Deformed solutions of the Yang-Baxter equation coming from skew braces

Marzia Mazzotta

Università del Salento

Joint work with Bernard Rybołowicz and Paola Stefanelli

UNIVERSITÀ DEL SALENTO

L'Ateneo tra i due mari

LOOP'S 2023 - 30th June 2023, Będlewo

Solutions of the Yang-Baxter equation

If S is a set, a map $r : S \times S \longrightarrow S \times S$ satisfying the braid relation

$$(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)$$

is called *set-theoretic solution*, or briefly *solution*, of the Yang-Baxter equation.

For a solution r, we introduce two maps $\lambda_a, \rho_b : S \to S$ and write

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in S$. In particular, the solution r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in S$;
- *right non-degenerate* if ρ_b is bijective, for every $b \in S$;
- *non-degenerate* if r is both left and right non-degenerate.

Solutions of the Yang-Baxter equation

If S is a set, a map $r : S \times S \longrightarrow S \times S$ satisfying the braid relation

$$(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)$$

is called *set-theoretic solution*, or briefly *solution*, of the Yang-Baxter equation.

For a solution r, we introduce two maps $\lambda_a, \rho_b : S \to S$ and write

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in S$. In particular, the solution r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in S$;
- *right non-degenerate* if ρ_b is bijective, for every $b \in S$;
- *non-degenerate* if r is both left and right non-degenerate.

Lu-Yan-Zhu conditions

Theorem (Lu, Yan, Zhu - 2000)

Let G be a group, $\lambda, \rho : G \to \mathrm{Sym}_G$ *maps and set* $\lambda_a(b) := \lambda(a)(b)$, $\rho_b(a) := \rho(b)(a)$, *for all* $a, b \in G$. *If* $\lambda, \rho : G \to \mathrm{Sym}_G$ *are a left action and a right action of G on itself, respectively, and*

$$\forall\ a, b \in G \quad ab = \lambda_a(b)\ \rho_b(a),$$

then the map $r(a,b) = (\lambda_a(b), \rho_b(a))$ *is a non-degenerate bijective solution on G.*

Venkov solutions

If G is a group and, for all $a, b \in G$, set $\lambda_a = \mathrm{id}_G$ and $\rho_b(a) = b^{-1}ab$, then

$$r(a,b) = \left(b,\ b^{-1}ab\right)$$

is a non-degenerate bijective solution on G.

Lu-Yan-Zhu conditions

Theorem (Lu, Yan, Zhu - 2000)

Let G be a group, $\lambda, \rho : G \to \mathrm{Sym}_G$ *maps and set* $\lambda_a(b) := \lambda(a)(b)$, $\rho_b(a) := \rho(b)(a)$, *for all* $a, b \in G$*. If* $\lambda, \rho : G \to \mathrm{Sym}_G$ *are a left action and a right action of G on itself, respectively, and*

$$\forall\ a, b \in G \quad ab = \lambda_a(b)\ \rho_b(a),$$

then the map $r(a, b) = (\lambda_a(b), \rho_b(a))$ *is a non-degenerate bijective solution on G.*

Venkov solutions

If G is a group and, for all $a, b \in G$, set $\lambda_a = \mathrm{id}_G$ and $\rho_b(a) = b^{-1}ab$, then

$$r(a, b) = \left(b,\ b^{-1}ab\right)$$

is a non-degenerate bijective solution on G.

Skew left braces

[Rump - 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*.

[Rump- 2007, Guarnieri, Vendramin - 2017]

A triple $(B, +, \circ)$ is called *skew (left) brace* if

- $(B, +)$ and $(B, \circ)$ are groups
- $\forall a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$

If $(B, +)$ is abelian, then $(B, +, \circ)$ is a *(left) brace*.

- The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.
- Similarly, we can define the structures of skew (right) braces. Skew braces simultaneously satisfying the left and the right axioms are called *two-sided skew braces*.

Skew left braces

[Rump - 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*.

[Rump- 2007, Guarnieri, Vendramin - 2017]

A triple $(B, +, \circ)$ is called *skew (left) brace* if

- $(B, +)$ and $(B, \circ)$ are groups
- $\forall a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$

If $(B, +)$ is abelian, then $(B, +, \circ)$ is a *(left) brace*.

- The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.
- Similarly, we can define the structures of skew (right) braces. Skew braces simultaneously satisfying the left and the right axioms are called *two-sided skew braces*.

Skew left braces

[Rump - 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*.

