Varieties of quasigroups with invertibility property

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Preliminaries

An algebra $(Q; \cdot; \overset{\ell}{\cdot}; \overset{r}{\cdot})$ with identities

$$(x \cdot y) \overset{\ell}{\cdot} y = x, \quad (x \overset{\ell}{\cdot} y) \cdot y = x, \quad x \overset{r}{\cdot} (x \cdot y) = y, \quad x \cdot (x \overset{r}{\cdot} y) = y \qquad (1)$$

is called a *quasigroup*;

The operation $(\cdot)$ is called *main*, $(\overset{\ell}{\cdot})$, $(\overset{r}{\cdot})$ are called *left* and *right divisions*.

Basic definitions

Let $(Q;\cdot)$ be a quasigroup, L_a, R_a, M_a are *left*, *right* and *middle translations* respectively, if

$$L_a(x) := a \cdot x, \qquad R_a(x) := x \cdot a, \qquad M_a(x) := x \overset{r}{\cdot} a. \tag{2}$$

Therefore,

$$L_a^{-1}(x) = a \overset{r}{\cdot} x, \qquad R_a^{-1}(x) = x \overset{\ell}{\cdot} a, \qquad M_a^{-1}(x) = a \overset{\ell}{\cdot} x. \tag{3}$$

Quasigroups with inverse properties

A *left IP quasigroup* and a *right IP quasigroup* [1] are defined by

$$\lambda(x) \cdot (x \cdot y) = y \quad (4)$$

$$(y \cdot x) \cdot \rho(x) = y \quad (5)$$

i.e. $L_x^{-1} = L_{\lambda x}$, $R_x^{-1} = R_{\rho x}$ for all $x \in Q$. In other words,

$$\{L_x^{-1} \mid x \in Q\} = \{L_x \mid x \in Q\}, \quad (6)$$

$$\{R_x^{-1} \mid x \in Q\} = \{R_x \mid x \in Q\}. \quad (7)$$

[1] Belousov V.D. Foundations of the theory of quasigroups and loops. Nauka (1967), 222 (Russian)

The problem

Every element a of a quasigroup $(Q;\cdot)$ defines six bijections:

$$\mathscr{M}_a := \{M_a, M_a^{-1}, L_a, L_a^{-1}, R_a, R_a^{-1}\}. \tag{8}$$

Problem

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Which quasigroup classes are defined by all equalities of the translation sets?

Methods of parastrophy symmetry

Parastrophes of an invertible operation

Each inverse of an invertible operation $(\cdot)$ is also invertible. All such operations are called *parastrophes* of $(\cdot)$ and they are defined by

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} :\Leftrightarrow x_1 \cdot x_2 = x_3, \tag{9}$$

where $\sigma \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, $\ell := (13)$, $r := (23)$, $s := (12)$.

$${}^{\sigma}(\overset{\tau}{\cdot}) = (\overset{\sigma\tau}{\cdot}) \tag{10}$$

holds for all $\sigma, \tau \in S_3$, thus S_3 acts on the set of all invertible binary operations defined on the carrier set Q. This action will be called the *parastrophy action.*

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Definition

For an element $k \in Q$ its stabilizer group $\mathrm{Ps}(k)$ is called a *parastrophic symmetry group* and its orbit $\mathrm{Po}(k)$ is a *parastrophy orbit*.

$$\mathrm{Ps}(k) := \{\sigma \mid {}^{\sigma}k = k\}, \qquad \mathrm{Po}(k) := \{k_1 \mid (\exists \sigma \in S_3)\; k_1 = {}^{\sigma}k\}. \tag{11}$$

Let P be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$.

Parastrophes of a propositions

A proposition ${}^{\sigma}P$ is said to be a *σ-parastrophe* of a proposition P, if it can be obtained from P by replacing the main operation with its σ^{-1}-parastrophe.

Corollary (F. Sokhatsky, 2016)

Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$.

Sokhatsky F.M. Parastrophic symmetry in quasigroup theory. *Visnyk DonNU, A: natural Sciences*. 2016. Vol.1-2. P. 70–83.

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Theorem (F. Sokhatsky, 2016)

Let P be true in a class of quasigroups $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all $\sigma \in \mathrm{Ps}(\mathfrak{A})$.

