Dihedral solutions of the set-theoretic Yang-Baxter equation

Alex W. Nowak
Howard University
Joint w. C. Buell, A. Zamojska-Dzienio, J.D.H. Smith

Loops 2023
Mathematical Conference and Research Center, Będlewo Poland

June 30, 2023

Summary

Set-theoretic Yang-Baxter equation

- A map $r : X^2 \to X^2; (x, y) \mapsto (x \circ y, x \bullet y)$ is a set-theoretic solution of the Yang-Baxter equation (SYBE) if, on X^3,

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}.$$

- Call (X, r) a braided set.
- We'll be dealing with Latin solutions, which means $(X, \circ)$ is a quasigroup and $(X, \bullet)$ is a right-quasigroup.

Set-theoretic Yang-Baxter equation

- A map $r : X^2 \to X^2; (x, y) \mapsto (x \circ y, x \bullet y)$ is a set-theoretic solution of the Yang-Baxter equation (SYBE) if, on X^3,

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}.$$

- Call (X, r) a braided set.
- We'll be dealing with Latin solutions, which means $(X, \circ)$ is a quasigroup and $(X, \bullet)$ is a right-quasigroup.

Set-theoretic Yang-Baxter equation

- A map $r : X^2 \to X^2; (x, y) \mapsto (x \circ y, x \bullet y)$ is a set-theoretic solution of the Yang-Baxter equation (SYBE) if, on X^3,

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}.$$

- Call (X, r) a braided set.
- We'll be dealing with Latin solutions, which means $(X, \circ)$ is a quasigroup and $(X, \bullet)$ is a right-quasigroup.

Derived solutions and derived racks

- A derived solution has the form $r : (x, y) \mapsto (x \circ y, x)$.
  - This makes $(X, \circ)$ a rack.
  - Conversely, any rack yields a derived solution.
- To any left non-degenerate solution $r : (x, y) \mapsto (x \circ y, x \bullet y)$, we may associate the derived (left) rack of the solution

$$x \triangleright_r y = x \circ (y \bullet (y \backslash_\circ x)).$$

Derived solutions and derived racks

- A derived solution has the form $r : (x, y) \mapsto (x \circ y, x)$.
  - This makes $(X, \circ)$ a rack.
  - Conversely, any rack yields a derived solution.
- To any left non-degenerate solution $r : (x, y) \mapsto (x \circ y, x \bullet y)$, we may associate the derived (left) rack of the solution

$$x \triangleright_r y = x \circ (y \bullet (y \backslash_\circ x)).$$

Derived solutions and derived racks

- A derived solution has the form $r : (x, y) \mapsto (x \circ y, x)$.
  - This makes $(X, \circ)$ a rack.
  - Conversely, any rack yields a derived solution.
- To any left non-degenerate solution $r : (x, y) \mapsto (x \circ y, x \bullet y)$, we may associate the derived (left) rack of the solution

$$x \triangleright_r y = x \circ (y \bullet (y \backslash_\circ x)).$$

Involutive quandles

Definition: Involutive quandle (Takasaki, 1942)

An involutive quandle $(X, \cdot)$ is one for which the left multiplication maps are involutions, ie, $x \cdot (x \cdot y) = y$.

- We've seen the dihedral quandles $x \cdot y = 2x - y \mod n$.
  - Represent the set of reflections in D_n under conjugation.
- A commutative, involutive quandle is a distrbiutive Steiner quasigroup (STS).

Involutive quandles

Definition: Involutive quandle (Takasaki, 1942)

An involutive quandle $(X, \cdot)$ is one for which the left multiplication maps are involutions, ie, $x \cdot (x \cdot y) = y$.

- We've seen the dihedral quandles $x \cdot y = 2x - y \mod n$.
  - Represent the set of reflections in D_n under conjugation.
- A commutative, involutive quandle is a distrbiutive Steiner quasigroup (STS).

Involutive quandles

Definition: Involutive quandle (Takasaki, 1942)

An involutive quandle $(X, \cdot)$ is one for which the left multiplication maps are involutions, ie, $x \cdot (x \cdot y) = y$.