[Rump- 2007, Guarnieri, Vendramin - 2017]

A triple $(B, +, \circ)$ is called *skew (left) brace* if

- $(B, +)$ and $(B, \circ)$ are groups
- $\forall a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$

If $(B, +)$ is abelian, then $(B, +, \circ)$ is a *(left) brace*.

- The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.
- Similarly, we can define the structures of skew (right) braces. Skew braces simultaneously satisfying the left and the right axioms are called *two-sided skew braces*.

Skew left braces

[Rump - 2007] traced a novel research direction for finding solutions by introducing the algebraic structure of *brace*.

[Rump- 2007, Guarnieri, Vendramin - 2017]

A triple $(B, +, \circ)$ is called *skew (left) brace* if

- $(B, +)$ and $(B, \circ)$ are groups
- $\forall a, b, c \in B \qquad a \circ (b + c) = a \circ b - a + a \circ c.$

If $(B, +)$ is abelian, then $(B, +, \circ)$ is a *(left) brace*.

- The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.
- Similarly, we can define the structures of skew (right) braces. Skew braces simultaneously satisfying the left and the right axioms are called *two-sided skew braces*.

Solutions associated to skew (left) braces

Given a skew (left) brace $(B, +, \circ)$ and set, for all $a, b \in B$,

$$\lambda_a(b) := -a + a \circ b \quad \text{and} \quad \rho_b(a) := (-a + a \circ b)^- \circ a \circ b,$$

then the maps λ and ρ satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Consequently, the map

$$r_B(a, b) = \left(-a + a \circ b, (-a + a \circ b)^- \circ a \circ b\right)$$

is a *non-degenerate bijective solution*.

The solution r_B is *involutive*, i.e., $r^2 = \mathrm{id}_{B\times B}$, if and only if $(B, +, \circ)$ is a brace.

Solutions associated to skew (left) braces

Given a skew (left) brace $(B, +, \circ)$ and set, for all $a, b \in B$,

$$\lambda_a(b) := -a + a \circ b \quad \text{and} \quad \rho_b(a) := (-a + a \circ b)^- \circ a \circ b,$$

then the maps λ and ρ satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Consequently, the map

$$r_B(a, b) = \left(-a + a \circ b, (-a + a \circ b)^- \circ a \circ b\right)$$

is a *non-degenerate bijective solution*.

The solution r_B is *involutive*, i.e., $r^2 = \mathrm{id}_{B \times B}$, if and only if $(B, +, \circ)$ is a brace.

Solutions associated to skew (left) braces

Given a skew (left) brace $(B, +, \circ)$ and set, for all $a, b \in B$,

$$\lambda_a(b) := -a + a \circ b \quad \text{and} \quad \rho_b(a) := (-a + a \circ b)^- \circ a \circ b,$$

then the maps λ and ρ satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Consequently, the map

$$r_B(a, b) = \left(-a + a \circ b, (-a + a \circ b)^- \circ a \circ b\right)$$

is a *non-degenerate bijective solution*.

The solution r_B is *involutive*, i.e., $r^2 = \mathrm{id}_{B \times B}$, if and only if $(B, +, \circ)$ is a brace.

Deformed solutions

[Doikou, Rybołowicz - 2022] introduced a new family of solutions coming from skew braces that can be obtained by *"deforming"* the classic map r_B by certain parameters.

Theorem (Doikou, Rybołowicz - 2022)

Let $(B, +, \circ)$ be a skew left brace and $z \in B$ such that the identity

$$(a - b + c) \circ z = a \circ z - b \circ z + c \circ z \qquad (*)$$

holds, for all $a, b, c \in B$. Then, the map $r_z : B \times B \to B \times B$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, \left(-a \circ z + a \circ b \circ z\right)^{-} \circ a \circ b\right)$$

a *non-degenerate bijective solution*.

Deformed solutions

[Doikou, Rybołowicz - 2022] introduced a new family of solutions coming from skew braces that can be obtained by *"deforming"* the classic map r_B by certain parameters.

Theorem (Doikou, Rybołowicz - 2022)

Let $(B, +, \circ)$ be a skew left brace and $z \in B$ such that the identity

$$(a - b + c) \circ z = a \circ z - b \circ z + c \circ z \qquad (*)$$

holds, for all $a, b, c \in B$. Then, the map $r_z : B \times B \to B \times B$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, \left(-a \circ z + a \circ b \circ z\right)^{-} \circ a \circ b\right)$$

a *non-degenerate bijective solution*.

Deformed solutions

[Doikou, Rybołowicz - 2022] introduced a new family of solutions coming from skew braces that can be obtained by *"deforming"* the classic map r_B by certain parameters.