Let ${}^{\sigma}\mathfrak{A}$ denote the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$.

A set of all pairwise parastrophic classes is called *a parastrophy orbit of the class* $\mathfrak{A}$:

$$\mathrm{Po}(\mathfrak{A}) := \{{}^{\sigma}\mathfrak{A} \mid \sigma \in S_3\} = \{\mathfrak{A}, {}^{\ell}\mathfrak{A}, {}^{r}\mathfrak{A}, {}^{s}\mathfrak{A}, {}^{s\ell}\mathfrak{A}, {}^{sr}\mathfrak{A}\}. \qquad (12)$$

The parastrophy orbit of varieties is uniquely defined by one of its varieties.

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Translations

It is well known that each element defines the same set of bijections in each parastrophe of a quasigroup. But what is the dependence between translations in a quasigroup and translations in σ-parastrophe of the quasigroup?
Belousov V.(1969), Duplak J.(2000), Shcherbakov V.(1991) parastrophes of translations presented in the table

	ι	s	ℓ	r	$s\ell$	sr
L_a	L_a	R_a	M_a^{-1}	L_a^{-1}	R_a^{-1}	M_a
R_a	R_a	L_a	R_a^{-1}	M_a	M_a^{-1}	L_a^{-1}
M_a	M_a	M_a^{-1}	L_a^{-1}	R_a	L_a	R_a^{-1}
L_a^{-1}	L_a^{-1}	R_a^{-1}	M_a	L_a	R_a	M_a^{-1}
R_a^{-1}	R_a^{-1}	L_a^{-1}	R_a	M_a^{-1}	M_a	L_a
M_a^{-1}	M_a^{-1}	M_a	L_a	R_a^{-1}	L_a^{-1}	R_a

[2] V.D. Belousov. *The group associated with a quasigroup*. Math. Issed. 4(1969), no.3, 21–39 (in Russian).
[3] J. Duplák, A parastrophic equivalence in quasigroups, Quasigroups Relat. Syst. 7(2000), 7–14
[4] Shcherbakov V. *О линейных квазигруппах и их группах автоморфизмов*. Матем. исследования. Кишинев,

The definition of the σ-*parastrophe* of left, right and middle translation are:

$$^{\sigma}L_a(x) := a \overset{\sigma^{-1}}{\cdot} x, \qquad ^{\sigma}R_a(x) := x \overset{\sigma^{-1}}{\cdot} a, \qquad ^{\sigma}M_a(x) := x \overset{r\sigma^{-1}}{\cdot} a, \quad (13)$$

where $\sigma \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$.

If a translation or its inverse defined by a has the direction τ in σ-parastrophe, then it has the direction $\nu\tau$ in the $\nu\sigma$-parastrophe

$$^{\tau}(^{\sigma}L_a) = {}^{\tau\sigma}L_a, \quad ^{\tau}(^{\sigma}R_a) = {}^{\tau\sigma}R_a, \quad ^{\tau}(^{\sigma}M_a) = {}^{\tau\sigma}M_a. \qquad (14)$$

F. M. Sokhatsky, A. V. Lutsenko, Classification of quasigroups according to directions of translations I. Commentationes Mathematicae Universitatis Carolinae. 2020. Vol. 61, No. 4. P. 567–579.

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Directions of translations

Proposition

In all parastrophes of a quasigroup $(Q;\cdot)$ an element a defines the same set of translations $\mathscr{M}_a$ (see (8)). Moreover,

$$\mathscr{M}_a = \{{}^{\iota}M_a, {}^{s}M_a, {}^{\ell}M_a, {}^{r}M_a, {}^{\ell s}M_a, {}^{rs}M_a\} = \{{}^{\sigma}M_a \mid \sigma \in S_3\}. \tag{15}$$

Namely, the following equalities are true

$$\begin{array}{lll} {}^{\iota}M_a = M_a, & {}^{\ell s}M_a = L_a, & {}^{r}M_a = R_a, \\ {}^{s}M_a = M_a^{-1}, & {}^{\ell}M_a = L_a^{-1}, & {}^{rs}M_a = R_a^{-1}. \end{array} \tag{16}$$

Directions of translations

The transformation ${}^{\kappa}M_a$ will be called *κ-translation* or *κ-parastrophe of the translation M_a*, the permutation κ will be called the *direction* of ${}^{\kappa}M_a$.