- We've seen the dihedral quandles $x \cdot y = 2x - y \mod n$.
  - Represent the set of reflections in D_n under conjugation.
- A commutative, involutive quandle is a distrbiutive Steiner quasigroup (STS).

Motivating example

Example (Smith 2015):

If $(X, +, 0)$ is a CML$_3$, then $r : (x, y) \mapsto (-x + y, -x)$ is a Latin SYBE solution.

- Some observations:
  1. The derived rack of r is the dihedral quandle $x \cdot y = -x - y$.
  2. $r : (x, y) \mapsto (x \cdot (e \cdot y), e \cdot x) = (S_x(S_e(y)), S_e(x))$
  3. $r^3 = (tr)^2 = 1$, where $t : (x, y) \mapsto (y, x)$ is the trivial solution. That is, we have D_3-symmetry.
- Is the coupling of D_3-symmetry with the dihedral quandle a coincidence?

Motivating example

Example (Smith 2015):

If $(X, +, 0)$ is a CML_3, then $r : (x, y) \mapsto (-x + y, -x)$ is a Latin SYBE solution.

- Some observations:
  1. The derived rack of r is the dihedral quandle $x \cdot y = -x - y$.
  2. $r : (x, y) \mapsto (x \cdot (e \cdot y), e \cdot x) = (S_x(S_e(y)), S_e(x))$
  3. $r^3 = (tr)^2 = 1$, where $t : (x, y) \mapsto (y, x)$ is the trivial solution. That is, we have D_3-symmetry.
- Is the coupling of D_3-symmetry with the dihedral quandle a coincidence?

Motivating example

Example (Smith 2015):

If $(X, +, 0)$ is a CML_3, then $r : (x, y) \mapsto (-x + y, -x)$ is a Latin SYBE solution.

- Some observations:
  1. The derived rack of r is the dihedral quandle $x \cdot y = -x - y$.
  2. $r : (x, y) \mapsto (x \cdot (e \cdot y), e \cdot x) = (S_x(S_e(y)), S_e(x))$
  3. $r^3 = (tr)^2 = 1$, where $t : (x, y) \mapsto (y, x)$ is the trivial solution. That is, we have D_3-symmetry.
- Is the coupling of D_3-symmetry with the dihedral quandle a coincidence?

Motivating example

Example (Smith 2015):

If $(X,+,0)$ is a CML_3, then $r:(x,y)\mapsto(-x+y,-x)$ is a Latin SYBE solution.

- Some observations:
  1. The derived rack of r is the dihedral quandle $x\cdot y=-x-y$.
  2. $r:(x,y)\mapsto(x\cdot(e\cdot y),e\cdot x)=(S_x(S_e(y)),S_e(x))$
  3. $r^3=(tr)^2=1$, where $t:(x,y)\mapsto(y,x)$ is the trivial solution. That is, we have D_3-symmetry.
- Is the coupling of D_3-symmetry with the dihedral quandle a coincidence?

Motivating example

Example (Smith 2015):

If $(X,+,0)$ is a CML$_3$, then $r:(x,y)\mapsto(-x+y,-x)$ is a Latin SYBE solution.

- Some observations:
  1. The derived rack of r is the dihedral quandle $x\cdot y=-x-y$.
  2. $r:(x,y)\mapsto(x\cdot(e\cdot y),e\cdot x)=(S_x(S_e(y)),S_e(x))$
  3. $r^3=(tr)^2=1$, where $t:(x,y)\mapsto(y,x)$ is the trivial solution. That is, we have D_3-symmetry.
- Is the coupling of D_3-symmetry with the dihedral quandle a coincidence?

Latin braided sets with triality

Definition: (B.N.S.Z.-D.)

Let $r : (x, y) \mapsto (x \circ y, x \bullet y)$ be a Latin braiding. If $r^3 = (tr)^2 = \mathrm{id}_{X^2}$, then (X, r) is a Latin, braided set with triality (LBST).

- Question: How much can LBST stray from our motivating example?
- Answer: Not very far!

Result 1: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

Result 2: B.N.S.Z.-D.