Theorem (Doikou, Rybołowicz - 2022)

Let $(B, +, \circ)$ be a skew left brace and $z \in B$ such that the identity

$$(a - b + c) \circ z = a \circ z - b \circ z + c \circ z \qquad (*)$$

holds, for all $a, b, c \in B$. Then, the map $r_z : B \times B \to B \times B$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, (-a \circ z + a \circ b \circ z)^- \circ a \circ b\right)$$

a *non-degenerate bijective solution*.

Some considerations

Clearly, if B is a skew brace: $(a - b + c) \circ 0 = a \circ 0 - b \circ 0 + c \circ 0$, hence $r_0 = r_B$ is a solution.

For all $a, b \in B$, we write the components of r_z as:

$$\sigma_a^z(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau_b^z(a) := (-a \circ z + a \circ b \circ z)^{-} \circ a \circ b.$$

If $z \in B$ satisfies $(*)$, then the following hold for the map r_z:

- $\forall\ a, b \in B \quad a \circ b = \sigma_a^z(b) \circ \tau_b^z(a)$;
- $\tau^z : (B, \circ) \to \mathrm{Sym}_B, b \mapsto \tau_b^z$ is a group anti-homomorphism;
- σ^z is a homomorphism if and only if $\forall\ a \in B \quad a \circ z = z + a$.

Consequence: In general, the maps σ^z and τ^z do not satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Some considerations

Clearly, if B is a skew brace: $(a - b + c) \circ 0 = a \circ 0 - b \circ 0 + c \circ 0$, hence $r_0 = r_B$ is a solution.

For all $a, b \in B$, we write the components of r_z as:

$$\sigma_a^z(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau_b^z(a) := (-a \circ z + a \circ b \circ z)^- \circ a \circ b.$$

If $z \in B$ satisfies $(*)$, then the following hold for the map r_z:

- $\forall\ a, b \in B \quad a \circ b = \sigma_a^z(b) \circ \tau_b^z(a)$;
- $\tau^z : (B, \circ) \to \mathrm{Sym}_B, b \mapsto \tau_b^z$ is a group anti-homomorphism;
- σ^z is a homomorphism if and only if $\forall\ a \in B \quad a \circ z = z + a$.

Consequence: In general, the maps σ^z and τ^z do not satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Some considerations

Clearly, if B is a skew brace: $(a - b + c) \circ 0 = a \circ 0 - b \circ 0 + c \circ 0$, hence $r_0 = r_B$ is a solution.

For all $a, b \in B$, we write the components of r_z as:

$$\sigma_a^z(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau_b^z(a) := (-a \circ z + a \circ b \circ z)^- \circ a \circ b.$$

If $z \in B$ satisfies $(*)$, then the following hold for the map r_z:

- $\forall\ a, b \in B \quad a \circ b = \sigma_a^z(b) \circ \tau_b^z(a)$;
- $\tau^z : (B, \circ) \to \mathrm{Sym}_B, b \mapsto \tau_b^z$ is a group anti-homomorphism;
- σ^z is a homomorphism if and only if $\forall\ a \in B \quad a \circ z = z + a$.

Consequence: In general, the maps σ^z and τ^z do not satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Some considerations

Clearly, if B is a skew brace: $(a - b + c) \circ 0 = a \circ 0 - b \circ 0 + c \circ 0$, hence $r_0 = r_B$ is a solution.

For all $a, b \in B$, we write the components of r_z as:

$$\sigma^z_a(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau^z_b(a) := (-a \circ z + a \circ b \circ z)^- \circ a \circ b.$$

If $z \in B$ satisfies $(*)$, then the following hold for the map r_z:

- $\forall\ a, b \in B \quad a \circ b = \sigma^z_a(b) \circ \tau^z_b(a)$;
- $\tau^z : (B, \circ) \to \mathrm{Sym}_B,\ b \mapsto \tau^z_b$ is a group anti-homomorphism;
- σ^z is a homomorphism if and only if $\forall\ a \in B \quad a \circ z = z + a$.

Consequence: In general, the maps σ^z and τ^z do not satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Some considerations

Clearly, if B is a skew brace: $(a - b + c) \circ 0 = a \circ 0 - b \circ 0 + c \circ 0$, hence $r_0 = r_B$ is a solution.

For all $a, b \in B$, we write the components of r_z as:

$$\sigma_a^z(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau_b^z(a) := (-a \circ z + a \circ b \circ z)^- \circ a \circ b.$$

If $z \in B$ satisfies $(*)$, then the following hold for the map r_z:

- $\forall\ a, b \in B \quad a \circ b = \sigma_a^z(b) \circ \tau_b^z(a)$;
- $\tau^z : (B, \circ) \to \mathrm{Sym}_B,\ b \mapsto \tau_b^z$ is a group anti-homomorphism;
- σ^z is a homomorphism if and only if $\forall\ a \in B \quad a \circ z = z + a$.