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New classes of quasigroup with inverse properties

There are 9 quasigroup classes:

1. middle, left and right IP quasigroups:

$$xy = \mu(yx); \qquad \lambda(x) \cdot xy = y; \qquad yx \cdot \rho(x) = y;$$

2. middle, left and right CIP quasigroups:

$$xy \cdot \alpha(x) = y; \qquad yx \cdot y = \beta(x); \qquad y \cdot xy = \gamma(x);$$

3. middle, left and right mirror quasigroups:

$$\varphi(x) \cdot y = y \cdot x; \qquad y \cdot yx = \delta(x); \qquad xy \cdot y = \xi(x).$$

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Inverse property quasigroups

Theorem 1

Each equality of two translation sets of different directions determines exactly one class of quasigroups. Namely,
1) the parastrophy orbit of (middle, left, right) inverse property quasigroups:

$^{\iota}\mathcal{M} = {}^{s}\mathcal{M}$	$M_x^{-1} = M_{\mu(x)}$	$yz = \mu(zy)$	MIP quas.
$^{\ell}\mathcal{M} = {}^{\ell s}\mathcal{M}$	$L_x^{-1} = L_{\lambda(x)}$	$\lambda(x) \cdot xy = y$	LIP quas.
$^{r}\mathcal{M} = {}^{rs}\mathcal{M}$	$R_x^{-1} = R_{\rho(x)}$	$yx \cdot \rho(x) = y$	RIP quas.

Cross inverse quasigroups

2) the parastrophy orbit of cross inverse quasigroups:

${}^{\ell}\mathcal{M} = {}^{r}\mathcal{M}$ ${}^{rs}\mathcal{M} = {}^{\ell s}\mathcal{M}$	$L_x^{-1} = R_{\alpha(x)}$ $R_x^{-1} = L_{\alpha(x)}$	$\alpha(x) \cdot yx = y$	CIP quas. $MCIP$ quas.
${}^{\iota}\mathcal{M} = {}^{rs}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{r}\mathcal{M}$	$R_x^{-1} = M_{\beta(x)}$ $M_x^{-1} = R_{\beta(x)}$	$xy \cdot x = \beta(y)$	$LCIP$ quas.
${}^{\ell s}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{\ell}\mathcal{M}$	$L_x = M_{\gamma(x)}$ $M_x^{-1} = L_{\gamma(x)}^{-1}$	$x \cdot yx = \gamma(y)$	$RCIP$ quas.

Mirror quasigroups

3) the parastrophy orbit of mirror quasigroups:

${}^{sr}\mathcal{M} = {}^{r}\mathcal{M}$ ${}^{\ell}\mathcal{M} = {}^{rs}\mathcal{M}$	$L_x = R_{\varphi(x)}$ $R_x^{-1} = L_{\varphi(x)}^{-1}$	$\varphi(x) \cdot y = y \cdot x$	MMP
${}^{r}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{rs}\mathcal{M}$	$R_x = M_{\delta(x)}$ $M_x^{-1} = R_{\delta(x)}^{-1}$	$x \cdot xy = \delta(y)$	LMP
${}^{\ell}\mathcal{M} = {}^{\iota}\mathcal{M}$ ${}^{s}\mathcal{M} = {}^{\ell s}\mathcal{M}$	$L_x^{-1} = M_{\xi(x)}$ $M_x^{-1} = L_{\xi(x)}$	$xy \cdot y = \xi(x)$	RMP

Theorem 2(A. Lutsenko, F. Sokhatsky)

Each variety of parastrophic orbit of IP quasigroups can be described by the following identities:

Variety	Defining formula	Defining identity
$\mathfrak{I} = {}^{s}\mathfrak{I}$	$xy = \mu(yx)$	$yx = z \cdot (xy \overset{\ell}{\cdot} z)$
${}^{\ell}\mathfrak{I} = {}^{sr}\mathfrak{I}$	$\lambda(x) \cdot xy = y$	$(z \overset{\ell}{\cdot} xz) \cdot xy = y$
${}^{r}\mathfrak{I} = {}^{s\ell}\mathfrak{I}$	$yx \cdot \rho(x) = y$	$yx \cdot (zx \overset{r}{\cdot} z) = y$

