# Quasi-projections and factorizations of monoids

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# General motivation

It was realized a long time ago that the problem of deciding whether a given mathematical object has a particular property can be solved by means of reducing it to the problem of representing the object as a ”union” of two (usually simpler) subobjects with minimal intersection and then to solve the problem for these subobjects for which one has techniques that were not available to begin with. Such a representation is called a *factorization* of the object.

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It was realized a long time ago that the problem of deciding whether a given mathematical object has a particular property can be solved by means of reducing it to the problem of representing the object as a "union" of two (usually simpler) subobjects with minimal intersection and then to solve the problem for these subobjects for which one has techniques that were not available to begin with. Such a representation is called a *factorization* of the object.

The factorization problem can be divided into two parts. The first part is to classify all the factorizations of a given mathematical object. The second part is to study the properties of a mathematical object that has already been factorized with respect to its two sub-objects.

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**DEFINITION**

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When this is the case, $A$ is said to be *factorized* by its submonoids $X$ and $B$. $X$ and $B$ are called, respectively, the *first and second factors* of this factorization. If $A$ is a group, then $A$ is factorizable if in addition $X$ and $B$ are subgroups of $A$.

# Factorizable monoids

Thus, a monoid $A$ is factorizable by its submonoids $X$ and $B$ if each element $a \in A$ may be written uniquely in the form $a = xb$ with $x \in X$ and $b \in B$. The relation $a = xb$ is called the $(X, B)$-*decomposition* of the element $a \in A$.

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For a given monoid $A$, we write $\mathrm{FAC}(A)$ for the set of all factorizations of $A$:

$$\mathrm{FAC}(A) = \{(X, B) : X, B \leq A \text{ and } A \text{ factorizes by } X \text{ and } B\}.$$

Note that $\mathrm{FAC}(A)$ always contains at least two elements, namely $(\{1\}, A)$ and $(A, \{1\})$. They are called the *trivial factorizations of A*. For a fixed submonoid $X$ of $A$, let $\mathrm{FAC}(A/X)$ be denote the set of those submonoids $B \leqslant A$ for which the pair $(X, B)$ is a monoid factorization of $A$.

# DESCENT COHOMOLOGY SETS

Let $\imath : X \to A$ be an (injective) homomorphism of monoids. We call a map $q : A \to X$ a *(left) 1-descent cocycle of the monoid A with coefficients in the monoid X* if the following conditions are satisfied:

(1) $q(1) = 1$;

(2) $q(\imath(x) \cdot a) = x \cdot q(a) \quad (\forall a \in A, \forall x \in X)$, and

(3) $q(a \cdot a') = q(a \cdot (\imath q)(a')) \quad (\forall a, a' \in A)$.

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- The set of the left 1-descent cocycles of $A$ with coefficients in $X$ is denoted by $\mathcal{Z}_l^1(A/X, \imath)$.
- Two 1-cocyles $q_1, q_2 \in \mathcal{Z}_l^1(A/X, \imath)$ are called *cohomologous* if there exists an element $x_0 \in U(X)$ such that for any $a \in A$, $q_1(a) \cdot x_0 = q_2(a \cdot \imath(x_0))$.

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- The cohomology relation is an equivalence relation on $\mathcal{Z}_l^1(A/X, \imath)$. The set of equivalence class of cohomologous left 1–descent cocycles is denoted by $\mathscr{D}_l^1(A/X, \imath)$.

Symmetrically, one has the set $\mathcal{Z}_r^1(A/X, \imath)$ of the right 1-descent cocycles of $A$ with coefficients in $X$ is denoted by $\mathcal{Z}_l^1(A/X, \imath)$ and the set $\mathscr{D}_r^1(A/X, \imath)$ equivalence class of cohomologous right 1–descent cocycles.

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Typical examples of 1-descent cocycles arise from monoid factorizations as follows. Let a monoid $A$ factorizes by its submonoids $X$ and $B$. Write $q_{X,B}$ and $p_{X,B}$ for the compositions $A \xrightarrow{[\imath_X, \imath_B]^{-1}} X \times B \xrightarrow{\pi_X} X$, $A \xrightarrow{[\imath_X, \imath_B]^{-1}} X \times B \xrightarrow{\pi_B} B$, respectively. Then $q_{XB} \in \mathcal{Z}^1_l(A/X, \imath_X)$ and $p_{XB} \in \mathcal{Z}^1_r(A/B, \imath_B)$.