If $\mathrm{Sq}_\circ : x \mapsto x \circ x$ is homomorphic, (X, r) is a split extension of a derived solution by one of the form $(x, y) \mapsto (-x + y, -x)$ for $(X, +)$ a CML_3.

Latin braided sets with triality

Definition: (B.N.S.Z.-D.)

Let $r : (x, y) \mapsto (x \circ y, x \bullet y)$ be a Latin braiding. If $r^3 = (tr)^2 = \mathrm{id}_{X^2}$, then (X, r) is a Latin, braided set with triality (LBST).

- Question: How much can LBST stray from our motivating example?
- Answer: Not very far!

Result 1: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

Result 2: B.N.S.Z.-D.

If $\mathrm{Sq}_\circ : x \mapsto x \circ x$ is homomorphic, (X, r) is a split extension of a derived solution by one of the form $(x, y) \mapsto (-x + y, -x)$ for $(X, +)$ a CML_3.

Latin braided sets with triality

Definition: (B.N.S.Z.-D.)

Let $r : (x, y) \mapsto (x \circ y, x \bullet y)$ be a Latin braiding. If $r^3 = (tr)^2 = \mathrm{id}_{X^2}$, then (X, r) is a Latin, braided set with triality (LBST).

- Question: How much can LBST stray from our motivating example?
- Answer: Not very far!

Result 1: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

Result 2: B.N.S.Z.-D.

If $\mathrm{Sq}_\circ : x \mapsto x \circ x$ is homomorphic, (X, r) is a split extension of a derived solution by one of the form $(x, y) \mapsto (-x + y, -x)$ for $(X, +)$ a CML_3.

Latin braided sets with triality

Definition: (B.N.S.Z.-D.)

Let $r : (x, y) \mapsto (x \circ y, x \bullet y)$ be a Latin braiding. If $r^3 = (tr)^2 = \mathrm{id}_{X^2}$, then (X, r) is a Latin, braided set with triality (LBST).

- Question: How much can LBST stray from our motivating example?
- Answer: Not very far!

Result 1: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

Result 2: B.N.S.Z.-D.

If $\mathrm{Sq}_\circ : x \mapsto x \circ x$ is homomorphic, (X, r) is a split extension of a derived solution by one of the form $(x, y) \mapsto (-x + y, -x)$ for $(X, +)$ a CML_3.

Latin braided sets with triality

Definition: (B.N.S.Z.-D.)

Let $r : (x, y) \mapsto (x \circ y, x \bullet y)$ be a Latin braiding. If $r^3 = (tr)^2 = \mathrm{id}_{X^2}$, then (X, r) is a Latin, braided set with triality (LBST).

- Question: How much can LBST stray from our motivating example?
- Answer: Not very far!

Result 1: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

Result 2: B.N.S.Z.-D.

If $\mathrm{Sq}_\circ : x \mapsto x \circ x$ is homomorphic, (X, r) is a split extension of a derived solution by one of the form $(x, y) \mapsto (-x + y, -x)$ for $(X, +)$ a CML_3.

Identities in LBST

For any LBST (X, r), we have the following:

- $x \bullet y = x \backslash_\circ x$, ie, $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$ (using Prover9).
    - Because $x \bullet y = x \backslash_\circ y$, we'll abbreviate $\backslash_\circ$ to $\backslash$.
- $(X, \backslash)$ is a commutative quasigroup in which the squaring map $\mathrm{Sq}_\backslash : x \mapsto x \backslash x$ is an involutive endomorphism.
    - In particular, $(x \backslash x) \backslash (x \backslash x) = x$
- Proving commutativity of $(X, \backslash)$ also “required” Prover9.

Identities in LBST

For any LBST (X, r), we have the following:

- $x \bullet y = x \backslash_\circ x$, ie, $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$ (using Prover9).
  - Because $x \bullet y = x \backslash_\circ y$, we'll abbreviate $\backslash_\circ$ to $\backslash$.
- $(X, \backslash)$ is a commutative quasigroup in which the squaring map $\mathrm{Sq}_\backslash : x \mapsto x \backslash x$ is an involutive endomorphism.
  - In particular, $(x \backslash x) \backslash (x \backslash x) = x$
- Proving commutativity of $(X, \backslash)$ also "required" Prover9.