Consequence: In general, the maps σ^z and τ^z do not satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Some considerations

Clearly, if B is a skew brace: $(a - b + c) \circ 0 = a \circ 0 - b \circ 0 + c \circ 0$, hence $r_0 = r_B$ is a solution.

For all $a, b \in B$, we write the components of r_z as:

$$\sigma_a^z(b) := -a \circ z + a \circ b \circ z \quad \text{and} \quad \tau_b^z(a) := (-a \circ z + a \circ b \circ z)^- \circ a \circ b.$$

If $z \in B$ satisfies $(*)$, then the following hold for the map r_z:

- $\forall\ a, b \in B \quad a \circ b = \sigma_a^z(b) \circ \tau_b^z(a)$;
- $\tau^z : (B, \circ) \to \mathrm{Sym}_B, b \mapsto \tau_b^z$ is a group anti-homomorphism;
- σ^z is a homomorphism if and only if $\forall\ a \in B \quad a \circ z = z + a$.

Consequence: In general, the maps σ^z and τ^z do not satisfy *Lu-Yan-Zhu conditions* on $(B, \circ)$.

Non-involutive solutions

Left braces determine also non-involutive solutions.

Example (Doikou, Rybołowicz - 2022)

Consider $\mathrm{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z} \right\}$ and the structure of brace $(\mathrm{Odd}, +_1, \circ)$ where the binary operation $+_1$ and $\circ$ are given by

$$\forall\, a, b \in \mathrm{Odd} \qquad a +_1 b := a - 1 + b \qquad \text{and} \qquad a \circ b := a \cdot b$$

with $+, \cdot$ are the usual addition and the multiplication of rational numbers, respectively. Then, for every $z \neq 1$, the solution r_z is not involutive.

Non-involutive solutions

Left braces determine also non-involutive solutions.

Example (Doikou, Rybołowicz - 2022)

Consider $\mathrm{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z} \right\}$ and the structure of brace $(\mathrm{Odd}, +_1, \circ)$ where the binary operation $+_1$ and $\circ$ are given by

$$\forall\, a, b \in \mathrm{Odd} \qquad a +_1 b := a - 1 + b \qquad \text{and} \qquad a \circ b := a \cdot b$$

with $+, \cdot$ are the usual addition and the multiplication of rational numbers, respectively. Then, for every $z \neq 1$, the solution r_z is not involutive.

Non-involutive solutions

Left braces determine also non-involutive solutions.

Example (Doikou, Rybołowicz - 2022)

Consider $\mathrm{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z} \right\}$ and the structure of brace $(\mathrm{Odd}, +_1, \circ)$ where the binary operation $+_1$ and $\circ$ are given by

$$\forall\, a, b \in \mathrm{Odd} \qquad a +_1 b := a - 1 + b \qquad \text{and} \qquad a \circ b := a \cdot b$$

with $+, \cdot$ are the usual addition and the multiplication of rational numbers, respectively. Then, for every $z \neq 1$, the solution r_z is not involutive.

The study of parameters

Question: If B is a skew (left) brace, which are all the parameters $z \in B$ giving rise to a solution r_z?

Definition (M., Rybołowicz, Stefanelli - 2023)

Let $(B, +, \circ)$ be a skew left brace. Then, we call the set

$$\mathcal{D}_r(B) = \{z \in B \mid \forall\, a, b \in B \quad (a + b) \circ z = a \circ z - z + b \circ z\},$$

the *right distributor* of B.

Theorem

Let $(B, +, \circ)$ be a skew (left) brace and $z \in B$. Then, the map r_z is a solution if and only if $z \in \mathcal{D}_r(B)$.

The study of parameters

Question: If B is a skew (left) brace, which are all the parameters $z \in B$ giving rise to a solution r_z?

Definition (M., Rybołowicz, Stefanelli - 2023)

Let $(B, +, \circ)$ be a skew left brace. Then, we call the set

$$\mathcal{D}_r(B) = \{z \in B \mid \forall\, a, b \in B \quad (a + b) \circ z = a \circ z - z + b \circ z\},$$

the *right distributor* of B.

Theorem

Let $(B, +, \circ)$ be a skew (left) brace and $z \in B$. Then, the map r_z is a solution if and only if $z \in \mathcal{D}_r(B)$.