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Note that if a monoid $A$ factorizes through its submonoids $X$ and $B$, $\mathcal{Z}_l^1(A/X,\imath_X)$ and $\mathcal{Z}_r^1(A/B,\imath_B)$ can be regarded as pointed sets, whose base-points are $q_{XB}$ and $p_{XB}$, respectively.

# Non-abelian cohomology pointed sets of monoids

Let $X, B$ be monoids. A *left action* of $B$ on $X$ is a homomorphism $\Phi : B \to \mathrm{End}(X)$ of monoids. In this situation, one also say that $B$ *acts on* $X$ *from the left*, and for all $b \in B$ and $x \in X$, one writes

$$b \bullet x = \Phi(b)(x).$$

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A map $\chi : B \to X$ is called a (*non-abelian*) *1-cocycle*, if $\chi(1) = 1$ and for arbitrary $b_1, b_2 \in B$:

$$\chi(b_1 \cdot b_2) = \chi(b_1) \cdot (b_1 \bullet \chi(b_2)).$$

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We denote the set of all 1-cocycles $B \to X$ with $\mathcal{Z}^1(B, X)$. One turns $\mathcal{Z}^1(B, X)$ into a pointed set by defining the distinguished element to be the map $0_{B,X}$.

Two 1-cocyles $\chi, \chi' : B \to X$ are called *equivalent* if there exists an invertible element $x_0 \in U(X)$ such that $\chi(b)(b \bullet x_0) = x_0\chi'(b)$ for all $b \in B$.

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This provides an equivalence relation on $\mathcal{Z}^1(B, X)$. The set of equivalence classes of 1-cocycles is called the *first non-abelian 1-cohomology set* of $B$ with coefficients in $X$ and is denoted by $\mathbf{H}^1(B, X)$. It is a pointed set with base-point the equivalence class of the map $0_{B,X}$.

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Note that in the case when both $X$ and $B$ are groups, the pointed set $\mathbf{H}^1(B, X)$ coincide with the ones introduced by S. Serre.

The following theorem shows that non-abelian cohomology of monoids can be seen as a particular instance of descent cohomology of monoids.

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### THEOREM

*Let a monoid B acts on a monoid X from the left and let* $X \rtimes B$ *be the corresponding semi-direct product. Then the assignment* $q \to \imath_B q$ *yields bijections*

$$\mathcal{Z}_l^1(X \rtimes B/X, \imath_X) \simeq \mathcal{Z}^1(B, X)$$

*and*

$$\mathscr{D}_l^1(X \rtimes B/X, \imath_X) \simeq \mathbf{H}^1(B, X)$$

### Definition

A *left quasi-projection on a monoid $A$* is a set map $\mathbf{l} : A \to A$ such that for all $a, a' \in A$:

(L1) $\mathbf{l}(1) = 1$.

(L2) $\mathbf{l}(a) \cdot \mathbf{l}(a') = \mathbf{l}(\mathbf{l}(a) \cdot a')$.

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Symmetrically, a *right quasi-projection on a monoid A* is a set map $\mathbf{r} : A \to A$ satisfying the following conditions:

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# QUASI-PROJECTIONS ON MONOIDS

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Condition L3 (resp. R3) implies that a left (resp. right) quasi-projection is an idempotent map.

# QUASI-PROJECTIONS ON MONOIDS

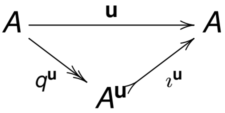
Let $A$ be a monoid. To each map $\mathbf{u} : A \to A$ of sets, one can associate two subsets of $A$:

$$A^{\mathbf{u}} = \{a \in A : \mathbf{u}(a) = a\}$$

and

$$A_{\mathbf{u}} = Ker(\mathbf{u}) = \{a \in A : \mathbf{u}(a) = 1\}.$$

We write $\imath^{\mathbf{u}} : A^{\mathbf{u}} \rightarrowtail A$ and $\imath_{\mathbf{u}} : A_{\mathbf{u}} \rightarrowtail A$ for the canonical inclusions. If, in addition, $\mathbf{u}$ is an idempotent map, then it factorizes through $A^{\mathbf{u}}$:



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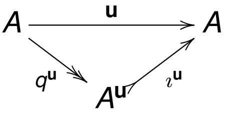