Corollary 1

For each variety of parastrophic orbit of IP quasigroups, the invertibility function will have the form:

$$\mathfrak{I} = {}^{s}\mathfrak{I}: \quad (\forall z)\ \mu = L_z R_z^{-1} = {}^{\ell s}M_z {}^{rs}M_z;$$

$$ {}^{\ell}\mathfrak{I} = {}^{sr}\mathfrak{I}: \quad (\forall z)\ \lambda = M_z^{-1} R_z = {}^{s}M_z {}^{r}M_z;$$

$$ {}^{r}\mathfrak{I} = {}^{s\ell}\mathfrak{I}: \quad (\forall z)\ \rho = M_z L_z = {}^{\iota}M_z {}^{\ell s}M_z$$

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Theorem 3

Each variety of the parastrophic orbit of CIP quasigroups can be described by the following identities:

Variety	Defining formula	Defining identity
$\mathfrak{C} = {}^s\mathfrak{C}$	$\alpha(x) \cdot yx = y$	$xy \cdot (xz \overset{r}{\cdot} z) = y$
${}^\ell\mathfrak{C} = {}^{sr}\mathfrak{C}$	$yx \cdot y = \beta(x)$	$yx \cdot y = zx \cdot z$
${}^r\mathfrak{C} = {}^{s\ell}\mathfrak{C}$	$y \cdot xy = \gamma(x)$	$y \cdot xy = z \cdot xz$

Corollary 2

For each variety of parastrophic orbit of CIP quasigroups, the invertibility function will have the form:

$$\mathfrak{C} = {}^s\mathfrak{C}: \quad (\forall z)\ \alpha = M_z R_z = {}^\iota M_z {}^r M_z;$$

$$ {}^\ell\mathfrak{C} = {}^{sr}\mathfrak{C}: \quad (\forall z)\ \beta = R_z L_z = {}^r M_z {}^{\ell s} M_z;$$

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$${}^{r}\mathfrak{C} = {}^{s\ell}\mathfrak{C}: \quad (\forall z)\ \gamma = L_z R_z = {}^{\ell s}M_z {}^{r}M_z$$

Theorem 4

Each variety of the parastrophic orbit of mirror quasigroups can be described by the following identities:

Variety	Defining formula	Defining identity
$\mathfrak{M} = {}^{s}\mathfrak{M}$	$\varphi(x) \cdot y = y \cdot x$	$(zx \overset{\ell}{\cdot} z) \cdot y = yx$
${}^{\ell}\mathfrak{M} = {}^{sr}\mathfrak{M}$	$y \cdot yx = \delta(x)$	$y \cdot yx = z \cdot zx$
${}^{r}\mathfrak{M} = {}^{s\ell}\mathfrak{M}$	$xy \cdot y = \xi(x)$	$xy \cdot y = xz \cdot z$

Corollary 3

For each variety of parastrophic orbit of mirror quasigroups, the invertibility function will have the form:

$$\mathfrak{M} = {}^{s}\mathfrak{M}: \quad (\forall z)\ \varphi = R_z^{-1} L_z = {}^{rs}M_z\, {}^{\ell s}M_z;$$

$${}^{\ell}\mathfrak{M} = {}^{sr}\mathfrak{M}: \quad (\forall z)\ \delta = L_z^2 = ({}^{\ell s}M_z)^2;$$

$${}^{r}\mathfrak{M} = {}^{s\ell}\mathfrak{M}: \quad (\forall z)\ \xi = R_z^2 = ({}^{r}M_z)^2$$

Theorem 4

Each variety of the parastrophic orbit of mirror quasigroups can be described by the following identities:

