BARYCENTRIC ALGEBRAS and BARYCENTRIC COORDINATES

A.B. Romanowska[1], J.D.H. Smith[2], A. Zamojska-Dzienio[1]

[1]Warsaw University of Technology, Warsaw, Poland
[2]Iowa State University, Ames, Iowa, USA

CONTENTS

AFFINE SUBSPACES and CONVEX SUBSETS of $\mathbb{R}^n$

$\mathbb{R}$ — the field of reals; $I^\circ :=]0,1[= (0,1) \subset \mathbb{R}$.

The **line** $L_{x,y}$ through $x, y \in \mathbb{R}^n$:
$L_{x,y} = \{xy\,\underline{p} = x(1-p) + yp \in \mathbb{R}^n \mid p \in \mathbb{R}\}$.

$A \subseteq \mathbb{R}^n$ is a (non-trivial) **affine subspace** of $\mathbb{R}^n$ if together with any two different points x and y it contains the line $L_{x,y}$.

The **line segment** $I_{x,y}$ joining the points x, y:
$I_{x,y} = \{xy\,\underline{p} = x(1-p) + yp \in \mathbb{R}^n \mid p \in I^\circ\}$.

$C \subseteq \mathbb{R}^n$ is a (non-trivial) **convex subset** of $\mathbb{R}^n$ if together with any two different points x and y it contains the line segment $I_{x,y}$.

AFFINE SPACES

R — a subfield of $\mathbb{R}$. An **affine space over** R (or **affine** R**-space**) — an algebra $(A, \underline{R})$, where

$$\underline{R} = \{\underline{p} \mid p \in R\}$$

and

$$xy\underline{p} = \underline{p}(x, y) = x(1 - p) + yp.$$

Note: $(A, \underline{R})$ is equivalent to the algebra

$$\Big(A, \sum_{i=1}^{n} x_i r_i \Big| \sum_{i=1}^{n} r_i = 1\Big).$$

THEOREM: The class of affine R-spaces is a variety (equationally defined class of algebras).

BARYCENTRIC ALGEBRAS

R — a subfield of $\mathbb{R}$; $I^\circ :=]0,1[= (0,1) \subset R$.

Barycentric algebra — an algebra $(A, \underline{I}^\circ)$,
with a binary operation $\underline{p}$ for each operator $p \in I^\circ$,
axiomatized by the following:

idempotence (I): $xx\underline{p} = x$,

skew-commutativity (SC):
$xy\underline{p} = xy\underline{1-p} =: xy\underline{p}'$,

skew-associativity (SA):
$[xy\underline{p}]\, z\, \underline{q} = x\,[yz\underline{q/(p \circ q)}]\, \underline{p \circ q}$

for all $p, q \in I^\circ$, where $p \circ q = (p'q')' = p + q - pq$.

Proposition: The class $\mathcal{B}$ of barycentric algebras is the smallest variety containing the class $\mathcal{C}$ of convex sets.

For all $p, q \in I^\circ$, $\mathcal{B}$ also satisfies:
entropicity (E): $[xy\underline{p}]\,[zt\underline{p}]\,\underline{q} = [xz\underline{q}]\,[yt\underline{q}]\,\underline{p}$, and

distributivity (D): $[xy\underline{p}]\,z\,\underline{q} = [xz\underline{q}]\,[yz\underline{q}]\,\underline{p}$,

$$x\,[yz\underline{p}]\,\underline{q} = [xy\underline{q}]\,[xz\underline{q}]\,\underline{p},$$

and $\mathcal{C}$ satisfies:
the **cancellation laws** (Cl): $(xy\underline{p} = xz\underline{p}) \to (y = z)$.

Proposition: $\mathcal{C}$ is the subquasivariety of the variety $\mathcal{B}$ defined by the cancellation laws.

EXAMPLES OF BARYCENTRIC ALGEBRAS

• **Convex subsets** of affine R-spaces under the operations

$$xy\underline{p} = xp' + yp = x(1-p) + yp$$

for each $p \in I^{\circ}$.

In particular,
• **Polytopes** - finitely generated convex sets.