Identities in LBST

For any LBST (X, r), we have the following:

- $x \bullet y = x \backslash_\circ x$, ie, $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$ (using Prover9).
  - Because $x \bullet y = x \backslash_\circ y$, we'll abbreviate $\backslash_\circ$ to $\backslash$.
- $(X, \backslash)$ is a commutative quasigroup in which the squaring map $\mathrm{Sq}_\backslash : x \mapsto x \backslash x$ is an involutive endomorphism.
  - In particular, $(x \backslash x) \backslash (x \backslash x) = x$
- Proving commutativity of $(X, \backslash)$ also "required" Prover9.

Identities in LBST

For any LBST (X, r), we have the following:

- $x \bullet y = x \backslash_\circ x$, ie, $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$ (using Prover9).
  - Because $x \bullet y = x \backslash_\circ y$, we'll abbreviate $\backslash_\circ$ to $\backslash$.
- $(X, \backslash)$ is a commutative quasigroup in which the squaring map $\text{Sq}_\backslash : x \mapsto x \backslash x$ is an involutive endomorphism.
  - In particular, $(x \backslash x) \backslash (x \backslash x) = x$
- Proving commutativity of $(X, \backslash)$ also "required" Prover9.

Identities in LBST

For any LBST (X, r), we have the following:

- $x \bullet y = x \backslash_\circ x$, ie, $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$ (using Prover9).
  - Because $x \bullet y = x \backslash_\circ y$, we'll abbreviate $\backslash_\circ$ to $\backslash$.
- $(X, \backslash)$ is a commutative quasigroup in which the squaring map $\mathsf{Sq}_\backslash : x \mapsto x \backslash x$ is an involutive endomorphism.
  - In particular, $(x \backslash x) \backslash (x \backslash x) = x$
- Proving commutativity of $(X, \backslash)$ also "required" Prover9.

The structure rack

Proposition: B.N.S.Z.-D.

The sturcture rack of an LBST (X, r) has form

$$x \triangleright_r y = x \circ (y \backslash y).$$

This is an affine STS, and it is principally isotopic to $(X, \circ)$

Corollary: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

The structure rack

Proposition: B.N.S.Z.-D.

The sturcture rack of an LBST (X, r) has form

$$x \triangleright_r y = x \circ (y \backslash y).$$

This is an affine STS, and it is principally isotopic to $(X, \circ)$

Corollary: B.N.S.Z.-D.

If (X, r) is a finite LBST, then $|X| = 3^n$.

Another squaring map and a structure theorem

- Consider $\mathsf{Sq}_\circ : x \mapsto x \circ x$
- This maps into a subquasigroup of idempotents of $(X, \circ, /, \backslash)$.
- $(\mathsf{Sq}_\circ(X), \circ)$ is a subquandle of $(X, \triangleright_r)$.
- For any $e \in X$, $X_e = \{x \in X \mid \mathsf{Sq}_\circ(x) = \mathsf{Sq}_\circ(e)\}$ is a subquasigroup of $(X, \circ, /, \backslash)$, and $(X_e, \backslash, e)$ is a CML_3.
  - Prover9 to the rescue again!

A Structure Theorem (B.N.S.Z.-D.)

Let (X, r) be an LBST. If $\mathsf{Sq}_\circ$ is an endomorphism of $(X, \circ)$, then we have a split exact sequence of quasigroups

$$\{*\} \to X_e \to X \to \mathsf{Sq}_\circ(X) \to \{*\}$$

Another squaring map and a structure theorem

- Consider $\mathsf{Sq}_\circ : x \mapsto x \circ x$
- This maps into a subquasigroup of idempotents of $(X, \circ, /, \backslash)$.
- $(\mathsf{Sq}_\circ(X), \circ)$ is a subquandle of $(X, \triangleright_r)$.
- For any $e \in X$, $X_e = \{x \in X \mid \mathsf{Sq}_\circ(x) = \mathsf{Sq}_\circ(e)\}$ is a subquasigroup of $(X, \circ, /, \backslash)$, and $(X_e, \backslash, e)$ is a CML_3.
  - Prover9 to the rescue again!