Variety	Defining formula	Defining identity
$\mathfrak{M} = {}^s\mathfrak{M}$	$\varphi(x) \cdot y = y \cdot x$	$(zx \overset{\ell}{\cdot} z) \cdot y = yx$
${}^\ell\mathfrak{M} = {}^{sr}\mathfrak{M}$	$y \cdot yx = \delta(x)$	$y \cdot yx = z \cdot zx$
${}^r\mathfrak{M} = {}^{s\ell}\mathfrak{M}$	$xy \cdot y = \xi(x)$	$xy \cdot y = xz \cdot z$

Corollary 3

For each variety of parastrophic orbit of mirror quasigroups, the invertibility function will have the form:

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$${}^r\mathfrak{M} = {}^{s\ell}\mathfrak{M}: \quad (\forall z)\ \xi = R_z^2 = ({}^r M_z)^2$$

About group isotopes with inverse properties

A quasigroup is called group isotope, if it is isotopic to a group.

Let $(Q;\circ)$ be a group isotope and let $0 \in Q$, then

$$x \circ y = \alpha x + a + \beta y \tag{17}$$

is called a *0-canonical decomposition*, if $(Q;+;0)$ is a group and $\alpha 0 = \beta 0 = 0$.

An arbitrary element of a group isotope uniquely defines its canonical decomposition[3].

A quasigroup $(Q;\cdot)$ is called *linear*, if it is a group isotope and coefficients of a canonical decomposition are automorphisms of the canonical decomposition group.

[3] Sokhatsky F.M. On group isotopes II, Ukrainian Math.J., 47(12) (1995), 1935–1948.

Theorem 5

Let $(Q;\circ)$ be a group isotope and

$$x \circ y = \alpha x + a + \beta y$$

be its canonical decomposition, then:

1) $(Q;\circ)$ is RIP quasigroup with invertibility function ρ if and only if α an involutive automorphism of $(Q;+)$ and $\rho(x) = J\beta^{-1}a + J\beta^{-1}\alpha\beta x + J\beta^{-1}\alpha a$,
2) $(Q;\circ)$ is LIP quasigroup with invertibility function λ if and only if β an involutive automorphism of $(Q;+)$ and $\lambda(x) = J\alpha^{-1}\beta a + J\alpha^{-1}\beta\alpha x + J\alpha^{-1}a$,
3) $(Q;\circ)$ is MIP quasigroup with invertibility function μ if and only if there exist anti-automorphism θ such that

$$\mu x = \theta x + c, \qquad \theta^2 = I_c^{-1}, \qquad \alpha = \theta\beta, \tag{18}$$

where $c := -\theta a + a$.

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where $c := -\theta a + a$.

Examples

Example

Consider the quasigroup $(\mathbb{Z}_8; \circ)$, where $\mathbb{Z}_8$ is a ring modulo 8 and

$$x \circ y := 5x + 4 + 3y.$$

The quasigroup $(\mathbb{Z}_8; \circ)$ is a left, right and middle IP-quasigroup with the invertibility functions $\lambda(x) = 5x$, $\rho(x) = 3x$, $\mu(x) = 7x$.

Example

Consider the quasigroup $(\mathbb{Z}_{15}; \circ)$ with the canonical decomposition

$$x \circ y = 2x + 3 + 4y.$$

have left, right and middle inverse properties, in particular $\lambda(x) = \rho(x) = \mu(x) = 11x$.

Examples

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Theorem 6

Let $(Q;\circ)$ be a group isotope and $x\circ y=\alpha x+a+\beta y$ be its canonical decomposition, then:

1) $(Q;\circ)$ is $MCIP$ quasigroup with invertibility function φ if and only if there exist anti-automorphism α of group $(Q;+)$ and $\beta=\alpha^{-1},\quad \varphi(x)=-\alpha^{-2}a-\alpha^{-3}x-\alpha^{-1}a,$
2) $(Q;\circ)$ is $LCIP$ quasigroup with invertibility function δ if and only if there exist anti-automorphism α of group $(Q;+)$ and $\beta=I_aJ\alpha^2,\quad \delta(x)=\alpha\beta x+\alpha a+a,$
3) $(Q;\circ)$ is $RCIP$ quasigroup with invertibility function γ if and only if there exist anti-automorphism β of group $(Q;+)$ and $\alpha=I_a^{-1}J\beta^2,\quad \gamma(x)=a+\beta a+\beta\alpha x.$

Theorem 7

If a group isotope has two of the three invertibility properties: $LCIP$, $RCIP$, $MCIP$, then it also satisfies the third property.