The minimal set of generators of a polytope P is the set of its vertices (extreme points).

In particular:

• **Simplices**

Proposition: The n-dimensional simplex Δ_n is the free barycentric algebra on $n+1$ free generators $x_0, x_1, \ldots, x_n$ — the vertices of Δ_n.

The elements of Δ_n may be expressed in the **standard form**:

$$(\ldots((x_0 x_1 \underline{p}_1) x_2 \underline{p}_2) \ldots) x_n \underline{p}_n$$

for $p_i \in I$, or as **convex combinations**:

$$x_0 q_0 + \cdots + x_n q_n,$$

where $q_i \in I$ and $\sum_{i=0}^{n} q_i = 1$.

Δ_n is the $\underline{I}^o$-subreduct of the free affine R-space R^n over the same set of generators.

• **Semilattices**

"Stammered" semilattices $(S, \cdot)$ - barycentric algebras with the operation $x \cdot y = xy\underline{p}$ for all $p \in I^\circ$.

Proposition: Stammered semilattices form the only non-trivial proper subvariety $\mathcal{SL}$ of $\mathcal{B}$, defined by

$$xy\underline{p} = xy\underline{q}$$

for all $p, q \in I^\circ$.

• **Semilattice sums**

Lemma: Each barycentric algebra A has a homomorphism ϱ onto a (stammered) semilattice S, with open convex sets A_s as the congruence classes $\varrho^{-1}(s)$ for $s \in S$.

S is the **semilattice replica** of A.
And we say that A is a **semilattice sum** of A_s.

THEOREM: Each barycentric algebra is a semilattice sum of open convex sets.

WALLS

A **wall** of a barycentric algebra $(B, \underline{I}^o)$ — a subset W of B such that

$$\forall\ a, b \in B\,,\ \forall\ \ p \in I^o,\ ab\underline{p} \in W \ \ \Leftrightarrow\ \ a \in W \text{ and } b \in W.$$

The walls of a polytope P are precisely its faces. (0-dimensional faces — its vertices, 1-dimensional faces — its edges.)

The faces of a polytope are again polytopes, and under inclusion, they form a lattice.

A polytope P is the union of its (relative) **boundary** (the union of proper faces) and its (relative) **interior**.

BARYCENTRIC COORDINATES IN A POLYTOPE

Simplex Δ_n in $\mathbb{R}^n$ with ordered set $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ of vertices.

Each element $\mathbf{x}$ of Δ_n may be presented uniquely as the convex combination

$$\mathbf{x} = \mathbf{v}_0 p_0 + \cdots + \mathbf{v}_n p_n,$$

with $p_i \in I$ and $\sum_{i=0}^{n} p_i = 1$.

If $\mathbf{x}$ and $\mathbf{v}_i$ are given by Cartesian coordinates of $\mathbb{R}^n$,
the barycentric coordinates p_i may be calculated by solving the above equation.

Every polytope P with $n+1$ vertices is a homomorphic image of the simplex Δ_n.
Hence each of its elements can also be presented by the above convex combination, however not in a unique way.

A problem which appears in many applications of polytopes:

Given the set V of vertices v_i of a polytope P, find some specific barycentric coordinates of any x of P in some homogeneous way.

One looks for a function that assigns to each point $\mathbf{x} \in P$, the barycentric coordinates $p(\mathbf{x}, \mathbf{v})$ so that $\sum_{\mathbf{v} \in V} p(\mathbf{x}, \mathbf{v}) = 1$ and

$$\mathbf{x} = \sum_{\mathbf{v} \in V} p(\mathbf{x}, \mathbf{v})\mathbf{v},$$

with some specific choice of $p(\mathbf{x}, \mathbf{v}) \in I$.

Some of the methods of solving this problem are based on a decomposition of a polytope into the union of some simplices.

A sample method

DECOMPOSITION THEOREM: P — a k-dimensional polytope with set V of $n+1$ vertices. Fix $\mathbf{v} \in V$.
Then P is the union of simplices isomorphic to Δ_k,
each generated by a $(k+1)$-element subset of V containing $\mathbf{v}$.

