

$$\gamma_1\gamma_2 - \gamma_3^2, \qquad \gamma_1 + \gamma_2 \tag{17}$$

are invertible-valued over the automorphisms $\varphi_1, \varphi_2, \varphi_3$ of the group $(Q, +)$.

Theorem 10 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12) is strongly self-orthogonal if and only if the mappings (16) and

$$\begin{array}{lll} \varphi_2\varphi_3 - \varphi_1^2, & \varphi_1\varphi_3 - \varphi_2^2, & \varphi_1\varphi_2 - \varphi_3^2, \\ \varphi_1 + \varphi_2, & \varphi_1 + \varphi_3, & \varphi_2 + \varphi_3 \end{array} \tag{18}$$

are automorphisms of the group $(Q, +)$.

* Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

Ternary case ($n = 3$)

Self-orthogonal medial quasigroups

Lemma 3 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12) is strongly self-orthogonal if and only if the polynomials (15) and

$$\gamma_1\gamma_2 - \gamma_3^2, \qquad \gamma_1 + \gamma_2 \tag{17}$$

are invertible-valued over the automorphisms $\varphi_1, \varphi_2, \varphi_3$ of the group $(Q, +)$.

Theorem 10 (I. Fryz, F. Sokhatsky, 2022*).

A ternary medial quasigroup (Q, f) with (12) is strongly self-orthogonal if and only if the mappings (16) and

$$\begin{matrix} \varphi_2\varphi_3 - \varphi_1^2, & \varphi_1\varphi_3 - \varphi_2^2, & \varphi_1\varphi_2 - \varphi_3^2, \\ \varphi_1 + \varphi_2, & \varphi_1 + \varphi_3, & \varphi_2 + \varphi_3 \end{matrix} \tag{18}$$

are automorphisms of the group $(Q, +)$.

* Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

Examples

Let $\mathbb{Z}_m$ be a ring of integers modulo m and f have the decomposition

$$f(x, y, z) := x + 2y + 3z.$$

If m is relatively prime to 6, then $(\mathbb{Z}_m; f)$ is a quasigroup.

1. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup if m is not divisible by 6;
2. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, but it is not strongly self-orthogonal if m is not divisible by 6 and m is divisible by 5 or 7;
3. $(\mathbb{Z}_m; f)$ is a strongly self-orthogonal ternary quasigroup if m is not divisible by 2, 3, 5 and 7.

Examples

Let $\mathbb{Z}_m$ be a ring of integers modulo m and f have the decomposition

$$f(x, y, z) := x + 2y + 3z.$$

If m is relatively prime to 6, then $(\mathbb{Z}_m; f)$ is a quasigroup.

1. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup if m is not divisible by 6;
2. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, but it is not strongly self-orthogonal if m is not divisible by 6 and m is divisible by 5 or 7;
3. $(\mathbb{Z}_m; f)$ is a strongly self-orthogonal ternary quasigroup if m is not divisible by 2, 3, 5 and 7.

Examples

Let $\mathbb{Z}_m$ be a ring of integers modulo m and f have the decomposition

$$f(x, y, z) := x + 2y + 3z.$$

If m is relatively prime to 6, then $(\mathbb{Z}_m; f)$ is a quasigroup.

1. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup if m is not divisible by 6;
2. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, but it is not strongly self-orthogonal if m is not divisible by 6 and m is divisible by 5 or 7;
3. $(\mathbb{Z}_m; f)$ is a strongly self-orthogonal ternary quasigroup if m is not divisible by 2, 3, 5 and 7.

Examples

Let $\mathbb{Z}_m$ be a ring of integers modulo m and f have the decomposition

$$f(x, y, z) := x + 2y + 3z.$$

If m is relatively prime to 6, then $(\mathbb{Z}_m; f)$ is a quasigroup.

1. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup if m is not divisible by 6;
2. $(\mathbb{Z}_m; f)$ is a self-orthogonal ternary quasigroup, but it is not strongly self-orthogonal if m is not divisible by 6 and m is divisible by 5 or 7;
3. $(\mathbb{Z}_m; f)$ is a strongly self-orthogonal ternary quasigroup if m is not divisible by 2, 3, 5 and 7.

$$d_{\vec{\nu}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \begin{vmatrix} \gamma_{1\nu_1} & \gamma_{2\nu_1} & \gamma_{3\nu_1} \\ \gamma_{1\nu_2} & \gamma_{2\nu_2} & \gamma_{3\nu_2} \\ \gamma_{1\nu_3} & \gamma_{2\nu_3} & \gamma_{3\nu_3} \end{vmatrix}.$$

Lemma 4 (F. Sokhatsky, I. Fryz).

A medial ternary quasigroup $(Q; f)$ with

$$f(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + a,$$

is a top-quasigroup if and only if each polynomial $d_{\vec{\nu}}$ is invertible-valued over the set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, where $\varphi_4 := I$.