### PROPOSITION

*If $\mathbf{l}$ (resp. $\mathbf{r}$) is a left (resp. right) quasi-projection on a monoid $A$, then $A^{\mathbf{l}}$ and $A_{\mathbf{l}}$ (resp. $A^{\mathbf{r}}$ and $A_{\mathbf{r}}$) are both submonoids of $A$. If, in addition, $A$ is a group, then $A^{\mathbf{l}}$ and $A_{\mathbf{l}}$ (resp. $A^{\mathbf{r}}$ and $A_{\mathbf{r}}$) are subgroups of $A$.*

# QUASI-PROJECTIONS ON MONOIDS AND DESCENT COHOMOLOGY

### PROPOSITION

*Let $A$ be a monoid. A map $\boldsymbol{u} : A \to A$ is a left (resp. right) quasi-projection on $A$ if and only if the inclusion $\imath^{\boldsymbol{u}} : A^{\boldsymbol{u}} \to A$ makes $A^{\boldsymbol{u}}$ a submonoid of $A$ and there is (necessarily) a unique element $q^{\boldsymbol{u}} \in \mathcal{Z}_l^1(A/A^{\boldsymbol{u}}, \imath^{\boldsymbol{u}})$ (resp. $p^{\boldsymbol{u}} \in \mathcal{Z}_r^1(A/A^{\boldsymbol{u}}, \imath^{\boldsymbol{u}})$) such that $\boldsymbol{u} = \imath^{\boldsymbol{u}} q^{\boldsymbol{u}}$ (resp. $\boldsymbol{u} = \imath^{\boldsymbol{u}} p^{\boldsymbol{u}}$).*

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For a given monoid $A$, we write $\mathrm{LQ}(A)$ (resp. $\mathrm{RQ}(A)$) for the set of all left (resp. right) quasi-projections on $A$.

# Quasi-projections on monoids and descent cohomology

## Proposition

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## Proposition

*For any submonoid $X \subseteq A$ of a monoid $A$, the assignment $q \longmapsto \imath_X q$ yields an injective map*

$$\kappa_X : \mathcal{Z}^1_l(A/X, \imath_X) \to LQ(A).$$

**THEOREM**

*For any monoid A, the map*

$$\bigsqcup_{X \leqslant A} \kappa_X : \bigsqcup_{X \leqslant A} \mathcal{Z}^1_l(A/X, \imath_X) \to LQ(A)$$

*is bijective. Its inverse takes $\boldsymbol{l} \in LQ(A)$ to $q^{\boldsymbol{l}}$, where $q^{\boldsymbol{l}}$ is the first factor in the factorization $\boldsymbol{l} = \imath^{\boldsymbol{l}} q^{\boldsymbol{l}}$ of the idempotent map $\boldsymbol{l}$.*

**DEFINITION**

Let $\mathbf{l}$ and $\mathbf{r}$ be respectively left and right quasi-projections on a monoid $A$. We say that $(\mathbf{l}, \mathbf{r})$ is a *complementary pair of quasi-projections on A* provided the following conditions are satisfied:

(I) $\mathbf{lr} = \mathbf{rl} = 0_{A,A}$ and

(II) $\mathbf{l} * \mathbf{r} = 1_A$,

where the map $\mathbf{l} * \mathbf{r} : A \to A$ is defined by $(\mathbf{l} * \mathbf{r})(a) = \mathbf{l}(a)\mathbf{r}(a)$.

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Given a monoid $A$, we write $\mathrm{CP}(A)$ for the set of complementary pair of quasi-projections an $A$.

**PROPOSITION**

*Let $A$ be a monoid and $(\boldsymbol{l}, \boldsymbol{r}) \in CP(A)$. Then $A$ factorizes through $A^{\boldsymbol{l}}$ and $A_{\boldsymbol{l}}$; i.e., the map*

$$[\imath^{\boldsymbol{l}}, \imath_{\boldsymbol{l}}] : A^{\boldsymbol{l}} \times A_{\boldsymbol{l}} \to A$$

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### Theorem

*For any monoid $A$, the correspondence*

$$(\boldsymbol{l}, \boldsymbol{r}) \longmapsto (A^{\boldsymbol{l}}, A^{\boldsymbol{r}})$$

*yields a bijection*

$$CP(A) \simeq FAC(A).$$

Let $A$ be a monoid and $\mathbf{I} : A \to A$ a set-theoretical map such that $A^{\mathbf{I}}$ is a subgroup of $A$. Then since $\mathbf{I}(a) \in A^{\mathbf{I}}$, it follows that $\mathbf{I}(a)^{-1} \in A^{\mathbf{I}}$. Hence it makes sense to define a map $\mathbf{I}^{\triangleright} : A \to A$ by $\mathbf{I}^{\triangleright}(a) = \mathbf{I}(a)^{-1}\, a$.