Note: Any two simplices of the decomposition D_v of the Decomposition Theorem have a common wall that is a simplex containing $\mathbf{v}$.

Choose a simplex S of D_v. Then each point $\mathbf{a}$ of S
is the convex combination of some vertices of S.
The coefficients of the remaining vertices of P are 0.

Presentation of points of P as affine or convex combination

• The generators of S freely generate the affine space $\mathbb{R}^k$ as well. So one can represent any point of P as an affine combination of the vertices of S. However some of the coordinates p_i may be negative.

• To find convex coordinates of any point $\mathbf{a}$ of P, one needs a method of deciding to which simplex S of D_v the point $\mathbf{a}$ belongs.

BARYCENTRIC COORDINATES IN A POLYGON

A polygon Π will be decomposed as a union of triangles.

Areal coordinates in a triangle

τ_{123} - a triangle spanned by affinely independent elements $\mathbf{v}_1 < \mathbf{v}_2 < \mathbf{v}_3$ of $\mathbb{R}^2$ in counterclockwise order.

Each $\mathbf{x} \in \mathbb{R}^2$ has a unique represention as an **affine combination**

$$\mathbf{x} = \mathbf{v}_1 p_1 + \mathbf{v}_2 p_2 + \mathbf{v}_3 p_3, \tag{1}$$

with $p_1 = 1 - p_2 - p_3$.

The unique solution of (1) with respect to p_1, p_2 and p_3 is given by

$$p_j = \frac{A\left(\mathbf{v}_{j-1}, \mathbf{x}, \mathbf{v}_{j+1}\right)}{A\left(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right)}$$

The suffix addition is taken modulo 3 here, and $A(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the area of the triangle spanned by counterclockwise ordered $\mathbf{a} < \mathbf{b} < \mathbf{c}$.

A point $\mathbf{x}$ different from a vertex belongs to τ_{123} if at least one of p_i is positive and $0 \le p_1, p_2, p_3 < 1$.

If all $p_i > 0$, one obtains classical **areal coordinates** of interior points $\mathbf{x}$ of τ_{123} (Möbius, 1827 and Muggeridge, 1901). If one of p_i is zero, then $\mathbf{x}$ belongs to a side of τ_{123}.

Points outside of τ_{123} have at least one negative coordinate. E.g. $\mathbf{x}$ lies to the left of the line $\mathcal{L}_{12}$ through $\mathbf{v}_1$ and $\mathbf{v}_2$, precisely when $A(\mathbf{v}_2, \mathbf{x}, \mathbf{v}_1) > 0$, and $\mathbf{x}$ lies to the right of the line $\mathcal{L}_{12}$ if $A(\mathbf{v}_2, \mathbf{x}, \mathbf{v}_1) < 0$.

The case of a general polygon

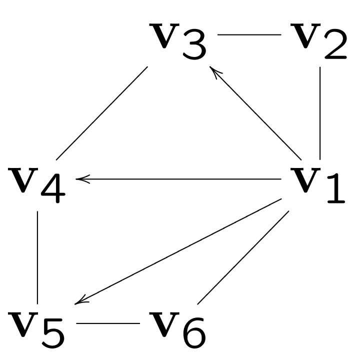
Π - a polygon spanned by counterclockwise ordered vertices $\mathbf{v}_1 < \cdots < \mathbf{v}_n$.