The study of parameters

Question: If B is a skew (left) brace, which are all the parameters $z \in B$ giving rise to a solution r_z?

Definition (M., Rybołowicz, Stefanelli - 2023)

Let $(B, +, \circ)$ be a skew left brace. Then, we call the set

$$\mathcal{D}_r(B) = \{z \in B \mid \forall\, a, b \in B \quad (a + b) \circ z = a \circ z - z + b \circ z\},$$

the *right distributor* of B.

Theorem

Let $(B, +, \circ)$ be a skew (left) brace and $z \in B$. Then, the map r_z is a solution if and only if $z \in \mathcal{D}_r(B)$.

Two-sided skew braces

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace. Then, $\mathcal{D}_r(B) = B$. In other words, r_z is a deformed solution for every $z \in B$.

Question: Let B a two-sided skew brace and $z, w \in B$. Under which conditions are r_z and r_w equivalent?

[Trappeniers - 2023]: Given a two-sided skew brace $(B, +, \circ)$, all the inner automorphisms of $(B, \circ)$ are skew brace automorphisms of B.

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, r_z and r_w are equivalent.

The converse is not true.

Two-sided skew braces

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace. Then, $\mathcal{D}_r(B) = B$. In other words, r_z is a deformed solution for every $z \in B$.

Question: Let B a two-sided skew brace and $z, w \in B$. Under which conditions are r_z and r_w equivalent?

[Trappeniers - 2023]: Given a two-sided skew brace $(B, +, \circ)$, all the inner automorphisms of $(B, \circ)$ are skew brace automorphisms of B.

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, r_z and r_w are equivalent.

The converse is not true.

Two-sided skew braces

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace. Then, $\mathcal{D}_r(B) = B$. In other words, r_z is a deformed solution for every $z \in B$.

Question: Let B a two-sided skew brace and $z, w \in B$. Under which conditions are r_z and r_w equivalent?

[Trappeniers - 2023]: Given a two-sided skew brace $(B, +, \circ)$, all the inner automorphisms of $(B, \circ)$ are skew brace automorphisms of B.

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, r_z and r_w are equivalent.

The converse is not true.

Two-sided skew braces

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace. Then, $\mathcal{D}_r(B) = B$. In other words, r_z is a deformed solution for every $z \in B$.

Question: Let B a two-sided skew brace and $z, w \in B$. Under which conditions are r_z and r_w equivalent?

[Trappeniers - 2023]: Given a two-sided skew brace $(B, +, \circ)$, all the inner automorphisms of $(B, \circ)$ are skew brace automorphisms of B.

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, r_z and r_w are equivalent.

The converse is not true.

Some properties of the right distributor

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then,

$$Z(B, \circ) \leq (\mathcal{D}_r(B), \circ) \leq (B, \circ).$$

In general, $(\mathcal{D}_r(B), +) \nleq (B, +)$, unless we get into particular cases.

If $(B, +, \circ)$ is a (left) brace, then $\mathcal{D}_r(B)$ is a two-sided subbrace of B.

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then, $\mathrm{Fix}(B) \subseteq \mathcal{D}_r(B)$ and $\mathrm{Ann}(B) \subseteq \mathcal{D}_r(B)$.

We recall that

$$\mathrm{Fix}(B) = \{a \in B \mid \forall\, x \in B\ \lambda_x(a) = a\} \quad \& \quad \mathrm{Ann}(B) = \mathrm{Soc}(B) \cap Z(B, \circ),$$

with $\mathrm{Soc}(B) = \{a \in B \mid \forall\, b \in B \quad a + b = a \circ b\ \wedge\ a + b = b + a\}$.

Some properties of the right distributor

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then,

$$Z(B, \circ) \leq (\mathcal{D}_r(B), \circ) \leq (B, \circ).$$

In general, $(\mathcal{D}_r(B), +) \nleq (B, +)$, unless we get into particular cases.

If $(B, +, \circ)$ is a (left) brace, then $\mathcal{D}_r(B)$ is a two-sided subbrace of B.

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then, $\operatorname{Fix}(B) \subseteq \mathcal{D}_r(B)$ and $\operatorname{Ann}(B) \subseteq \mathcal{D}_r(B)$.

We recall that

$$\operatorname{Fix}(B) = \{a \in B \mid \forall\, x \in B\ \lambda_x(a) = a\} \quad \& \quad \operatorname{Ann}(B) = \operatorname{Soc}(B) \cap Z(B, \circ),$$

with $\operatorname{Soc}(B) = \{a \in B \mid \forall\, b \in B \quad a + b = a \circ b \ \wedge\ a + b = b + a\}$.