Polynomials over the set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$:

$$
\begin{aligned}
&p_1(\gamma_1,\gamma_2) := \gamma_1 - \gamma_2,\\
&p_2(\gamma_1,\gamma_2,\gamma_3) := \gamma_1 + \gamma_2 + \gamma_3,\\
&p_3(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - \gamma_1\gamma_2 - \gamma_1\gamma_3 - \gamma_2\gamma_3,\\
&p_4(\gamma_1,\gamma_2) := \gamma_1 + \gamma_2,\\
&p_5(\gamma_1,\gamma_2,\gamma_3) := \gamma_1^2 - \gamma_2\gamma_3,\\
&p_6(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1\gamma_2 - \gamma_3\gamma_4,\\
&p_7(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1^3 + \gamma_2^2\gamma_4 + \gamma_2\gamma_3^2 - \gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_2^2 - \gamma_1\gamma_3\gamma_4,\\
&p_8(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1^3 + 2\gamma_2\gamma_3\gamma_4 - \gamma_1\gamma_3^2 - \gamma_1\gamma_2^2 - \gamma_1\gamma_4^2,\\
&p_9(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1^2 - \gamma_2^2 + \gamma_1\gamma_3 - \gamma_2\gamma_4,\\
&p_{10}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1^3 + \gamma_2\gamma_4^2 + \gamma_2\gamma_3^2 - 2\gamma_1\gamma_3\gamma_4 - \gamma_1\gamma_2^2,\\
&p_{11}(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := \gamma_1^2\gamma_3 + \gamma_2\gamma_4^2 + \gamma_1\gamma_2\gamma_3 - \gamma_1\gamma_3\gamma_4 - \gamma_1^2\gamma_4 - \gamma_2^2\gamma_3.
\end{aligned}
\tag{19}
$$

Theorem 11 (F. Sokhatsky, I. Fryz).

A medial ternary quasigroup $(Q; f)$ with

$$f(x_1, x_2, x_3) = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + a,$$

is a top-quasigroup if and only if each of the polynomials from (19) are invertible-valued over the set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$, where $\varphi_4 := I$.

Theorem 12 (F. Sokhatsky, Ie. Pirus, 2015).*

Let $\mathbb{Z}_m$ be a ring of residues and let

$$f(x, y, z) := 2x + 8y + 11z. \tag{20}$$

If the least prime factor of m is greater than 107, then $(\mathbb{Z}_m; f)$ is an asymmetric top-quasigroup of order m.

Theorem 13 (F. Sokhatsky, Ie. Pirus, 2015).*

A ternary medial asymmetric top-quasigroup over an m-ordered cyclic group exists if and only if the least prime factor of m is greater than 19.

* Sokhatsky F., Pirus Ie. About parastrophically orthogonal quasigroups, Book of extended abstracts of the International Mathematical Conference on Quasigroups and Loops “Loops'15”, 28 June - 04 July 2015, Ohrid, Macedonia, 46-47.

Theorem 12 (F. Sokhatsky, Ie. Pirus, 2015).*

Let $\mathbb{Z}_m$ be a ring of residues and let

$$f(x, y, z) := 2x + 8y + 11z. \tag{20}$$

If the least prime factor of m is greater than 107, then $(\mathbb{Z}_m; f)$ is an asymmetric top-quasigroup of order m.

Theorem 13 (F. Sokhatsky, Ie. Pirus, 2015).*

A ternary medial asymmetric top-quasigroup over an m-ordered cyclic group exists if and only if the least prime factor of m is greater than 19.

* Sokhatsky F., Pirus Ie. About parastrophically orthogonal quasigroups, Book of extended abstracts of the International Mathematical Conference on Quasigroups and Loops "Loops'15", 28 June - 04 July 2015, Ohrid, Macedonia, 46-47.

Table of Contents

References

1. Belousov V.D. *n-ary quasigroups*, Chishinau: Stiintsa, 1972. (in Russian)
2. Syrbu P. Orthogonal and Self-orthogonal n-Operations (Ph.D. thesis), Academy of Science of Moldova SSR, 1990 (in Russian).
3. Kirnasovsky O.U. *Linear isotopes of small orders groups*, Quasigroups Related Systems, **2** (1995), No. 1(2), 51–82.
4. Belyavskaya G., Mullen G.L. *Strongly orthogonal and uniformly orthogonal many-placed operations*, Algebra Discrete Math., Vol. 5, №1, 2006, pp.1-17.
5. Belyavskaya G. B., Popovich T. V. *Totally conjugate orthogonal quasigroups and complete graphs*, J. Math. Sci., Vol. 185 (2012), No. 2, 184-191. DOI: https://doi.org/10.1007/s10958-012-0907-z
6. Sokhatsky F.M., Fryz I.V. *Invertibility criterion of composition of two multiary quasigroups*, Comment. Math. Univ. Carolin., Vol. 53 (2012), No. 3, 429-445.
7. Ethier E.T., Mullen G.L. *Strong forms of orthogonality for sets of hypercubes*, Discrete Math., Vol. 321 (2012), Iss. 12-13, 2050-2061. DOI: https://doi.org/10.1016/j.disc.2012.03.008

References

8. Shcherbacov V.A. *Orthogonality of linear (alinear) quasigroups and their parastrophes*, arXiv:1212.1804v1 [math.GR] 8 Dec 2012, 1-23.

9. Sokhatsky F., Pirus Ie. *About top-quasigroups*, Proceedings of the Third Conference of Mathematical Society of Moldova IMCS-50, August 19-23, 2014, Chisinau, Republic of Moldova, 162-165.

10. Sokhatsky F., Pirus Ie. *About parastrophically orthogonal quasigroups*, Book of extended abstracts of the International Mathematical Conference on Quasigroups and Loops "Loops'15", 28 June - 04 July 2015, Ohrid, Macedonia, 46-47.

11. Krainichuk H. *Classification of group isotopes according to their symmetry groups*, Folia Mathematica, Vol. 19 (2017), No. 1, 84-98.

12. Iryna Fryz, Fedir Sokhatsky. *Construction of medial ternary self-orthogonal quasigroups*. Bul. Acad. Stiinte Repub. Mold. Mat. 2022. №3(100). P. 41-55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41

THANK YOU FOR YOUR ATTENTION!