A. Lutsenko, Classification of group isotopes according to their inverse properties. Applied problems of mechanics

Theorem 6

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Example

1) The quasigroup $(\mathbb{Z}_5; \circ)$ where

$$x \circ y := 2x + 2 + 3y.$$

is a middle CIP quasigroup and it is neither left CIP quasigroup nor right CIP quasigroup.
2) Consider the quasigroup $(\mathbb{Z}_{11}; \circ)$ with the canonical decomposition

$$x \circ y = 3x + 5 + 4y.$$

$(\mathbb{Z}_{11}; \circ)$ is a middle CIP-quasigroup with the invertibility function $\gamma x = 2x - 1$.

Theorem 8

Let $(Q;\cdot)$ be a group isotope and (17) be its canonical decomposition, then:

1) $(Q;\cdot)$ is a middle mirror quasigroup with invertibility function φ if and only if $(Q;+)$ is abelian and $\beta = \alpha$, $\varphi = \iota$;
2) $(Q;\cdot)$ is left mirror quasigroup with invertibility function δ if and only if $(Q;+)$ is abelian and $\beta = -\iota$, $\delta = \iota$;
3) $(Q;\cdot)$ is right mirror quasigroup with invertibility function ξ if and only if $(Q;+)$ is abelian and $\alpha = -\iota$, $\xi = \iota$.

Corollary

If a group isotope has at least two of the following properties: left mirror, right mirror, middle mirror, then it also satisfies the third one.

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Let $(Q;\cdot)$ be a group isotope and (17) be its canonical decomposition, then:

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Corollary

In the variety of group isotopes the following assertions are true:

1) the subvariety of middle mirror quasigroups coincides with the subvariety of commutative quasigroups;
2) the subvariety of left mirror quasigroups coincides with the subvariety of left symmetric quasigroups;
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1) the quasigroup $(\mathbb{Z}_7; \cdot)$, where

$$x \cdot y = 4x + 2 + 4y$$

over the field $\mathbb{Z}_7$ belongs to the variety of middle mirror quasigroups;
2)the quasigroup $(\mathbb{Z}_5; *)$, where

$$x * y = 5x + 3 + 4y$$

over the field $\mathbb{Z}_5$ belongs to the variety of left mirror quasigroups;
3)the quasigroup $(\mathbb{Z}_9; \circ)$, where

$$x \circ y = 8x + 1 + 3y$$

over the ring $\mathbb{Z}_9$ belongs to the variety of right mirror quasigroups.

Classification group isotope with inverse properties

Variety	Group isotope	Conditions of canonical decomposition (17)
$\mathfrak{I}$	MIP	$\alpha = \theta\beta$, $\mu(x) = \theta x - \theta a + a$, $\theta^2 = I_c^{-1}, \quad c := -\theta a + a$
${}^{\ell}\mathfrak{I}$	LIP	$\beta^2 = \iota$, $\lambda(x) = J\alpha^{-1}\beta a + J\alpha^{-1}\beta\alpha x + J\alpha^{-1}a$
${}^{r}\mathfrak{I}$	RIP	$\alpha^2 = \iota$, $\rho(x) = J\beta^{-1}a + J\beta^{-1}\alpha\beta x + J\beta^{-1}\alpha a$
$\mathfrak{C}$	$MCIP$	$\beta = \alpha^{-1}, \quad \gamma(x) = -\alpha^{-2}a - \alpha^{-3}x - \alpha^{-1}a,$
${}^{\ell}\mathfrak{C}$	$LCIP$	$\beta = I_aJ\alpha^2, \quad \gamma(x) = \alpha\beta x + \alpha a + a$
${}^{r}\mathfrak{C}$	$RCIP$	$\alpha = I_a^{-1}J\beta^2, \quad \gamma(x) = a + \beta a + \beta\alpha x$
$\mathfrak{M}$	MMP	$(Q;+)$ – Abelian group, $\beta = \alpha, \quad \phi = \iota,$
${}^{\ell}\mathfrak{M}$	LMP	$(Q;+)$ – Abelian group, $\beta = -\iota$ $(\beta = I_aJ), \phi = \iota$
${}^{r}\mathfrak{M}$	RMP	$(Q;+)$ – Abelian group, $\alpha = -\iota$ $(\alpha = I_a^{-1}J), \phi = \iota$