A Structure Theorem (B.N.S.Z.-D.)

Let (X, r) be an LBST. If $\mathsf{Sq}_\circ$ is an endomorphism of $(X, \circ)$, then we have a split exact sequence of quasigroups

$$\{*\} \to X_e \to X \to \mathsf{Sq}_\circ(X) \to \{*\}$$

Another squaring map and a structure theorem

- Consider $\mathsf{Sq}_\circ : x \mapsto x \circ x$
- This maps into a subquasigroup of idempotents of $(X, \circ, /, \backslash)$.
- $(\mathsf{Sq}_\circ(X), \circ)$ is a subquandle of $(X, \triangleright_r)$.
- For any $e \in X$, $X_e = \{x \in X \mid \mathsf{Sq}_\circ(x) = \mathsf{Sq}_\circ(e)\}$ is a subquasigroup of $(X, \circ, /, \backslash)$, and $(X_e, \backslash, e)$ is a CML_3.
  - Prover9 to the rescue again!

A Structure Theorem (B.N.S.Z.-D.)

Let (X, r) be an LBST. If $\mathsf{Sq}_\circ$ is an endomorphism of $(X, \circ)$, then we have a split exact sequence of quasigroups

$$\{*\} \to X_e \to X \to \mathsf{Sq}_\circ(X) \to \{*\}$$

Another squaring map and a structure theorem

- Consider $\mathsf{Sq}_\circ : x \mapsto x \circ x$
- This maps into a subquasigroup of idempotents of $(X, \circ, /, \backslash)$.
- $(\mathsf{Sq}_\circ(X), \circ)$ is a subquandle of $(X, \triangleright_r)$.
- For any $e \in X$, $X_e = \{x \in X \mid \mathsf{Sq}_\circ(x) = \mathsf{Sq}_\circ(e)\}$ is a subquasigroup of $(X, \circ, /, \backslash)$, and $(X_e, \backslash, e)$ is a CML_3.
  - Prover9 to the rescue again!

A Structure Theorem (B.N.S.Z.-D.)

Let (X, r) be an LBST. If $\mathsf{Sq}_\circ$ is an endomorphism of $(X, \circ)$, then we have a split exact sequence of quasigroups

$$\{*\} \to X_e \to X \to \mathsf{Sq}_\circ(X) \to \{*\}$$

Another squaring map and a structure theorem

- Consider $\mathsf{Sq}_\circ : x \mapsto x \circ x$
- This maps into a subquasigroup of idempotents of $(X, \circ, /, \backslash)$.
- $(\mathsf{Sq}_\circ(X), \circ)$ is a subquandle of $(X, \triangleright_r)$.
- For any $e \in X$, $X_e = \{x \in X \mid \mathsf{Sq}_\circ(x) = \mathsf{Sq}_\circ(e)\}$ is a subquasigroup of $(X, \circ, /, \backslash)$, and $(X_e, \backslash, e)$ is a CML_3.
  - Prover9 to the rescue again!

A Structure Theorem (B.N.S.Z.-D.)

Let (X, r) be an LBST. If $\mathsf{Sq}_\circ$ is an endomorphism of $(X, \circ)$, then we have a split exact sequence of quasigroups

$$\{*\} \to X_e \to X \to \mathsf{Sq}_\circ(X) \to \{*\}$$

The structure theorem cont.

- Question: Is $\mathrm{Sq}_\circ$ always a homomorphism?
  - A naive Prover9 attempt ran out of memory.
  - A naive Mace4 attempt found no counterexample of order 3, 9, or 27. Ran out of memory at 81.
- Any linear LBST is of the form

$$(x, y) \mapsto (-x + \varphi^{-1}(y), 2\varphi(x)),$$

where $2\varphi^3 + \varphi^2 + \varphi - 1 = 0$.
  - $\mathrm{Sq}_\circ$ is a homomorphism here.