Some properties of the right distributor

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then,

$$Z(B, \circ) \leq (\mathcal{D}_r(B), \circ) \leq (B, \circ).$$

In general, $(\mathcal{D}_r(B), +) \nleq (B, +)$, unless we get into particular cases.

If $(B, +, \circ)$ is a (left) brace, then $\mathcal{D}_r(B)$ is a two-sided subbrace of B.

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then, $\mathrm{Fix}(B) \subseteq \mathcal{D}_r(B)$ and $\mathrm{Ann}(B) \subseteq \mathcal{D}_r(B)$.

We recall that

$$\mathrm{Fix}(B) = \{a \in B \mid \forall\ x \in B\ \lambda_x(a) = a\} \quad \& \quad \mathrm{Ann}(B) = \mathrm{Soc}(B) \cap Z(B, \circ),$$

with $\mathrm{Soc}(B) = \{a \in B \mid \forall\ b \in B \quad a + b = a \circ b\ \wedge\ a + b = b + a\}$.

Some properties of the right distributor

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then,

$$Z(B, \circ) \leq (\mathcal{D}_r(B), \circ) \leq (B, \circ).$$

In general, $(\mathcal{D}_r(B), +) \nleq (B, +)$, unless we get into particular cases.

If $(B, +, \circ)$ is a (left) brace, then $\mathcal{D}_r(B)$ is a two-sided subbrace of B.

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then, $\mathrm{Fix}(B) \subseteq \mathcal{D}_r(B)$ and $\mathrm{Ann}(B) \subseteq \mathcal{D}_r(B)$.

We recall that

$$\mathrm{Fix}(B) = \{a \in B \mid \forall\ x \in B\ \lambda_x(a) = a\} \quad \& \quad \mathrm{Ann}(B) = \mathrm{Soc}(B) \cap Z(B, \circ),$$

with $\mathrm{Soc}(B) = \{a \in B \mid \forall\ b \in B \quad a + b = a \circ b\ \wedge\ a + b = b + a\}$.

Some properties of the right distributor

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then,

$$Z(B, \circ) \leq (\mathcal{D}_r(B), \circ) \leq (B, \circ).$$

In general, $(\mathcal{D}_r(B), +) \nleq (B, +)$, unless we get into particular cases.

If $(B, +, \circ)$ is a (left) brace, then $\mathcal{D}_r(B)$ is a two-sided subbrace of B.

Proposition

Let $(B, +, \circ)$ be a skew (left) brace. Then, $\mathrm{Fix}(B) \subseteq \mathcal{D}_r(B)$ and $\mathrm{Ann}(B) \subseteq \mathcal{D}_r(B)$.

We recall that

$\mathrm{Fix}(B) = \{a \in B \mid \forall\ x \in B\ \lambda_x(a) = a\}$ & $\mathrm{Ann}(B) = \mathrm{Soc}(B) \cap Z(B, \circ)$,

with $\mathrm{Soc}(B) = \{a \in B \mid \forall\ b \in B \quad a + b = a \circ b\ \wedge\ a + b = b + a\}$.

Weak braces

Definition (Catino, M., Miccoli, Stefanelli, 2022)

A *weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

- $\forall a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c,$
- $\forall a \in S \quad a \circ a^- = -a + a,$

where $-a$ and a^- denote the inverses of $(S, +)$ and $(S, \circ)$.

If $(S, +, \circ)$ is a weak brace, then the map

$$r_S(a, b) = \left(-a + a \circ b, (-a + a \circ b)^- \circ a \circ b\right),$$

for all $a, b \in S$, is a solution that has a *behaviour close to bijectivity*. The solution $r_{S^{op}}$ associated to the weak brace S^{op} is such that

$$r_S\, r_{S^{op}}\, r_S = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Weak braces

Definition (Catino, M., Miccoli, Stefanelli, 2022)

A *weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

- $\forall a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c$,
- $\forall a \in S \quad a \circ a^{-} = -a + a$,

where $-a$ and a^{-} denote the inverses of $(S, +)$ and $(S, \circ)$.

If $(S, +, \circ)$ is a weak brace, then the map

$$r_S(a, b) = \left(-a + a \circ b, (-a + a \circ b)^{-} \circ a \circ b\right),$$

for all $a, b \in S$, is a solution that has a *behaviour close to bijectivity*. The solution $r_{S^{op}}$ associated to the weak brace S^{op} is such that

$$r_S\, r_{S^{op}}\, r_S = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Weak braces

Definition (Catino, M., Miccoli, Stefanelli, 2022)

A *weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

- $\forall a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c,$
- $\forall a \in S \quad a \circ a^{-} = -a + a,$

where $-a$ and a^{-} denote the inverses of $(S, +)$ and $(S, \circ)$.