Matrix quasigroups

Let K be a commutative ring with a unit element and $K^n := K \times \ldots \times K$. The groupoid $(K^n; f)$ being defined by

$$f(\bar{x}, \bar{y}) = \bar{x}A + \bar{y}B + \bar{a}, \tag{19}$$

where A, $B \in M_n(K)$ and $\bar{a} \in K^n$, is called *matrix quasigroup over the ring* K if the matrix A, B are invertible.

The groupoid $(K^n; f)$ being defined by

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Each matrix quasigroup is central.

Each central quasigroup being isotopic to an elementary abelian group is isomorphic to a matrix quasigroup.

In the class of matrix quasigroups, it is sufficient to consider two parastrophic orbits since all mirror quasigroups are IP quasigroups. Classification of matrix IP and CIP quasigroups was considered in the work.

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Theorem 10

Let $(K^n; f, \bar{0})$ unitary matrix quasigroup and (20) be its canonical decomposition, then:

1) $(K^n; f, \bar{0})$ is a middle IP quasigroup if and only if there exists an invertible matrix C such that $C^2 = E$, $B = AC$. The invertibility function μ is calculated according to the formula $\mu(\bar{x}) = \bar{x}C$;
2) $(K^n; f, \bar{0})$ is a left IP quasigroup if and only if $B^2 = E$. The invertibility function λ is calculated according to the formula $\lambda(\bar{x}) = -\bar{x}ABA^{-1}$;
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The form of the one-sided, two-sided, three-sided IP quasigroup is found in the following theorem.

Theorem 12

Each matrix IP quasigroup over the ring K has the form:

middle IP quasigroup	$f(\bar{x}, \bar{y}) = \bar{x}A + \bar{y}AC + \bar{a}$
left IP quasigroup	$f(\bar{x}, \bar{y}) = \bar{x}A + \bar{y}C + \bar{a}$
right IP quasigroup	$f(\bar{x}, \bar{y}) = \bar{x}C + \bar{y}A + \bar{a}$
left-middle IP quasigroup	$f(\bar{x}, \bar{y}) = \bar{x}C_1C_2 + \bar{y}C_1 + \bar{a}$
right-middle IP quasigroup	$f(\bar{x}, \bar{y}) = \bar{x}C_1 + \bar{y}C_1C_2 + \bar{a}$
left-right IP quasigroup	$f(\bar{x}, \bar{y}) = \bar{x}C_1 + \bar{y}C_2 + \bar{a}$
left-right-middle sided IP quas.	$f(\bar{x}, \bar{y}) = \bar{x}C_1 + \bar{y}C_2 + \bar{a}$, $C_1C_2 = C_2C_1$

where the matrix A is invertible and $C^2 = C_1^2 = C_2^2 = E$; $\bar{a} \in K^n$.

Corollary 4

The number of different matrix IP quasigroups of the order 4 over the field $\mathbb{Z}_2$:

middle IP quasigroup	96
left IP	96
right IP quasigroup	96
left-middle IP quasigroup	64
right-middle IP quasigroup	64
left-right IP quasigroup	64
left-right-middle sided IP quasigroup	40
	136

Theorem 11

Let $(K^n; f, \bar{0})$ unitary matrix quasigroup and (20) be its canonical decomposition, then:

1) $(K^n; f, \bar{0})$ is a middle CIP quasigroup if and only if $B = A^{-1}$. The invertibility function γ is calculated according to the formula $\gamma(\bar{x}) = -\bar{x}A^{-3}$;
2) $(K^n; f, \bar{0})$ is a left CIP quasigroup if and only if $B = -A^2$. The invertibility function γ is calculated according to the formula $\gamma(\bar{x}) = \bar{x}BA$;
3) $(K^n; f, \bar{0})$ is a right CIP quasigroup if and only if $A = -B^2$. The invertibility function γ is calculated according to the formula $\gamma(\bar{x}) = \bar{x}AB$.

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Thank you for attention!