The structure theorem cont.

- Question: Is $\mathrm{Sq}_\circ$ always a homomorphism?
  - A naive Prover9 attempt ran out of memory.
  - A naive Mace4 attempt found no counterexample of order 3, 9, or 27. Ran out of memory at 81.
- Any linear LBST is of the form

$$(x, y) \mapsto (-x + \varphi^{-1}(y), 2\varphi(x)),$$

where $2\varphi^3 + \varphi^2 + \varphi - 1 = 0$.

  - $\mathrm{Sq}_\circ$ is a homomorphism here.

The structure theorem cont.

- Question: Is $\mathrm{Sq}_\circ$ always a homomorphism?
  - A naive Prover9 attempt ran out of memory.
  - A naive Mace4 attempt found no counterexample of order 3, 9, or 27. Ran out of memory at 81.
- Any linear LBST is of the form

$$(x, y) \mapsto (-x + \varphi^{-1}(y), 2\varphi(x)),$$

where $2\varphi^3 + \varphi^2 + \varphi - 1 = 0$.
  - $\mathrm{Sq}_\circ$ is a homomorphism here.

The structure theorem cont.

- Question: Is $\mathrm{Sq}_\circ$ always a homomorphism?
  - A naive Prover9 attempt ran out of memory.
  - A naive Mace4 attempt found no counterexample of order 3, 9, or 27. Ran out of memory at 81.
- Any linear LBST is of the form

$$(x, y) \mapsto (-x + \varphi^{-1}(y), 2\varphi(x)),$$

where $2\varphi^3 + \varphi^2 + \varphi - 1 = 0$.
  - $\mathrm{Sq}_\circ$ is a homomorphism here.

Generalizations of LBST

- Both
  1. $r : \mathbb{Z}_n^2 \to \mathbb{Z}_n^2; (x, y) \mapsto (2x - y, x)$
  2. $r : \mathbb{Z}_n^2 \to \mathbb{Z}_n^2; (x, y) \mapsto (2x + y, -x)$

  are SYBE solutions satisfying D_n-relations $r^n = (tr)^2 = \mathrm{id}_{X^2}$.
- Given any pointed involutive quandle, $(X, \cdot, e)$,

$$r : (x, y) \mapsto (x \cdot (e \cdot y), e \cdot x) = (S_x(S_e(y)), S_e(x))$$

is a SYBE solution with D_∞-symmetry: $(tr)^2 = \mathrm{id}_{X^2}$.

Generalizations of LBST

- Both
  1. $r : \mathbb{Z}_n^2 \to \mathbb{Z}_n^2; (x, y) \mapsto (2x - y, x)$
  2. $r : \mathbb{Z}_n^2 \to \mathbb{Z}_n^2; (x, y) \mapsto (2x + y, -x)$

  are SYBE solutions satisfying D_n-relations $r^n = (tr)^2 = \mathrm{id}_{X^2}$.
- Given any pointed involutive quandle, $(X, \cdot, e)$,

$$r : (x, y) \mapsto (x \cdot (e \cdot y), e \cdot x) = (S_x(S_e(y)), S_e(x))$$

  is a SYBE solution with D_∞-symmetry: $(tr)^2 = \mathrm{id}_{X^2}$.

Braided dihedral sets

Definition:

Let (X, r) be a Latin, braided set. If $(tr)^2 = \mathrm{id}_{X^2}$, (X, r) is a Latin, braided, dihedral set (LBDS).

Proposition: B.N.S.Z.-D.

Let (X, r) be a LBDS. Then

1. $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$;
2. the structure rack $(X, \triangleright_r)$ is an involutive quandle.

- Because, $\backslash_\circ$ is not necessarily commutative, they lack the rigidity of LBST.

Braided dihedral sets

Definition:

Let (X, r) be a Latin, braided set. If $(tr)^2 = \mathrm{id}_{X^2}$, (X, r) is a Latin, braided, dihedral set (LBDS).

Proposition: B.N.S.Z.-D.