If $(S, +, \circ)$ is a weak brace, then the map

$$r_S(a, b) = \left(-a + a \circ b, (-a + a \circ b)^{-} \circ a \circ b\right),$$

for all $a, b \in S$, is a solution that has a *behaviour close to bijectivity*.
The solution $r_{S^{op}}$ associated to the weak brace S^{op} is such that

$$r_S\, r_{S^{op}}\, r_S = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Weak braces

Definition (Catino, M., Miccoli, Stefanelli, 2022)

A *weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are inverse semigroups satisfying

- $\forall a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c,$
- $\forall a \in S \quad a \circ a^{-} = -a + a,$

where $-a$ and a^{-} denote the inverses of $(S, +)$ and $(S, \circ)$.

If $(S, +, \circ)$ is a weak brace, then the map

$$r_S(a, b) = \left(-a + a \circ b, (-a + a \circ b)^{-} \circ a \circ b\right),$$

for all $a, b \in S$, is a solution that has a *behaviour close to bijectivity*. The solution $r_{S^{op}}$ associated to the weak brace S^{op} is such that

$$r_S\, r_{S^{op}}\, r_s = r_S, \qquad r_{S^{op}}\, r_S\, r_{S^{op}} = r_{S^{op}}, \qquad \text{and} \qquad r_S r_{S^{op}} = r_{S^{op}} r_S.$$

Structural properties of weak braces

Theorem

Let $(S, +, \circ)$ be a weak brace. Then, $(S, +)$ is a Clifford semigroup.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

In general, $(S, \circ)$ is not Clifford. If it is, we call $(S, +, \circ)$ *dual weak brace*.
In this case, the solution r_S has also a *behaviour close to the non-degeneracy* in the sense that

$$\lambda_a \lambda_{a^-} \lambda_a = \lambda_a, \qquad \lambda_{a^-} \lambda_a \lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \quad \lambda_a \lambda_{a^-} = \lambda_{a^-} \lambda_a$$

$$\rho_a \rho_{a^-} \rho_a = \rho_a, \qquad \rho_{a^-} \rho_a \rho_{a^-} = \rho_{a^-}, \quad \text{and} \quad \rho_a \rho_{a^-} = \rho_{a^-} \rho_a$$

for every $a \in S$.

Structural properties of weak braces

Theorem

Let $(S,+,\circ)$ be a weak brace. Then, $(S,+)$ is a Clifford semigroup.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

In general, $(S,\circ)$ is not Clifford. If it is, we call $(S,+,\circ)$ *dual weak brace*.
In this case, the solution r_S has also a *behaviour close to the non-degeneracy* in the sense that

$$\lambda_a\lambda_{a^-}\lambda_a = \lambda_a, \qquad \lambda_{a^-}\lambda_a\lambda_{a^-} = \lambda_{a^-}, \qquad \text{and} \qquad \lambda_a\lambda_{a^-} = \lambda_{a^-}\lambda_a$$

$$\rho_a\rho_{a^-}\rho_a = \rho_a, \qquad \rho_{a^-}\rho_a\rho_{a^-} = \rho_{a^-}, \qquad \text{and} \qquad \rho_a\rho_{a^-} = \rho_{a^-}\rho_a$$

for every $a \in S$.

Theorem

Let $(S, +, \circ)$ be a weak brace. Then, $(S, +)$ is a Clifford semigroup.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

In general, $(S, \circ)$ is not Clifford. If it is, we call $(S, +, \circ)$ *dual weak brace*.

In this case, the solution r_S has also a *behaviour close to the non-degeneracy* in the sense that

$$\lambda_a\lambda_{a^-}\lambda_a = \lambda_a, \qquad \lambda_{a^-}\lambda_a\lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \quad \lambda_a\lambda_{a^-} = \lambda_{a^-}\lambda_a$$

$$\rho_a\rho_{a^-}\rho_a = \rho_a, \qquad \rho_{a^-}\rho_a\rho_{a^-} = \rho_{a^-}, \quad \text{and} \quad \rho_a\rho_{a^-} = \rho_{a^-}\rho_a$$

for every $a \in S$.

Structural properties of weak braces

Theorem

Let $(S, +, \circ)$ be a weak brace. Then, $(S, +)$ is a Clifford semigroup.

An inverse semigroup S is a *Clifford semigroup* if it has central idempotents.