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**PROPOSITION**

*Suppose that $\mathbf{I}$ is a left quasi-projection on a monoid $A$. Then the map $\mathbf{I}^{\triangleright} : A \to A$ is a right quasi-projection on $A$ and the pair $(\mathbf{I}, \mathbf{I}^{\triangleright})$ is complementary.*

## THEOREM

*For any group A, we have a bijection*

$$LQ(A) \to FAC(A)$$

*given by* $\boldsymbol{l} \longmapsto (A^{\boldsymbol{l}}, A_{\boldsymbol{l}})$. *The inverse map is given by*

$$(X, B) \longmapsto (A \xrightarrow{[\imath_X, \imath_B]^{-1}} X \times B \xrightarrow{\pi_X} X \xrightarrow{\imath_X} A).$$

*Symmetrically, the correspondence*

$$\boldsymbol{r} \longmapsto (A_{\boldsymbol{r}}, A^{\boldsymbol{r}})$$

*yields a bijection* $RQ(A) \to FAC(A)$, *whose inverse takes* $(X, B) \in FAC(A)$ *to the composite* $A \xrightarrow{[\imath_X, \imath_B]^{-1}} X \times B \xrightarrow{\pi_B} B \xrightarrow{\imath_B} A$.

# The case of semi-direct product

Recall that any group $A$ acts from the left on itself by conjugation, namely we have a left action

$$\bullet : A \times A \to A, \;\; a \bullet b = aba^{-1}.$$

Thus one can define the set of all non-abelian 1-cocycles $\mathcal{Z}^1(A, A)$.

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Thus one can define the set of all non-abelian 1-cocycles $\mathcal{Z}^1(A, A)$.

**PROPOSITION**

*Let A be a group. The following are equivalent for an idempotent map* $\boldsymbol{I} : A \to A$ *of sets:*

## The case of semi-direct product

Recall that any group $A$ acts from the left on itself by conjugation, namely we have a left action

$$\bullet : A \times A \to A, \;\; a \bullet b = aba^{-1}.$$

Thus one can define the set of all non-abelian 1-cocycles $\mathcal{Z}^1(A, A)$.

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3. *$\boldsymbol{l}$ is a left quasi-projection on $A$ such that $A^{\boldsymbol{l}}$ is a normal subgroup of $A$.*

*When these hold, $A$ is isomorphic to the semi-direct product $A^{\boldsymbol{l}} \rtimes A^{\boldsymbol{l}^{\triangleright}}$.*

### THEOREM

*For any group A, the assignment*

$$(X, B) \longmapsto (A \xrightarrow{[\imath_X, \imath_B]^{-1}} X \times B \xrightarrow{\pi_X} X \xrightarrow{\imath_X} A),$$

*establishes a bijection*

$$FAC^{\rtimes}(A) \simeq \mathcal{Z}^1_{Id}(A, A).$$

*Here, $FAC^s(A)$ is the subset of $FAC^{\rtimes}(A)$ containing of semi-direct factorizations of A, while $\mathcal{Z}^1_{Id}(A, A)$ is the subset of $\mathcal{Z}^1(A, A)$ whose elements are idempotent non-abelian 1-cocycles $A \to A$.*

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Two submonoids $B$ and $B'$ of a monoid $A$ are called *conjugate* if $B' = aBa^{-1}$ for some invertible $a \in A$. When this is the case, one says that $a$ conjugates $B$ to $B'$.

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*Let a monoid B acts from the left on a monoid B and $X \rtimes B$ be the corresponding semi-direct product. Then the assignment*

$$\chi \to \{(\chi(b), b) : b \in B\}$$

*yields an isomorphism*

$$\boldsymbol{H}^1(B, U(A)) \simeq FAC(X \rtimes B/X)/_\sim$$

*of pointed sets.*

Let $X$ be a normal subgroup of a group $A$. Then $(a, x) \longmapsto axa^{-1}$ yields a left action of the group $A$ on the group $X$ and so we may define the pointed set $\mathcal{Z}^1(A, X)$. We write $\mathcal{Z}^1_\bullet(A, X)$ for the subset of $\mathcal{Z}^1(A, X)$ containing of the maps $\chi : A \to X$ such that $\chi(x) = x^{-1}$ for all $x \in X$.

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