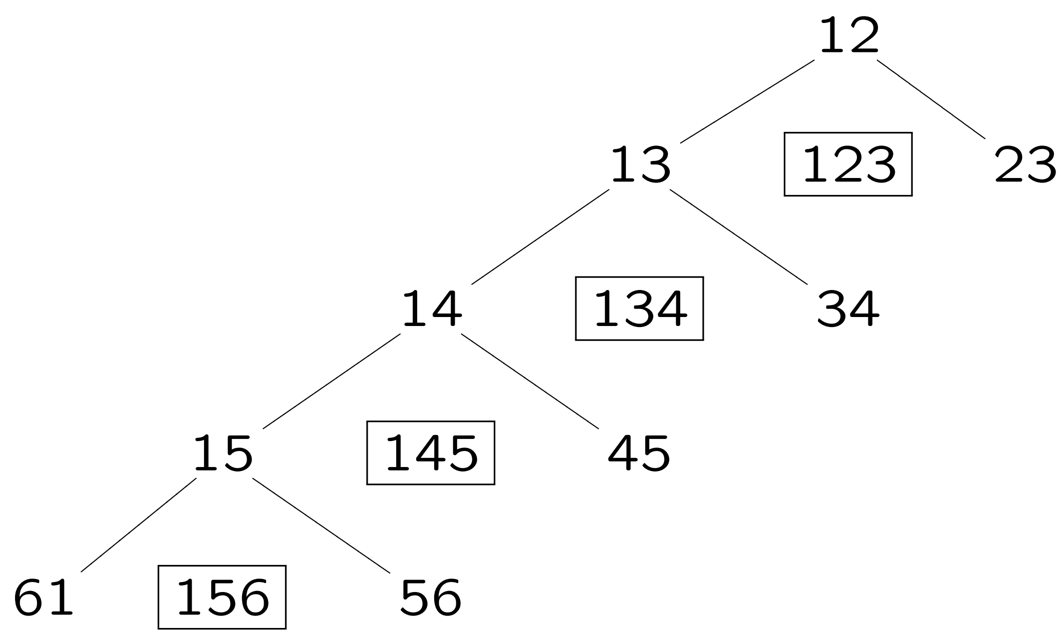
Decomposition Theorem provides the decomposition $\mathcal{D}_1 = \mathcal{D}_{v_1}$ of Π into the union of the triangles:

$$\tau_{123}, \tau_{134}, \cdots, \tau_{1n-1n}. \tag{2}$$

Let $\mathbf{a} \in \Pi$. To find to which triangle $\mathbf{a}$ belongs, one calculates the affine coordinates of $\mathbf{a}$ consecutively in triangles of (2). The first triple i, j, k for which the coordinates are nonnegative provides the triangle τ_{ijk} containing $\mathbf{a}$, and the coordinates p_i, p_j, p_k, all other coordinates are 0.

Chordal decompositions, parsing trees: hexagon examples





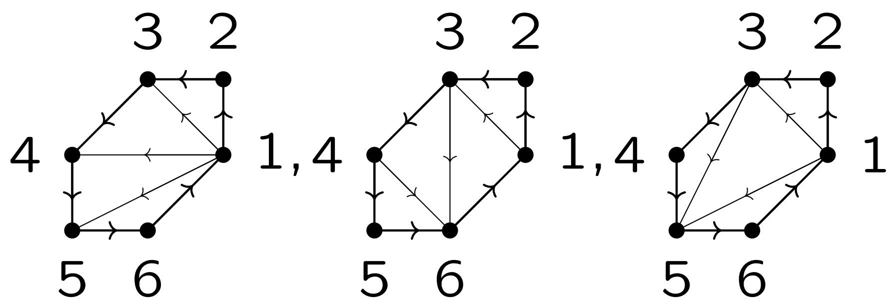
Chordal coordinates in a polygon

The **skeleton** of the polygon Π is the cyclic graph C_n constituted by the vertices and undirected edges of the polygon.
In the cyclic graph C_n, a **chord** is an edge connecting vertices which are not adjacent in C_n.

A **chordal decomposition** of the polygon Π with ordered vertex set $V = \{\mathbf{v}_1 < \mathbf{v}_2 < \cdots < \mathbf{v}_n\}$ is a system of $n - 3$ non-crossing chords of C_n that decompose Π as a union of $n - 2$ simplices (triangles) whose vertices are vertices of Π.

Given a chordal decomposition, one obtains others by the action of the dihedral group D_n.

The hexagon as a representative example



Three distinct types provide a full set of representatives for the orbits of the group D_n on the chordally subdivided graph C_6.

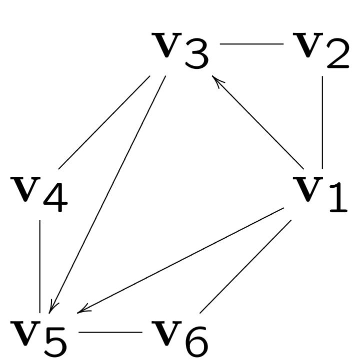
The number of all decompositions equals 14 (**Catalan number**).