In general, $(S, \circ)$ is not Clifford. If it is, we call $(S, +, \circ)$ *dual weak brace*.
In this case, the solution r_S has also a *behaviour close to the non-degeneracy* in the sense that

$$\lambda_a\lambda_{a^-}\lambda_a = \lambda_a, \qquad \lambda_{a^-}\lambda_a\lambda_{a^-} = \lambda_{a^-}, \quad \text{and} \quad \lambda_a\lambda_{a^-} = \lambda_{a^-}\lambda_a$$

$$\rho_a\rho_{a^-}\rho_a = \rho_a, \qquad \rho_{a^-}\rho_a\rho_{a^-} = \rho_{a^-}, \quad \text{and} \quad \rho_a\rho_{a^-} = \rho_{a^-}\rho_a$$

for every $a \in S$.

Deformed solutions on dual weak braces

Theorem (M., R., S. - 2023)

Let $(S, +, \circ)$ be a dual weak brace and $z \in S$. Then, the map $r_z : S \times S \to S \times S$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, (-a \circ z + a \circ b \circ z)^- \circ a \circ b\right),$$

for all $a, b \in S$, is a solution if and only if $z \in \mathcal{D}_r(S)$.

Note that $\mathrm{E}(S) \subseteq \mathcal{D}_r(S)$.

Example

Let $X = \{e, x, y\}$ and $(S, \circ)$ the commutative inverse monoid on X with identity e satisfying the relations $x \circ x = y \circ y = x$ and $x \circ y = y$. Considered the trivial weak brace on S, then the solutions $r_e = r$ and r_x are not equivalent.

Deformed solutions on dual weak braces

Theorem (M., R., S. - 2023)

Let $(S, +, \circ)$ be a dual weak brace and $z \in S$. Then, the map $r_z : S \times S \to S \times S$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, (-a \circ z + a \circ b \circ z)^{-} \circ a \circ b\right),$$

for all $a, b \in S$, is a solution if and only if $z \in \mathcal{D}_r(S)$.

Note that $\mathrm{E}(S) \subseteq \mathcal{D}_r(S)$.

Example

Let $X = \{e, x, y\}$ and $(S, \circ)$ the commutative inverse monoid on X with identity e satisfying the relations $x \circ x = y \circ y = x$ and $x \circ y = y$. Considered the trivial weak brace on S, then the solutions $r_e = r$ and r_x are not equivalent.

Deformed solutions on dual weak braces

Theorem (M., R., S. - 2023)

Let $(S, +, \circ)$ be a dual weak brace and $z \in S$. Then, the map $r_z : S \times S \to S \times S$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, (-a \circ z + a \circ b \circ z)^- \circ a \circ b\right),$$

for all $a, b \in S$, is a solution if and only if $z \in \mathcal{D}_r(S)$.

Note that $\mathrm{E}(S) \subseteq \mathcal{D}_r(S)$.

Example

Let $X = \{e, x, y\}$ and $(S, \circ)$ the commutative inverse monoid on X with identity e satisfying the relations $x \circ x = y \circ y = x$ and $x \circ y = y$. Considered the trivial weak brace on S, then the solutions $r_e = r$ and r_x are not equivalent.

Some references

- **A. Doikou, B. Rybołowicz**: *Novel non-involutive solutions of the Yang-Baxter equation from (skew) braces*, preprint
- **L. Guarnieri, L. Vendramin**: *Skew braces and the Yang-Baxter equation*, Math. Comp. 86(307), (2017) 2519-2534
- **Lu, M. Yan, Y.-C. Zhu:** *On the set-theoretical Yang-Baxter equation*, Duke Math. J. 104(1) (2000), 1–18.
- **M. Mazzotta, B. Rybołowicz, P. Stefanelli**: *Deformed solutions of the Yang-Baxter equation coming from dual weak braces and unital near-trusses*, preprint
- **W. Rump**: *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra 307(1) (2007) 153-170

Thank you!

Some references

- **A. Doikou, B. Rybołowicz**: *Novel non-involutive solutions of the Yang-Baxter equation from (skew) braces*, preprint
- **L. Guarnieri, L. Vendramin**: *Skew braces and the Yang-Baxter equation*, Math. Comp. 86(307), (2017) 2519-2534
- **Lu, M. Yan, Y.-C. Zhu:** *On the set-theoretical Yang-Baxter equation*, Duke Math. J. 104(1) (2000), 1–18.
- **M. Mazzotta, B. Rybołowicz, P. Stefanelli**: *Deformed solutions of the Yang-Baxter equation coming from dual weak braces and unital near-trusses*, preprint
- **W. Rump**: *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra 307(1) (2007) 153-170

Thank you!