Superquasigroups and their multiplication groups

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In other words: the parity map $p\colon (Q,\cdot,/,\backslash) \to (\mathbb{Z}/_2,+,+,+)$ is a quasigroup homomorphism: $|x\cdot y| = |x/y| = |x\backslash y| = |x| + |y|$.

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Using the "'tautological" superset $1_{\mathbb{Z}/_2} \colon \mathbb{Z}/_2 \to \mathbb{Z}/_2$, have

$$[(T_0 \times \{0\}) \uplus (T_1 \times \{1\})] \uplus [(T_0 \times \{1\}) \uplus (T_1 \times \{0\})]$$

from any superset $T = T_0 \uplus T_1$. Creates new superquasigroups.

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Multiplication supergroup $\mathrm{MLT}\, Q$ of superquasigroup Q

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Example: $C_4 = \langle i \rangle = \{\pm 1\} \uplus \{\pm i\}$ with transversal $\{1\} \uplus \{i\}$.

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Example: Dihedral group $D_4 = B_4 \widehat{\times} B_4$ from the Boolean group (Klein Vierergruppe) $B_4 = \{\pm 1\} \uplus \{\pm i\}$ on transversal

	1	i
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i	i	1

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$*$	1	i	-1	$-i$	e	ie	$-e$	$-ie$
1	1	i	-1	$-i$	e	ie	$-e$	$-ie$
i	i	-1	$-i$	1	ie	e	$-ie$	$-e$
-1	-1	$-i$	1	i	$-e$	$-ie$	e	ie
$-i$	$-i$	1	i	-1	$-ie$	$-e$	ie	e
e	e	ie	$-e$	$-ie$	1	$-i$	-1	i
ie	ie	$-e$	$-ie$	e	$-i$	-1	i	1
$-e$	$-e$	$-ie$	e	ie	-1	i	1	$-i$
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- Respective Frobenius-Schur indicators, expected value of $\chi(x^2)$, are $-1, 0, 1$ for the nonlinear character.

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even els., odd perms!

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- $|\mathrm{Mlt}(S,*)| = 64$. Which group of order 64, degree 8?
- $\mathrm{MLT}(S,*)_0 \cong C_2^2 \wr C_2$: unique order 32, subgroup C_2^4, degree 8.

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Theorem: The quatedral loop has multiplication group $C_2 \wr C_2^2$.

Thank you for your attention!