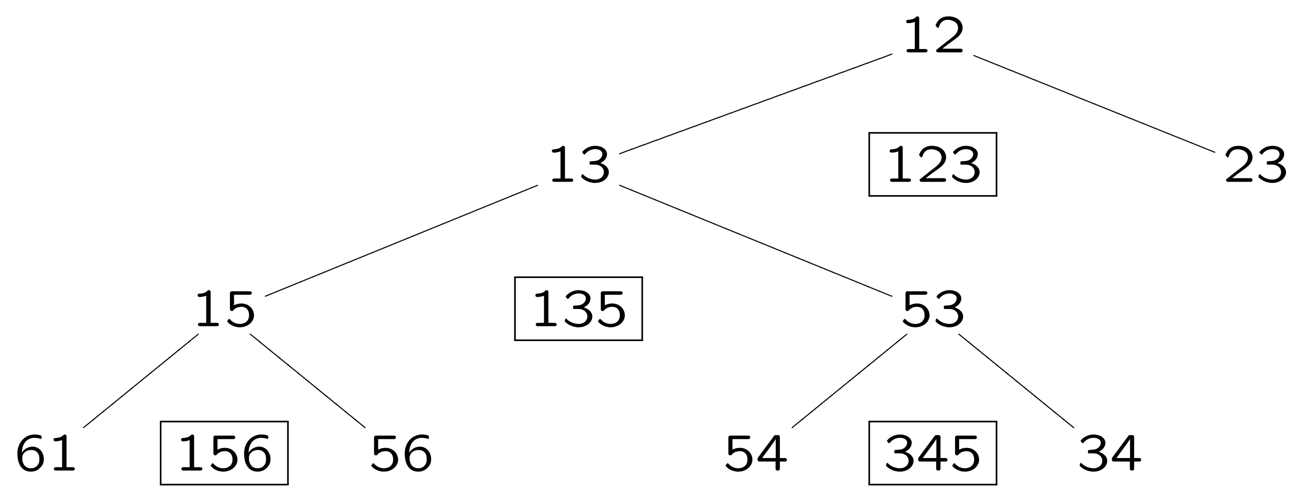
Chordal decompositions and parsing trees

To each chordal decomposition of Π, one assigns a parsing tree. This is shown on the example of a hexagon.

The trees provide a basis for a recursive procedure for triangle identification and orientation including the location of a given point within a triangle. The procedure is founded on the correspondence between chordal decompositions of Π and rooted binary trees.

Chordal decompositions, parsing trees: hexagon examples





Cartographic coordinates

If a point $\mathbf{a}$ belongs to a triangle τ_{ijk} of a chordal decomposition δ of Π, then the **chordal coordinates** p_i, p_j, p_k are the areal coordinates in τ_{ijk}, and all other coordinates are 0.

Any bias introduced by a particular decomposition may subsequently be removed taking the average of a point's coordinates in each of the decompositions appearing in the orbit of a dihedral group. In this way one obtains **cartographic coordinates**.

Some references

• Floater M.S.: Generalized barycentric coordinates and applications, Acta Numer. **24** (2015), 161-–214.
• Möbius A.F.: Der Baryzentrische Calcül, Barth, Leipzig, 1827.
• Muggeridge G.D.: Areal coordinates, Math. Gaz. **2** (1901), 45–51.
• Neumann, W.D.: On the quasivariety of convex subsets of affine spaces, Arch. Math. (Basel) **21** (1970), 11–16.
• Romanowska, A.: Convex sets and barycentric algebras, Contemp. Math. **721** (2019), 243–259.
• Romanowska, A.B., Smith, J.D.H.: Modal Theory, Heldermann, Berlin, 1985.
• Romanowska, A.B., Smith, J.D.H.: Modes, World Scientific, Singapore, 2002.
• Warren J.: Barycentric cordinates for convex polytopes, Adv. Comput. Math. **6** (1996), 97–108.

Thank you for your attention!