Let (X, r) be a LBDS. Then

1. $r : (x, y) \mapsto (x \circ y, x \backslash_\circ x)$;
2. the structure rack $(X, \triangleright_r)$ is an involutive quandle.

- Because, $\backslash_\circ$ is not necessarily commutative, they lack the rigidity of LBST.

Conjugation in D_n

- In the conjugation quandle $\mathrm{Conj}(D_n)$, products take the form
  - $r^j s \triangleright r^k s = r^{2j-k} s$
  - $r^j \triangleright r^k = r^k$
  - $r^j \triangleright r^k s = r^{2j+k} s$
  - $r^j s \triangleright r^k = r^{-k}$
- All operations in LBST with homomorphic squaring map take one of these forms!
- Is there anything information about a braided set (X, r) hiding in the conjugation quandle of the group $\langle r, t \rangle$?

Conjugation in D_n

- In the conjugation quandle Conj(D_n), products take the form
  - $r^j s \triangleright r^k s = r^{2j-k} s$
  - $r^j \triangleright r^k = r^k$
  - $r^j \triangleright r^k s = r^{2j+k} s$
  - $r^j s \triangleright r^k = r^{-k}$
- All operations in LBST with homomorphic squaring map take one of these forms!
- Is there anything information about a braided set (X, r) hiding in the conjugation quandle of the group $\langle r, t \rangle$?

Conjugation in D_n

- In the conjugation quandle $\mathrm{Conj}(D_n)$, products take the form
  - $r^j s \triangleright r^k s = r^{2j-k} s$
  - $r^j \triangleright r^k = r^k$
  - $r^j \triangleright r^k s = r^{2j+k} s$
  - $r^j s \triangleright r^k = r^{-k}$
- All operations in LBST with homomorphic squaring map take one of these forms!
- Is there anything information about a braided set (X, r) hiding in the conjugation quandle of the group $\langle r, t \rangle$?

Conjugation in D_n

- In the conjugation quandle Conj(D_n), products take the form
  - $r^j s \triangleright r^k s = r^{2j-k} s$
  - $r^j \triangleright r^k = r^k$
  - $r^j \triangleright r^k s = r^{2j+k} s$
  - $r^j s \triangleright r^k = r^{-k}$
- All operations in LBST with homomorphic squaring map take one of these forms!
- Is there anything information about a braided set (X, r) hiding in the conjugation quandle of the group $\langle r, t \rangle$?

Conjugation in D_n

- In the conjugation quandle $\mathrm{Conj}(D_n)$, products take the form
  - $r^j s \triangleright r^k s = r^{2j-k} s$
  - $r^j \triangleright r^k = r^k$
  - $r^j \triangleright r^k s = r^{2j+k} s$
  - $r^j s \triangleright r^k = r^{-k}$
- All operations in LBST with homomorphic squaring map take one of these forms!
- Is there anything information about a braided set (X, r) hiding in the conjugation quandle of the group $\langle r, t \rangle$?

Conjugation in D_n

- In the conjugation quandle $\text{Conj}(D_n)$, products take the form
  - $r^j s \triangleright r^k s = r^{2j-k} s$
  - $r^j \triangleright r^k = r^k$
  - $r^j \triangleright r^k s = r^{2j+k} s$
  - $r^j s \triangleright r^k = r^{-k}$
- All operations in LBST with homomorphic squaring map take one of these forms!
- Is there anything information about a braided set (X, r) hiding in the conjugation quandle of the group $\langle r, t \rangle$?

Other avenues

- Expand the notion of LBDS to $(sr)^2 = 1$, where s is any involutive solution, not just the trivial one.
- Non-Latin dihedral solutions

References

- Lebed, V. *On structure groups of set-theoretic solutions to the Yang-Baxter equation*, Proc. Edinb. Math. Soc. **62** (2019), 683-717.
- Smith, J.D.H, *Quantum idempotence, distributivity, and the Yang-Baxter equation* Comment. Math. Univ. Carol. **57** (2016), 567-583
- Stanovský, D. and P. Vojtěchovský, *Idempotent solutions of the Yang-Baxter equation and twisted group division*, Fund. Math. **255** (2021) 51-68.